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Geometry of an end and absence of eigenvalues in the essential spectrum

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1 Abstract

In this note, we shall consider the Laplace operator on a Riemannian manifold and derive a growth estimate at infinity of solutions of eigenvalue equation. Then we assert the absence of eigenvalues in the essential spectrum under some curvature condition of an end. Indeed, we treat two cases;

(1) the curvature K of an end converges to a constant -1 at infinity

(2) the curvature K of an end converges to a constant 0 at infinity with the decay order $K = O(r^{-2})$

Moreover, we should note that the decay order $K + 1 = o(r^{-1})$ is sharp. Indeed, we can construct an example of an manifold with the curvature decay $K + 1 = O(r^{-1})$ and an eigenvalue $\dfrac{(n-1)^2}{4} + 1$ contained in the essential spectrum $[\dfrac{(n-1)^2}{4}, \infty)$.

This note is an announcement of the article [17]. The readers who want to know the details of our arguments should refer to [17].

2 Some facts on the Laplacian on a Riemannian manifold

The following sentence due to Kac seems to symbolize what the spectral geometry is:

Can one hear the shape of the drum ?

In this sentence, the word 'shape' symbolizes 'geometry' and 'drum' symbolizes 'Riemannian manifold'. Moreover, 'drum' gives us sounds and 'sounds' symbolizes 'spectrum of the Laplacian'. Thus the spectral geometry studies the relationship between analytic properties and geometric properties a Riemannian manifold has.

Now let us recall some basic facts about the spectrum of the Laplacian on a Riemannian manifold. Let $M = (M, g)$ be a Riemannian manifold. Then we have two analytic notions, that is, the Laplace operator Δ and Riemannian measure dv_M. These are defined as follows: Let $(x^1, x^2, \cdots, x^n)$ be a local coordinates of M. Then

$$\Delta u = \sum_{i,j} \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \left(\sqrt{G} g^{ij} \frac{\partial}{\partial x^j} u \right),$$
$$dv_M = \sqrt{G} dx^1 dx^2 \cdots dx^n,$$

where

$$(g^{ij}) = (g_{ij})^{-1}, \; g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right), \; G = \det(g_{ij}).$$

We should note that the curvatures controls the metric g, and then the latter controls the Laplacian Δ and measure dv_M.

When M is complete, then the Laplacian Δ defined on $C_0^\infty(M)$ is known to be essentially self-adjoit on the Hilbert space $L^2(M, dv_M)$. We write its self-adjoit extension by the same symbol Δ.

Now let us recall two typical examples of the spectrum of a noncompact Riemannian manifold M:

(1) When M is $\mathbf{H}^n(-1)$, then $\sigma(-\Delta) = [\frac{(n-1)^2}{4}, \infty)$ and $\sigma_p(-\Delta) = \emptyset$. Here $\mathbf{H}^n(-1)$ stands for the simply connected Riemannian manifold with constant sectional curvatures -1.

(2) When M is $\mathbf{R}^n$, then $\sigma(-\Delta) = [0, \infty)$ and $\sigma_p(-\Delta) = \emptyset$.

Next let us recall two facts on the spectrum of noncompact Riemannian manifolds:

(1) The essential spectrum of the Laplacian is invariant under any compact perturbation of the manifold.

(2) When $E_0 = \inf \sigma_{ess}(-\Delta) > 0$, a compact perturbation of the Riemannian metric can give rise to any finite number of point spectrum in the opening $(0, E_0)$.

The examples and facts above have cast the following question on us:

What is the optimal curvature condition that ensures
the absence of eigenvalues in the essential spectrum ?

This problem were studied by Donnelly, Donnelly and N. Garofalo, J. Escobar and A. Freire, L. Karp, M. A. Pinsky, T. Tayoshi. But their results are far from best.

We shall recall known results.

(I) The case that the curvature K of M converges to a constant -1 at infinity

Let us recall the decay conditions on $K + 1$ in the previous works which ensure the absence of eigenvalues greater than $\frac{(n-1)^2}{4}$. The former best results are as follows:

M. A. Pinsky(1979):

(1) $\dim M = 2$ and (M, g) is rotationally symmetric, that is, (M, g) can be written as $(M, g) = (\mathbf{R}^2, dr^2 + f(r)^2 g_{S(1)})$;

(2) $K \leq 0$;

(3) $K \leq -1$ $(r \geq r_0)$;

(4) $\displaystyle\int_{r_0}^{\infty} |K + 1| dr < \infty$,

where r stands the Euclidean distance to the origin.
A general case (i.e., not necessarily rotationally symmetric case) was presented by Donnelly:

Donnelly(1990):

(1) M is a simply connected negatively curved Riemannian manifold;

(2) $\displaystyle\int_{1}^{\infty} r^{\beta} |K + 1| dr < \infty$;

(3) $\displaystyle\lim_{r \to \infty} r^{\beta} |K + 1| = 0$,

where K stands for the radial curvature with respct to a fixed point and $\beta > 2$ is a constant.

Roughly speaking, this Donnelly's curvature condition is $K+1 = O(r^{-3-\varepsilon})$. The curvature condition imposed in the main theorem of this note is $K+1 = o(r^{-1})$. It not only improves the former results but also we can show that this condition $K+1 = o(r^{-1})$ is optimal by constructing an example.

(II) The case that the curvature K of simply connected complete manifold M is nonpositive and it goes to zero at infinity

Let us recall the decay conditions on K in the previous works which ensure the absence of eigenvalues. In this case, the earlier works treated only the case that dim $M = 2$, because their arguments require faster than quadratic decay for K which, in dimensions greater than two, would force M to be isometric with $\mathbf{R}^n$ due to Green-Wu's theorem. That is why this problem for higher dimensions remained a challenge so far. For example,

Donnelly (1993):

(1) $\displaystyle\int_1^\infty r^{\beta_1}|K|dr < \infty.;$

(2) $\displaystyle\lim_{r\to\infty} r^{\beta_2}|K| = 0,$

where $\beta_1 \geq 2$ and $\beta_2 \geq 3$ are constants.

Roughly speaking, this curvature condition is $K = O(r^{-3-\varepsilon})$. In this note, we shall treat manifolds of all dimensions under the assumption of some quadratic decay for the curvature, and show the absence of eigenvalues.

3 Main Theorem–negatively curved end–

In this section, we shall give one of the main theorem of this note. Let M be a noncompact complete Riemannian manifold of dimension n. Suppose that there exists an open subset U of M with compact boundary ∂U such that the outward pointing normal exponential map $\exp_{\partial U} : N^+(\partial U) \to M - \overline{U}$ induces a diffeomorphism. We note that U is not necessarily relatively compact. We shall say that a plane $\pi \subset T_x M$ ($x \in M - \overline{U}$) is a radial plane if π contains ∇r. The radial curvature means the restriction of the sectional curvature to all the radial planes.

In the sequel, we shall use the following notations:

$$B(s,t) = \{x \in M - \overline{U} \mid s < r(x) < t\} \text{ for } 0 \le s < t;$$
$$B(s,\infty) = \{x \in M - \overline{U} \mid s < r(x)\} \text{ for } 0 \le s;$$
$$S(t) = \{x \in M - \overline{U} \mid r(x) = t\} \text{ for } 0 \le t,$$

where we set $r(x) = \mathrm{dist}(\partial U, x)$ for $x \in M - \overline{U}$. Moreover, we denote the induced measure from dv_M on each $S(t)$ $(t > 0)$ simply by dA.

We shall consider the eigenvalue equation

$$\Delta f + \alpha f = 0$$

on an end $M - \overline{U}$ and drive a growth estimate at infinity of solutions f, from which will follow the absence of eigenvalues in the essential spectrum:

Theorem 3.1. *Let M and f be as above. Let an end $M - \overline{U}$ satisfy the following conditions: there exists $r_0 > 0$ such that*

$$\nabla dr \ge 0 \text{ on } S(r_0); \tag{1}$$
$$0 \ge \text{radial curvature} = -1 + o(r^{-1}) \text{ on } B(r_0,\infty).$$

If $\alpha > \dfrac{(n-1)^2}{4}$ and f is a not identically vanishing, then we have for any $\gamma > 0$

$$\liminf_{t\to\infty} t^\gamma \int_{S(t)} \left\{ \left(\frac{\partial f}{\partial r}\right)^2 + f^2 \right\} dA = \infty.$$

Corollary 3.1. *Let M be a complete Riemannian manifold and have at least one end as in Theorem 3.1. Then $[\frac{(n-1)^2}{4}, \infty) \subset \sigma_{ess}(-\Delta)$ and any $\alpha > \frac{(n-1)^2}{4}$ is not eigenvalue of $-\Delta$.*

Remark 3.1. *If M has finite number of ends and each end satisfies the curvature condition as in Theorem 3.1, then $\sigma_{\mathrm{ess}}(-\Delta) = [\frac{(n-1)^2}{4}, \infty)$.*

The following Proposition shows that the curvature decay condition $K + 1 = o(r^{-1})$ in Theorem 3.1 is sharp:

Proposition 3.1. *There exists a rotationally symmetric manifold $M = (\mathbf{R}^n, dr^2 + f^2(r) g_{S^{n-1}(1)})$ with the following properties:*

(1) $\lim_{r\to\infty} |\nabla dr - (g - dr \otimes dr)| = 0$, *and hence* $\sigma_{ess}(-\Delta) = [(n-1)^2/4, \infty)$;

(2) $\sigma_p(-\Delta) \cap (\frac{(n-1)^2}{4}, \infty) = \frac{(n-1)^2}{4} + 1$;

(3) $R + 1 = O(r^{-1})$ *as* $r \to \infty$, *where* R *stands for the radial curvature of* M.

4 Main theorem–asymptotically flat end–

In this section, we shall consider the asymptotically flat end case. The main theorem of this section is the following:

Theorem 4.1. *Let* M *be a complete Riemannian manifold of dimension* n *and suppose that there exists an open subset* U *of* M *with compact boundary* ∂U *such that the outward pointing normal exponential map* $\exp_{\partial U} : N^+(\partial U) \to M - \overline{U}$ *induces a diffeomorphism. We suppose that there exists* $r_0 > 0$ *such that*

$$(\nabla dr)|_{S(r_0)} \begin{cases} \geq \dfrac{a}{r_0}(g - dr \otimes dr) & \text{if } 0 < a < 1 \\ \geq \varepsilon(g - dr \otimes dr) & \text{if } a = 1 \end{cases} \tag{2}$$

$$-\frac{b(b-1)}{r^2} \leq \text{radial curvature} \leq \frac{a(1-a)}{r^2} \quad \text{on } B(r_0, \infty),$$

where $\varepsilon > 0$ *is a constant, and* $a \in (0, 1]$ *and* $b \geq 1$ *are also constants satisfying* $\frac{n+1}{n-1}a > b$. *Let* u *be a nontrivial solution to*

$$\Delta u + \lambda u = 0 \quad on \quad B(r_0, \infty),$$

where $\lambda > 0$ *is a constant. Then*

$$\liminf_{t\to\infty} t^a \int_{S(t)} \left\{ \left(\frac{\partial u}{\partial r}\right)^2 + u^2 \right\} dA \neq 0.$$

In particular, $\sigma(-\Delta) = [0, \infty)$ *and* $-\Delta$ *has no eigenvalue.*

The method of proofs of Theorem 3.1 and 4.1 is a modification of solutions of Eidus and Mochizuki to the analogous problem for the Schrödinger operator on Euclidian space.

5 Outline of the proof of Theorem 3.1

In this section, we shall give the outline of the proof of Theorem 3.1. Theorem 4.1 can be proved in a similar way.

(a) The transform of the Hilbert space and operator

We set $c = \dfrac{n-1}{2}$, $L = e^{2cr}(\Delta + c^2)e^{-cr}$, and $d\mu_c = e^{-2cr}dv_M$. Then we have the following equivalence by setting $u = e^{cr}f$:

$$\begin{cases} (\Delta + c^2)f + \lambda f = 0 \ \text{ on } U^c, \\ f \in L^2(U^c, dv_M) \end{cases}$$

$$\Updownarrow \ u = e^{cr}f$$

$$(*) \begin{cases} Lu + \lambda u = 0 \ \text{ on } U^c, \\ u \in L^2(U^c, d\mu_c) \end{cases}$$

(b) Geometric observation

Under the assumption of Theorem 3.1, we can show that

$$|\nabla dr - (g - dr \otimes dr)| = o\left(\frac{1}{r}\right) \ (r \to \infty)$$

by using the comparison theorem in Riemannian geometry.

(c) Combination of analysis and geometry

The following identity which combines analysis and geometry of M is a key identity in our proof:

$$-\frac{\partial \Delta r}{\partial r} = |\nabla dr|^2 + \mathrm{Ricci}(\nabla r, \nabla r),$$

where Ricci stands for the Ricci curvature of M.

(d) Classical analysis

We consider $(L, d\mu_c)$ instead of (Δ, dv_M) and use **(b)** and **(c)**. Then remaining arguments turn out to be purely analytic ones.

Let u satisfy $(*)$ with $\lambda > 0$ and

$$\liminf_{t\to\infty} t^\gamma \int_{S(t)} \left\{ \left(\frac{\partial u}{\partial r}\right)^2 + u^2 \right\} dA_c = 0$$

for some constant $\gamma > 0$. Then the proof of Theorem 3.1 is accomplished by going through the following four steps:

(1st step) For any $m > 0$,

$$\int_{B(r_0,\infty)} r^m(u^2 + |\nabla u|^2)d\mu_c < \infty.$$

(2nd step) For any $k > 0$,

$$\int_{B(r_0,\infty)} e^{kr}\{u^2 + |\nabla u|^2\} d\mu_c < \infty.$$

(3rd step)

$$u \equiv 0 \qquad \text{on} \;\; B(r_0, \infty).$$

(4th step) Recalling $f = e^{-cr}u$, we transform this result " 3rd step " into Theorem 3.1.

Remark We assume that the convexity assumption (1) or (2). But that is necessary for the absence of eigenvalues. Indeed, we have the following example:

Proposition 5.1. *Let ξ be a unit Killing vector field on the standard unit sphere $(S^3(1), g_0)$ which satisfies* $\ker f_* = \mathbf{R}\xi$, *where $f : S^3(1) \to \mathbf{C}P^1$ is the Hopf fibering. Let $M = (\mathbf{R}^4, g)$, where $g = dr^2 + e^{2r}(g_0 - \xi^* \otimes \xi^*) + e^{-2r}(\xi^* \otimes \xi^*)$ $(r \geq 1)$ and ξ^* is the dual 1-form of ξ on $(S^3(1), g_0)$. Then*

$$\begin{cases} \text{radial curvature} \equiv -1 \;\text{ on } B(1, \infty); \\ \sigma_{ess}(-\Delta) = [\frac{1}{4}, \infty); \\ \forall k \geq 1 : \;\sigma_p(-\Delta) \cap [k, \infty) \neq \emptyset. \end{cases}$$

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