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# On the Mourre estimates for three body Schrödinger operators in a constant magnetic field

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## 1 Introduction

In this article, we study the spectral theory for a three body quantum system in a constant magnetic field which consists of one neutral and two charged particles.

The scattering theory for $N$-body quantum systems in a constant magnetic field has been studied by Gérard-Łaba [GŁ1, GŁ2, GŁ3, GŁ4]). But they have assumed that all particles in the systems are charged, that is, there is no neutral particle in the systems under consideration, even if the systems consist of only two particles (see also [Ł1, Ł2]). Under this assumption, if there is no neutral proper subsystem, one has only to observe the behavior of all subsystems parallel to the magnetic field. However, if the system has neutral particles or clusters, the problem seems more difficult to be solved: For instance, neutral particles can move freely without being influenced by the magnetic field, but charged particles and clusters are bound in the directions perpendicular to the field. Hence one has to analyze these different motions of particles and clusters simultaneously. Here it should be noted that Gérard-Łaba [GŁ3] dealt with a three body system which has at least one proper neutral subsystem (see also [GŁ4]).

Skibsted [S2, S3] studied the scattering theory for $N$-body quantum systems in combined constant electric and magnetic fields, but his result needs the asymptotic completeness for the systems in a constant magnetic field *only*. By virtue of his works, we see that it is important to know whether the asymptotic completeness for $N$-body quantum systems holds or not in the presence of a constant magnetic field only.

For an $N$-body quantum system, we denote by $L$ the number of charged particles in the system. It is obvious that $N-L$ is the number of neutral particles in the system. In [A1, A2], we studied the scattering theory for an $N$-body quantum system with $L=1$ in a constant magnetic field. Even in this simple case, the problem was open till then. How to choose a conjugate operator for the Hamiltonian which governs the system was one of the keys in [A1, A2]. When $L=1$, it is important that the center of charge of the system coincides with the position of the only charged particle of the system. By virtue of this, we obtained the Mourre estimate and used it in order to obtain the so-called minimal velocity estimate which is one of useful propagation estimates. Our purpose is to remove the restriction on $L$. In this article, we will announce a result of [A3], in which under the assumption that $N=3$ and $L=2$, we have studied the spectral properties of the Hamiltonian under consideration. When the total charge of the system is non-zero, we have constructed a conjugate operator for the

Hamiltonian which governs the system and prove the Mourre estimate. The Mourre estimate is powerful also in studying the scattering theory for the Hamiltonian, as mentioned above. Our construction of a conjugate operator needs the simplicity of the geometric structure of three body systems.

For convenience in the arguments of later sections, we suppose that $N$ is equal to two or three, and that $N - L = 1$. We consider a system of $N$ particles moving in a given constant magnetic field $\boldsymbol{B} = (0,0,B) \in \boldsymbol{R}^3$, $B > 0$. In this article, we sometimes call the system by the set of all indices of particles of the system, such as for instance $\{1,\ldots,N\}$. For $j = 1,\ldots,N$, let $m_j > 0$ and $q_j \in \boldsymbol{R}$ be the mass and charge of the $j$-th particle, respectively. Suppose that the first particle is neutral and the rest are charged, that is,

$$q_1 = 0, \quad q_2, \ldots, q_N \neq 0. \tag{1.1}$$

We assume that the total charge of the system $q$ is non-zero:

$$q = \sum_{j=1}^{N} q_j \neq 0. \tag{1.2}$$

This assumption (1.2) is crucial in this article.

Denoting the space dimension by $d$, we will deal with both the case where $d = 2$ and the one where $d = 3$ in this article. In most cases, scattering pictures in a constant magnetic field depend on the space dimension. We first consider the case where $d = 2$. For $j = 1,\ldots,N$, let $y_j = (y_{j,1}, y_{j,2}) \in \boldsymbol{R}^2$ be the position vector of the $j$-th particle. The total Hamiltonian for the system is defined by

$$H = \frac{1}{2m_1} {D_{y_1}}^2 + \left( \sum_{j=2}^{N} \frac{1}{2m_j} (D_{y_j} - q_j \boldsymbol{A}(y_j))^2 \right) + V \tag{1.3}$$

acting on $L^2(\boldsymbol{R}^{2\times N})$, where the potential $V$ is the sum of the pair potentials $V_{jk}(y_j - y_k)$, that is,

$$V = \sum_{1 \leq j < k \leq N} V_{jk}(y_j - y_k),$$

$D_{y_j} = -i\nabla_{y_j}$, $j = 1,\ldots,N$, is the momentum operator of the $j$-th particle, and $\boldsymbol{A}(r) \in \boldsymbol{R}^2$ is the vector potential which is given by

$$\boldsymbol{A}(r) = \frac{B}{2}(-r_2, r_1), \quad r = (r_1, r_2) \in \boldsymbol{R}^2.$$

We equip the configuration space $Y = \boldsymbol{R}^{2\times N}$ with the metric

$$\langle y, \tilde{y} \rangle = \sum_{j=1}^{N} m_j y_j \cdot \tilde{y}_j, \quad |y|_1 = \sqrt{\langle y, y \rangle}$$

for $y = (y_1, \dots, y_N) \in Y$ and $\bar{y} = (\bar{y}_1, \dots, \bar{y}_N) \in Y$, where the dot $\cdot$ means the usual Euclidean metric.

Introducing the total pseudomomentum $k_{\text{total}}$ of the system which is defined by

$$k_{\text{total}} = D_{y_1} + \sum_{j=2}^{N}(D_{y_j} + q_j \boldsymbol{A}(y_j)), \tag{1.4}$$

one can remove the dependence on $k_{\text{total}}$ from the Hamiltonian $H$: It is well-known that $k_{\text{total}}$ commutes with $H$, and that since the total charge of this system $q$ is non-zero, the two components of the total pseudomomentum $k_{\text{total}}$ cannot commute with each other, but satisfy the Heisenberg commutation relation (see e.g. [AHS2]). Now we introduce the unitary operator

$$U = e^{-iD_{y_{\text{cm},1}}D_{y_{\text{cm},2}}/(qB)} e^{-iqBy_{\text{cm},1}y_{\text{cm},2}/2} e^{iy_{\text{cm}}\cdot q\boldsymbol{A}(y_{\text{cc}})} \tag{1.5}$$

on $L^2(Y)$ with the position vector of the center of mass of the system $y_{\text{cm}}$, the position vector of the center of charge of the system $y_{\text{cc}}$ and the total momentum of the system $D_{y_{\text{cm}}}$ defined by

$$y_{\text{cm}} = \frac{1}{M}\sum_{j=1}^{N} m_j y_j, \quad y_{\text{cc}} = \frac{1}{q}\sum_{j=1}^{N} q_j y_j, \quad D_{y_{\text{cm}}} = \sum_{j=1}^{N} D_{y_j}, \tag{1.6}$$

where $M = \sum_{j=1}^{N} m_j$ is the total mass of the system, and we wrote $y_{\text{cm}} = (y_{\text{cm},1}, y_{\text{cm},2})$ and $D_{y_{\text{cm}}} = (D_{y_{\text{cm},1}}, D_{y_{\text{cm},2}})$. Writing $k_{\text{total}} = (k_{\text{total},1}, k_{\text{total},2})$, we obtain

$$U k_{\text{total},1} U^* = D_{y_{\text{cm},1}}, \quad U k_{\text{total},2} U^* = qBy_{\text{cm},1}. \tag{1.7}$$

Then it is well-known that $UHU^*$ is independent of $(D_{y_{\text{cm},1}}, qBy_{\text{cm},1})$ (see e.g. [GŁ4]). We now introduce subspaces $Y_{a_{\max},1}$, $Y_{a_{\max},2}$ and $Y^{a_{\max}}$ of $Y$ as follows: We define $Y_{a_{\max},1}$ and $Y_{a_{\max},2}$ as

$$Y_{a_{\max},1} = \{y \in Y \mid y_j = y_k \text{ and } y_{j,2} = 0 \text{ for any } j,\, k\},$$
$$Y_{a_{\max},2} = \{y \in Y \mid y_j = y_k \text{ and } y_{j,1} = 0 \text{ for any } j,\, k\}.$$

It is seen that $Y_{a_{\max},j} \cong \boldsymbol{R}_{y_{\text{cm},j}}$. $Y_{a_{\max}} = Y_{a_{\max},1} \oplus Y_{a_{\max},2}$ is called the configuration space of the center of mass motion. $Y^{a_{\max}}$ is the configuration space of the system in the center of mass frame, which is defined by

$$Y^{a_{\max}} = \left\{ y \in Y \,\middle|\, \sum_{j=1}^{N} m_j y_j = 0 \right\}.$$

It is well-known that $Y = Y^{a_{\max}} \oplus Y_{a_{\max}}$ holds. Then one can identify the Hamiltonian $UHU^*$ acting on $UL^2(Y)$ with an operator $\hat{H}$ acting on $\mathcal{H} = L^2(Y^{a_{\max}} \oplus Y_{a_{\max},2})$, that is,

$$UHU^* = \hat{H} \otimes \text{Id} \tag{1.8}$$

on $UL^2(Y) = \mathcal{H} \otimes L^2(Y_{a_{\max},1})$. $U$ is called a reducing unitary transformation.

We next consider the case where $d = 3$. For $j = 1, \ldots, N$, let $x_j = (y_j, z_j) \in \boldsymbol{R}^3$ be the position vector of the $j$-th particle. The total Hamiltonian for the system is defined by

$$\tilde{H} = \left(\sum_{j=1}^{N} \frac{1}{2m_j} D_{z_j}{}^2\right) + \frac{1}{2m_1} D_{y_1}{}^2 + \left(\sum_{j=2}^{N} \frac{1}{2m_j} (D_{y_j} - q_j \boldsymbol{A}(y_j))^2\right) + V \tag{1.9}$$

acting on $L^2(\boldsymbol{R}^{3\times N})$, where the potential $V$ is the sum of the pair potentials $V_{jk}(x_j - x_k)$, that is,

$$V = \sum_{1 \leq j < k \leq N} V_{jk}(x_j - x_k),$$

$(D_{y_j}, D_{z_j}) = (-i\nabla_{y_j}, -i\partial_{z_j})$, $j = 1, \ldots, N$, is the momentum operator of the $j$-th particle. We equip $Z = \boldsymbol{R}^N$ with the metric

$$\langle z, \tilde{z} \rangle = \sum_{j=1}^{N} m_j z_j \cdot \tilde{z}_j, \quad |z|_1 = \sqrt{\langle z, z \rangle}$$

for $z = (z_1, \ldots, z_N) \in Z$ and $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_N) \in Z$. We introduce subspaces $Z_{a_{\max}}$ and $Z^{a_{\max}}$ of $Z$ as follows: We define $Z_{a_{\max}}$ as

$$Z_{a_{\max}} = \{z \in Z \mid z_j = z_k \text{ for any } j,\ k\}.$$

$Z_{a_{\max}}$ is called the configuration space of the center of mass motion parallel to the magnetic field $\boldsymbol{B}$. $Z^{a_{\max}}$ is the configuration space of the system parallel to the magnetic field $\boldsymbol{B}$ in the center of mass frame, which is defined by

$$Z^{a_{\max}} = \left\{ z = (z_1, \ldots, z_N) \in \boldsymbol{R}^N \,\middle|\, \sum_{j=1}^{N} m_j z_j = 0 \right\}.$$

It is well-known that $Z = Z^{a_{\max}} \oplus Z_{a_{\max}}$ holds. Then one can separate the center of mass motion of the system parallel to $\boldsymbol{B}$ from $\tilde{H}$:

$$\tilde{H} = H \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{z_{a_{\max}}}\right) \tag{1.10}$$

on $L^2(Y \times Z) = L^2(Y \times Z^{a_{\max}}) \otimes L^2(Z_{a_{\max}})$, where

$$H = -\frac{1}{2}\Delta_{z^{a_{\max}}} + \frac{1}{2m_1} D_{y_1}{}^2 + \left(\sum_{j=2}^{N} \frac{1}{2m_j} (D_{y_j} - q_j \boldsymbol{A}(y_j))^2\right) + V \tag{1.11}$$

on $L^2(Y \times Z^{a_{\max}})$, and $\Delta_{z^{a_{\max}}}$ and $\Delta_{z_{a_{\max}}}$ are the Laplace-Beltrami operators on $Z^{a_{\max}}$ and $Z_{a_{\max}}$, respectively.

Introducing the total pseudomomentum $k_{\text{total}}$ of the system perpendicular to $\boldsymbol{B}$ which is defined by (1.4), one can remove the dependence on $k_{\text{total}}$ from the Hamiltonian $H$ as in the case where $d = 2$: Introducing the reducing unitary transformation $U$ on $L^2(Y \times Z^{a_{\max}})$ which is defined by (1.5), one can identify the Hamiltonian $UHU^*$ acting on $UL^2(Y \times Z^{a_{\max}})$ with an operator $\hat{H}$ acting on $\mathcal{H} = L^2((Y^{a_{\max}} \oplus Y_{a_{\max},2}) \times Z^{a_{\max}})$, that is,

$$UHU^* = \hat{H} \otimes \mathrm{Id} \tag{1.12}$$

on $UL^2(Y \times Z^{a_{\max}}) = \mathcal{H} \otimes L^2(Y_{a_{\max},1})$.

Our goal in this article is to study the spectral theory for $\hat{H}$. Now we state the assumption on the pair potentials $V_{jk}$: Let $d$ be equal to two or three.

$(\mathrm{V})_d$ $V_{jk} = V_{jk}(r) \in C^\infty(\boldsymbol{R}^d)$, $1 \le j < k \le 3$, is a real-valued function that satisfies

$$|\partial_r^\alpha V_{jk}(r)| \le C_\alpha \langle r \rangle^{-\mu-|\alpha|}$$

for some $\mu > 0$, where $\langle r \rangle = \sqrt{1 + |r|^2}$.

*Remark.* In our talk, we assumed that $V_{12}$ and $V_{13}$, which are pair interactions between neutral and charged particles, are finite-range. However, since we have seen that the assumption may be relaxed as above in [A3] recently, we will here announce it. The local singularity of $V_{jk}$ like $|r|^{-\mu_0}$ with $0 < \mu_0 < d/2$ may be allowed.

Under this assumption $(\mathrm{V})_d$, the Hamiltonians $H$ and $\hat{H}$ are self-adjoint.

The main result of this article is the following theorem:

**Theorem 1.1.** *Suppose that $N = 3$, $L = 2$, $d$ is equal to two or three, and that the potential $V$ satisfies the condition $(\mathrm{V})_d$. Put*

$$d(\lambda) = \mathrm{dist}(\lambda, \Theta \cap (-\infty, \lambda])$$

*for $\lambda \ge \inf \Theta$, where $\Theta$ is the set of thresholds of $\hat{H}$. Then for any for $\lambda \ge \inf \Theta$, there exists a conjugate operator $\hat{A}$ for $\hat{H}$ at the energy $\lambda$ such that the following holds: For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ supported in the open interval $(\lambda - \delta, \lambda + \delta)$, there exists a compact operator $K$ on $\mathcal{H}$ such that*

$$f(\hat{H}) i[\hat{H}, \hat{A}] f(\hat{H}) \ge 2(d(\lambda) - \varepsilon) f(\hat{H})^2 + K \tag{1.13}$$

*holds.*

*Moreover, eigenvalues of $\hat{H}$ can accumulate only at $\Theta$, and $\Theta \cup \sigma_{\mathrm{pp}}(\hat{H})$ is a closed countable set.*

If one wants to study the scattering theory for the Hamiltonian $H$, the following corollary seems useful, which follows from the fact that $H = U^*(\hat{H} \otimes \mathrm{Id})U$ and a standard argument immediately (cf. [A1, A2]):

**Corollary 1.2.** *Suppose that $N = 3$, $L = 2$, $d$ is equal to two or three, and that the potential $V$ satisfies the condition $(\mathrm{V})_d$. Let $\lambda \in \boldsymbol{R} \setminus (\Theta \cup \sigma_{\mathrm{pp}}(H))$ be such that $\lambda \geq \inf \Theta$. Put $A = U^*(\hat{A} \otimes \mathrm{Id})U$, where $\hat{A}$ is a conjugate operator for $\hat{H}$ at $\lambda$ and $U$ is the reducing unitary transformation. Then there exist $\delta > 0$ and $c > 0$ such that for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ supported in the open interval $(\lambda - \delta, \lambda + \delta)$,*

$$f(H)i[H, A]f(H) \geq cf(H)^2 \tag{1.14}$$

*holds.*

## 2 The case where $d = 2$

In this section, we construct a conjugate operator for $\hat{H}$ and state an outline of the proof of Theorem 1.1 in the case where $d = 2$. Throughout this section, we assume the condition $(\mathrm{V})_2$.

We first introduce some notation that is used in many body scattering theory, in order to simplify the representation of the proofs below: Let $N = 3$. A non-empty subset of the set $\{1, 2, 3\}$ is called a cluster. Let $C_j$, $1 \leq j \leq j_0$, be clusters. If $\cup_{1 \leq j \leq j_0} C_j = \{1, 2, 3\}$ and $C_j \cap C_k = \emptyset$ for $1 \leq j < k \leq j_0$, $a = \{C_1, \ldots, C_{j_0}\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in $a$. We identify the pair $(j, k)$ with the two-cluster decomposition $\{\{j, k\}, \{l\}\}$, where $l$ satisfies $\{j, k, l\} = \{1, 2, 3\}$. We write $a_{\max} = \{\{1, 2, 3\}\}$ and $a_{\min} = \{\{1\}, \{2\}, \{3\}\}$. Then the set of all cluster decompositions $\mathcal{A}$ is written as

$$\mathcal{A} = \{a_{\max}, (1, 2), (1, 3), (2, 3), a_{\min}\}. \tag{2.1}$$

Let $a, b \in \mathcal{A}$. If each cluster in $b$ is a subset of a cluster in $a$, we say $b \subset a$.

The cluster Hamiltonian $H_a$, $a \in \mathcal{A}$, on $L^2(Y)$ is defined as follows:

$$\begin{aligned} H_{a_{\min}} &= H_0 = \frac{1}{2m_1} {D_{y_1}}^2 + \sum_{j=2}^{3} \frac{1}{2m_j} (D_{y_j} - q_j \boldsymbol{A}(y_j))^2, \\ H_{(j,k)} &= H_0 + V_{jk}(y_j - y_k), \quad H_{a_{\max}} = H. \end{aligned} \tag{2.2}$$

In particular, one has $H_a$ as well as $H$ does commute with the total pseudomomentum $k_{\mathrm{total}}$ of the system. Thus $UH_aU^*$ acting on $UL^2(Y)$ is reduced to $\hat{H}_a$ acting on $\mathcal{H}$ in the same way as in (1.8).

For two-cluster decomposition $a \in \mathcal{A}$, the cluster Hamiltonian $H_a$ is represented as the sum of innercluster Hamiltonians $H^{C_k}$ with $k = 1, 2$: We first consider $a = (1, j)$ with $j = 2, 3$. For $j = 2, 3$, we define the innercluster Hamiltonian $H^{\{1,j\}}$ on $L^2(\boldsymbol{R}^{2\times 2})$ as

$$\begin{aligned} H^{\{1,j\}} &= H_0^{\{1,j\}} + V_{1j}(y_1 - y_j), \quad H_0^{\{1,j\}} = H^{\{1\}} + H^{\{j\}}, \\ H^{\{1\}} &= \frac{1}{2m_1} {D_{y_1}}^2, \quad H^{\{j\}} = \frac{1}{2m_j}(D_{y_j} - q_j \boldsymbol{A}(y_j))^2. \end{aligned} \tag{2.3}$$

Then one has

$$H_{(1,2)} = H^{\{1,2\}} + H^{\{3\}}, \quad H_{(1,3)} = H^{\{1,3\}} + H^{\{2\}}. \tag{2.4}$$

We note that $H^{\{1,j\}}$ with $j = 2, 3$ is the Hamiltonian which was considered essentially in [A1]. Introducing the innercluster Hamiltonian $H^{\{2,3\}}$ on $L^2(\boldsymbol{R}^{2\times 2})$ as

$$H^{\{2,3\}} = H_0^{\{2,3\}} + V_{23}(y_2 - y_3), \quad H_0^{\{2,3\}} = H^{\{2\}} + H^{\{3\}}, \tag{2.5}$$

one has

$$H_{(2,3)} = H^{\{2,3\}} + H^{\{1\}}. \tag{2.6}$$

Applying the Weyl theorem for the reduced Hamiltonians of $H^{\{2,3\}}$ and $H_0^{\{2,3\}}$, it is seen that

$$\sigma(H^{\{2,3\}}) = \sigma_{\mathrm{pp}}(H^{\{2,3\}}) \text{ is countable,} \tag{2.7}$$

because

$$\sigma(H_0^{\{2,3\}}) = \sigma_{\mathrm{pp}}(H_0^{\{2,3\}}) = \tau_2 + \tau_3 \tag{2.8}$$

by virtue of $d = 2$ (see [AHS2] and [GŁ4]). Here $\tau_j$ is the set of the Landau levels for $j = 2, 3$:

$$\tau_j = \sigma(H^{\{j\}}) = \left\{ \frac{|q_j|B}{m_j}\left(n + \frac{1}{2}\right) \,\middle|\, n \in \boldsymbol{N} \cup \{0\} \right\}. \tag{2.9}$$

For convenience, we revisit the case where $N = 2$ and $L = 1$, which was already studied by the author [A1] when the space dimension $d$ was three. Begin with the following self-adjoint operator $A_1$ on $L^2(\boldsymbol{R}^{2\times 2})$ for $H^{\{1,2\}}$:

$$A_1 = \frac{1}{2}(y_1 \cdot D_{y_1} + D_{y_1} \cdot y_1). \tag{2.10}$$

By a straightforward computation, one can obtain the commutation relation

$$i[H_0^{\{1,2\}}, A_1] = \frac{1}{m_1} {D_{y_1}}^2 = 2H^{\{1\}} = 2(H_0^{\{1,2\}} - H^{\{2\}}). \tag{2.11}$$

By virtue of (2.9), the commutation relation (2.11) seems nice for studying the spectral theory for the reduced Hamiltonian $\hat{H}^{\{1,2\}}$. However, since $A_1$ does *not* commute with the total pseudomomentum $k_{\text{total}}^{\{1,2\}} = D_{y_1} + D_{y_2} + q_2\boldsymbol{A}(y_2)$ of the system $\{1,2\}$, $U^{\{1,2\}}A_1(U^{\{1,2\}})^*$ cannot be reduced to an operator on $\mathcal{H}^{\{1,2\}}$, where $U^{\{1,2\}}$ and $\mathcal{H}^{\{1,2\}}$ are equal to $U$ and $\mathcal{H}$ defined as in §1 with $N = 2$, respectively. In order to overcome this difficulty, we introduce the self-adjoint operator $\hat{A}^{\{1,2\}}$ on $\mathcal{H}^{\{1,2\}}$, which is obtained by removing the dependence on $U^{\{1,2\}}k_{\text{total}}^{\{1,2\}}(U^{\{1,2\}})^*$ from the operator $U^{\{1,2\}}A_1(U^{\{1,2\}})^*$. This $\hat{A}^{\{1,2\}}$ is a conjugate operator for the reduced Hamiltonian $\hat{H}^{\{1,2\}}$. In [A1], using the relative coordinates and the center of mass coordinates, we obtained this $\hat{A}^{\{1,2\}}$, but its representation was slightly complicated and unsuitable for generalizations to $N$-body systems. Now we follow the argument in [A2]: In [A2], it is obtained that the self-adjoint operator $(U^{\{1,2\}})^*(\hat{A}^{\{1,2\}} \otimes \mathrm{Id})U^{\{1,2\}}$ on $L^2(\boldsymbol{R}^{2\times 2})$ can be written as

$$(U^{\{1,2\}})^*(\hat{A}^{\{1,2\}} \otimes \mathrm{Id})U^{\{1,2\}} = \frac{1}{2}(w_1^{\{1,2\}} \cdot D_{y_1} + D_{y_1} \cdot w_1^{\{1,2\}}) \tag{2.12}$$

with

$$w_1^{\{1,2\}} = y_1 - \gamma_{\text{cc}}^{\{1,2\}}, \quad \gamma_{\text{cc}}^{\{1,2\}} = -\frac{2}{q_2 B^2}\boldsymbol{A}(k_{\text{total}}^{\{1,2\}}). \tag{2.13}$$

Since by a simple computation

$$\boldsymbol{A}(\boldsymbol{A}(r)) = -\frac{B^2}{4}r, \quad r \in \boldsymbol{R}^2,$$

we will often use the notation $\boldsymbol{A}^{-1}$ defined by

$$\boldsymbol{A}^{-1}(r) = -\frac{4}{B^2}\boldsymbol{A}(r), \quad r \in \boldsymbol{R}^2.$$

Then $\gamma_{\text{cc}}^{\{1,2\}}$ can be rewritten as

$$\gamma_{\text{cc}}^{\{1,2\}} = \frac{1}{2q_2}\boldsymbol{A}^{-1}(k_{\text{total}}^{\{1,2\}}). \tag{2.14}$$

$\gamma_{\text{cc}}^{\{1,2\}}$ is called the center of orbit of the center of charge of the system $\{1,2\}$ (see [AHS2] and [GŁ2, GŁ3, GŁ4]), although in [A1, A2] we did not notice this fact unfortunately. In this case, one knows that $q_2$ coincides with the total charge of the system $\{1,2\}$, of course. One of basic properties of $\gamma_{\text{cc}}^{\{1,2\}}$ is that

$$y_{\text{cc}}^{\{1,2\}} - \gamma_{\text{cc}}^{\{1,2\}} = y_2 - \gamma_{\text{cc}}^{\{1,2\}} = \frac{-1}{2q_2}\boldsymbol{A}^{-1}(D_{y_1} + (D_{y_2} - q_2\boldsymbol{A}(y_2))) \tag{2.15}$$

is $H^{\{1,2\}}$-bounded, where $y_{\text{cc}}^{\{1,2\}}$ is the position vector of the center of charge of the system $\{1,2\}$ and coincides with $y_2$. Since $y_{\text{cc}}^{\{1,2\}} - \gamma_{\text{cc}}^{\{1,2\}}$ does commute with $k_{\text{total}}^{\{1,2\}}$ by (2.15),

$U^{\{1,2\}}(y_{\rm cc}^{\{1,2\}} - \gamma_{\rm cc}^{\{1,2\}})(U^{\{1,2\}})^*$ is $\hat{H}^{\{1,2\}}$-bounded. Here $U^{\{1,2\}}(y_{\rm cc}^{\{1,2\}} - \gamma_{\rm cc}^{\{1,2\}})(U^{\{1,2\}})^*$ was identified with an operator acting on $\mathcal{H}^{\{1,2\}}$. Such identification will be used frequently below. We notice that one can write

$$\begin{aligned} i[V_{12}, \hat{A}^{\{1,2\}}] = &-(y_1 - y_2)\cdot(\nabla V_{12})(y_1 - y_2) \\ &-(U^{\{1,2\}}(y_2 - \gamma_{\rm cc}^{\{1,2\}})(U^{\{1,2\}})^*)\cdot(\nabla V_{12})(y_1 - y_2) \end{aligned}$$

on $\mathcal{H}^{\{1,2\}}$ since $V_{12}$ commutes with $k_{\rm total}^{\{1,2\}}$. By the assumption that $|\partial_r^\alpha V_{12}(r)| \le C_\alpha \langle r\rangle^{-\mu-|\alpha|}$ with some $\mu > 0$, $(\hat{H}_0^{\{1,2\}} + 1)^{-1} i[V_{12}, \hat{A}^{\{1,2\}}](\hat{H}_0^{\{1,2\}} + 1)^{-1}$ is compact on $\mathcal{H}^{\{1,2\}}$, because $|(y_1 - y_2)\cdot(\nabla V_{12})(y_1 - y_2)| \le C\langle y_1 - y_2\rangle^{-\mu}$ and $|(\nabla V_{12})(y_1 - y_2)| \le C\langle y_1 - y_2\rangle^{-\mu-1}$ hold, and $U^{\{1,2\}}(y_{\rm cc}^{\{1,2\}} - \gamma_{\rm cc}^{\{1,2\}})(U^{\{1,2\}})^*$ is $\hat{H}_0^{\{1,2\}}$-bounded. Thus for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ there exists a compact operator $K_1$ on $\mathcal{H}^{\{1,2\}}$ such that

$$f(\hat{H}^{\{1,2\}}) i[V_{12}, \hat{A}^{\{1,2\}}] f(\hat{H}^{\{1,2\}}) = K_1$$

holds. Since both $D_{y_1}$ and $k_{\rm total}^{\{1,2\}}$ commute with $H_0^{\{1,2\}}$, it is clear that

$$i[\hat{H}_0^{\{1,2\}}, \hat{A}^{\{1,2\}}] = 2(\hat{H}_0^{\{1,2\}} - U^{\{1,2\}} H^{\{2\}} (U^{\{1,2\}})^*) \tag{2.16}$$

holds by virtue of (2.11). By using these two estimates, we obtained the desirable Mourre estimate as in [A1].

Now we return to the present problem. First we define the set of thresholds $\Theta$ for $H$ (or $\hat{H}$). Put

$$\begin{aligned} &\theta_{a_{\min}} = \tau_2 + \tau_3, \quad \theta_{(2,3)} = (\tau_2 + \tau_3) \cup \sigma_{\rm pp}(H^{\{2,3\}}), \\ &\theta_{(1,2)} = (\tau_2 \cup \sigma_{\rm pp}(H^{\{1,2\}})) + \tau_3, \quad \theta_{(1,3)} = (\tau_3 \cup \sigma_{\rm pp}(H^{\{1,3\}})) + \tau_2, \end{aligned}$$

and define the set of thresholds $\Theta$ for $H$ (or $\hat{H}$) by

$$\Theta = \bigcup_{a \in \mathcal{A}\setminus\{a_{\max}\}} \theta_a. \tag{2.17}$$

Let $\lambda \ge \inf\Theta$. We will define the original operator $A = U^*(\hat{A}\otimes \mathrm{Id})U$ of a conjugate operator $\hat{A}$ for the reduced Hamiltonian $\hat{H}$ at $\lambda$. Following the above argument in the case where $N = 2$, a candidate for $A$ is

$$\begin{aligned} &A^{a_{\max}} = \frac{1}{2}(w_1 \cdot D_{y_1} + D_{y_1}\cdot w_1), \\ &w_1 = y_1 - \gamma_{\rm cc}^{\{1,2,3\}}, \quad \gamma_{\rm cc}^{\{1,2,3\}} = \frac{1}{2q}\boldsymbol{A}^{-1}(k_{\rm total}), \end{aligned} \tag{2.18}$$

which is a natural extension of (2.12) with (2.13) to the case where $N = 3$. In fact, if $V_{12} \equiv V_{13} \equiv 0$, $A^{a_{\max}}$ works well. However, by a simple computation, it is seen that in general,

$(H_0+1)^{-1}i[V, A^{a_{\max}}](H_0+1)^{-1}$ is *not* bounded on $L^2(Y)$ unfortunately. This implies the difference between the case where $L=1$ and the one where $L=2$. We here put

$$A^{(1,j)} = \frac{1}{2}(w_1^{\{1,j\}} \cdot D_{y_1} + D_{y_1} \cdot w_1^{\{1,j\}}),$$
$$w_1^{\{1,j\}} = y_1 - \gamma_{\mathrm{cc}}^{\{1,j\}}, \quad \gamma_{\mathrm{cc}}^{\{1,j\}} = \frac{1}{2q_j} A^{-1}(k_{\mathrm{total}}^{\{1,j\}}), \tag{2.19}$$

for $j = 2, 3$, where $k_{\mathrm{total}}^{\{1,j\}} = D_{y_1} + D_{y_j} + q_j A(y_j)$ is the total pseudomomentum of the subsystem $\{1, j\}$. $A^{(1,j)}$ is the original operator of a conjugate operator for $\hat{H}^{\{1,j\}}$ as seen above, and is also a candidate for $A$. In fact, if $V_{13} \equiv V_{23} \equiv 0$, $A^{(1,2)}$ works well as observed above, and if $V_{12} \equiv V_{23} \equiv 0$, $A^{(1,3)}$ works well. However, by a simple computation, it is seen that in general, $(H_0+1)^{-1}i[V, A^{(1,j)}](H_0+1)^{-1}$ is *not* bounded on $L^2(Y)$, either. In order to overcome this difficulty, we will patch these candidates together by introducing a partition of unity of the configuration space $Y^{a_{\max}}$.

To this end, we will make some preparations. We first introduce a family of projections $\{\pi_{a,q}\}_{a\in\mathcal{A}}$ of the configuration space $Y$ in terms of charge: For $y = (y_1, y_2, y_3) \in Y$,

$$\pi_{a_{\max},q}y = (y_{\mathrm{cc}}, y_{\mathrm{cc}}, y_{\mathrm{cc}}),$$
$$\pi_{(1,2),q}y = (y_2, y_2, y_3), \quad \pi_{(1,3),q}y = (y_3, y_2, y_3), \tag{2.20}$$
$$\pi_{(2,3),q}y = (y_1, y_{\mathrm{cc}}, y_{\mathrm{cc}}), \quad \pi_{a_{\min},q}y = (y_1, y_2, y_3).$$

We note that $y_j$, $j = 2, 3$, coincides with the position vector of the center of charge of the subsystem $\{1, j\}$, and that $y_{\mathrm{cc}}$ coincides with the position vector of the center of charge of the subsystem $\{2, 3\}$. We also notice that $\pi_{a_{\max},q}Y = Y_{a_{\max}}$. One can see easily that

$$\pi_{a,q}\pi_{a_{\max},q} = \pi_{a_{\max},q}\pi_{a,q} = \pi_{a_{\max},q}, \quad a \in \mathcal{A}, \tag{2.21}$$
$$\pi_{a_{\min},q} = \mathrm{Id}. \tag{2.22}$$

We set $\pi^{a,q} = \mathrm{Id} - \pi_{a,q}$ for $a \in \mathcal{A}$. In particular, $\pi^{a_{\min},q} = 0$ by (2.22). Now we note that for $y = (y_1, y_2, y_3) \in Y$,

$$\pi^{a_{\max},q}y = \left(y_1 - \frac{q_2y_2 + q_3y_3}{q}, \frac{q_3}{q}(y_2 - y_3), -\frac{q_2}{q}(y_2 - y_3)\right),$$
$$\pi^{(1,2),q}y = (y_1 - y_2, 0, 0), \quad \pi^{(1,3),q}y = (y_1 - y_3, 0, 0), \tag{2.23}$$
$$\pi^{(2,3),q}y = \left(0, \frac{q_3}{q}(y_2 - y_3), -\frac{q_2}{q}(y_2 - y_3)\right), \quad \pi^{a_{\min},q}y = (0, 0, 0),$$

by using $y_{\mathrm{cc}} = (q_2y_2 + q_3y_3)/q$. We denote by $\Pi^{a_{\max}}$ the orthogonal projection of $Y$ onto $Y^{a_{\max}}$. It is well-known that for $y \in Y$, $y^{a_{\max}} = \Pi^{a_{\max}}y$ is represented as

$$y^{a_{\max}} = (y_1 - y_{\mathrm{cm}}, y_2 - y_{\mathrm{cm}}, y_3 - y_{\mathrm{cm}}). \tag{2.24}$$

Then we have

$$\begin{aligned}
\pi^{a_{\max},q}y^{a_{\max}} &= \left(y_1 - \frac{q_2y_2+q_3y_3}{q}, \frac{q_3}{q}(y_2-y_3), -\frac{q_2}{q}(y_2-y_3)\right),\\
\pi^{(1,2),q}y^{a_{\max}} &= (y_1-y_2,0,0), \quad \pi^{(1,3),q}y^{a_{\max}} = (y_1-y_3,0,0),\\
\pi^{(2,3),q}y^{a_{\max}} &= \left(0, \frac{q_3}{q}(y_2-y_3), -\frac{q_2}{q}(y_2-y_3)\right),\\
\pi^{a_{\min},q}y^{a_{\max}} &= (0,0,0),
\end{aligned} \tag{2.25}$$

for $y^{a_{\max}} \in Y^{a_{\max}}$ by (2.23), (2.24) and a simple computation. (2.23) and (2.25) imply that $\pi^{a,q}|_{Y^{a_{\max}}}$, $a \in \mathcal{A}$, is a projection of $Y^{a_{\max}}$. Hence for $y^{a_{\max}} \in Y^{a_{\max}}$, we write $y^{a,q} = \pi^{a,q}|_{Y^{a_{\max}}}y^{a_{\max}}$.

Now we would like to introduce a version of a Graf partition of unity of $Y^{a_{\max}}$. To this end, we follow the argument of [Gr]: There exists a $\rho > 0$ such that $40\rho \leq 1$,

$$\rho \leq \frac{1}{2}\left(1 + \frac{18(q_2^2+q_3^2)}{q^2}\right)^{-1}$$

and

$$\begin{aligned}
&10\rho\left\{\left|y_1 - \frac{q_2y_2+q_3y_3}{q}\right|^2 + \frac{q_2^2+q_3^2}{q^2}\langle y_2-y_3\rangle^2|y_2-y_3|^2\right\}\\
&\leq |y_1-y_j|^2 + \langle y_2-y_3\rangle^2|y_2-y_3|^2
\end{aligned} \tag{2.26}$$

for $j = 2, 3$ (referring to (2.25)), by virtue of the simplicity of the geometric structure of three body systems.

Referring to (2.25), in order to measure the size of $y^{a,q}$, we now introduce a family of functions $\{\kappa^a(y^{a_{\max}})\}_{a\in\mathcal{A}\setminus\{(2,3)\}}$ on $Y^{a_{\max}}$ as follows:

$$\begin{aligned}
\kappa^{a_{\max}}(y^{a_{\max}}) &= \left|y_1 - \frac{q_2y_2+q_3y_3}{q}\right|^2 + \frac{q_2^2+q_3^2}{q^2}\langle y_2-y_3\rangle^2|y_2-y_3|^2,\\
\kappa^{a_{\min}}(y^{a_{\max}}) &\equiv 0, \quad \kappa^{(1,j)}(y^{a_{\max}}) = |y_1-y_j|^2, \quad j = 2, 3.
\end{aligned} \tag{2.27}$$

It seems appropriate to think that the size of $y^{(2,3),q}$ is used in order to define the weight $\langle y_2 - y_3\rangle^2$ in the definition of $\kappa^{a_{\max}}(y^{a_{\max}})$. By virtue of this family $\{\kappa^a(y^{a_{\max}})\}_{a\in\mathcal{A}\setminus\{(2,3)\}}$, one can know the *nearest* center of charge for the neutral particle among $y_2$, $y_3$ and $y_{cc}$: We define a family of sets $\{\Omega^a\}_{a\in\mathcal{A}\setminus\{(2,3)\}}$ as

$$\begin{aligned}
\Omega^a = \big\{y^{a_{\max}} \in Y^{a_{\max}} \mid \kappa^a(y^{a_{\max}}) - \rho^{\#(a)} < \kappa^b(y^{a_{\max}}) - \rho^{\#(b)}\\
\text{for any } b \in \mathcal{A}\setminus\{(2,3)\} \text{ such that } b \neq a\big\},
\end{aligned} \tag{2.28}$$

where $\rho^{\#(a_{\min})} \equiv 0$.

The following proposition is proved in the way quite similar to that in [Gr], [D] and [DG]. We here omit the proof (see [A3]).

**Proposition 2.1.** (1) *If* $a,\ b \in \mathcal{A}\setminus\{(2,3)\}$ *satisfy* $a \neq b$, $\overline{\Omega^a}\cap\overline{\Omega^b}$ *is a set of measure zero. Here* $\overline{\Omega^a}$ *is the closure of* $\Omega^a$. *The family of sets* $\{\Omega^a \mid a \in \mathcal{A}\setminus\{(2,3)\}\}$ *is a family of disjoint open sets in* $Y^{a_{\max}}$ *and one has*

$$\bigcup_{a\in\mathcal{A}\setminus\{(2,3)\}} \overline{\Omega^a} = Y^{a_{\max}}.$$

(2) *For* $y^{a_{\max}} \in \overline{\Omega^{a_{\min}}}$ *and* $j \in \{2,3\}$,

$$|y_1 - y_j|^2 \geq \rho^2$$

*holds.*
(3) *For* $y^{a_{\max}} \in \overline{\Omega^{a_{\max}}}$,

$$\langle y_2 - y_3\rangle^2 |y_2 - y_3|^2 \leq \rho^2$$

*holds.*
(4) *If* $\kappa^{a_{\max}}(y^{a_{\max}}) \geq (\rho-\rho^2)/2$ *and* $\kappa^{(1,j)}(y^{a_{\max}}) \leq 2\rho^2$ *with* $j \in \{2,3\}$, *then*

$$\begin{aligned}\langle y_2 - y_3\rangle^2 |y_2 - y_3|^2 &\geq 2\rho^2,\\ |y_1 - y_k|^2 &\geq \frac{q^2}{18(q_2^2+q_3^2)}\rho\end{aligned}$$

*hold for* $k \in \{2,3\}$ *such that* $k \neq j$.

Next we fix a function $\varphi \in C_0^\infty(Y^{a_{\max}})$ such that $\operatorname{supp}\varphi \subset \{y^{a_{\max}} \in Y^{a_{\max}} \mid |y^{a_{\max}}|_1 \leq \sigma\}$ with a sufficiently small $\sigma > 0$,

$$\varphi \geq 0, \quad \int_{Y^{a_{\max}}} \varphi(y^{a_{\max}})\, dy^{a_{\max}} = 1.$$

Then we define

$$\bar{\eta}_a(y^{a_{\max}}) = (1_{\Omega^a} * \varphi)(y^{a_{\max}}), \quad \tilde{\eta}_a(y^{a_{\max}}) = \frac{\bar{\eta}_a(y^{a_{\max}})}{\sqrt{\sum_{b\in\mathcal{A}\setminus\{(2,3)\}} \bar{\eta}_b^2(y^{a_{\max}})}} \tag{2.29}$$

for $a \in \mathcal{A}\setminus\{(2,3)\}$, where $1_{\Omega^a}$ is the characteristic function of the set $\Omega^a$.

The following proposition can also be shown in the same way as in [Gr], by virtue of Proposition 2.1. So we omit the proof.

**Proposition 2.2.** $\tilde{\eta}_a(y^{a_{\max}})$, $a \in \mathcal{A}\setminus\{(2,3)\}$, *are all bounded smooth functions on* $Y^{a_{\max}}$ *with bounded derivatives. One has*

$$\sum_{a\in\mathcal{A}\setminus\{(2,3)\}} \tilde{\eta}_a^2(y^{a_{\max}}) \equiv 1.$$

*Moreover, there exists a* $\sigma > 0$ *such that the following holds: For* $y^{a_{\max}} \in \operatorname{supp} \tilde{\eta}_{a_{\min}}$ *and* $j \in \{2, 3\}$,

$$|y_1 - y_j|^2 \geq \frac{1}{2}\rho^2$$

*holds. For* $y^{a_{\max}} \in \operatorname{supp} \tilde{\eta}_{a_{\max}}$,

$$\langle y_2 - y_3 \rangle^2 |y_2 - y_3|^2 \leq 2\rho^2$$

*holds. For* $y^{a_{\max}} \in \operatorname{supp} \tilde{\eta}_{(1,j)}$ *with* $j \in \{2, 3\}$,

$$\langle y_2 - y_3 \rangle^2 |y_2 - y_3|^2 \geq 2\rho^2,$$
$$|y_1 - y_k|^2 \geq \frac{q^2}{18(q_2^2 + q_3^2)}\rho$$

*hold for* $k \in \{2, 3\}$ *such that* $k \neq j$.

Next we will construct an original operator of a conjugate operator for $\hat{H}$: We put

$$g_{a,R}(y^{a_{\max}}) = \tilde{\eta}_a \left( \frac{y^{a_{\max}}}{R\langle y_2 - y_3 \rangle} \right) \tag{2.30}$$

with a parameter $R > 0$. We note that $g_{a,R}$ is a smooth function on $Y^{a_{\max}}$ and

$$|\partial^\alpha g_{a,R}(y^{a_{\max}})| \leq C_\alpha R^{-|\alpha|} \langle y_2 - y_3 \rangle^{-|\alpha|} \tag{2.31}$$

holds. Then we introduce an operator $A_R$ as follows: We put

$$A_R = \sum_{a \in \mathcal{A} \setminus \{(2,3)\}} g_{a,R}(y^{a_{\max}}) A^a g_{a,R}(y^{a_{\max}}), \tag{2.32}$$

where $A^{a_{\min}} = A^{a_{\max}}$. This definition is an extension of that of conjugate operator in the case where $N = 2$ and $L = 1$. We will often abbreviate $g_{a,R}(y^{a_{\max}})$ as $g_{a,R}$ below. One can check easily the fact that $A_R$ does commute with $k_{\text{total}}$. Then we denote by $\hat{A}_R$ the reduced operator of $UA_RU^*$ which acts on $\mathcal{H}$. Nelson's commutator theorem guarantees the self-adjointness of $\hat{A}_R$. Then we see that $(\hat{H}_0 + 1)^{-1} i[\hat{H}_0, \hat{A}_R](\hat{H}_0 + 1)^{-1}$ is bounded on $\mathcal{H}$ and

$$\begin{aligned} &(\hat{H}_0 + 1)^{-1} i[\hat{H}_0, \hat{A}_R](\hat{H}_0 + 1)^{-1} \\ &= (\hat{H}_0 + 1)^{-1} \left\{ 2 \left( \hat{H}_0 - U \left( \sum_{j=2}^{3} H^{\{j\}} \right) U^* \right) \right\} (\hat{H}_0 + 1)^{-1} \\ &\quad + O(R^{-1}), \end{aligned} \tag{2.33}$$

which is an important estimate in order to prove the Mourre estimate for $\hat{H}$.

Now we need the following lemma concerned with $i[V, \hat{A}_R]$. We here state an outline of its proof only (see [A3] for details).

**Lemma 2.3.** $(\hat{H}_0+1)^{-1}i[V,\hat{A}_R](\hat{H}_0+1)^{-1}$ *is bounded on* $\mathcal{H}$.

*Outline of the proof.* First we consider the charged-charged pair potential $V_{23}$. Since

$$\begin{aligned} i[V_{23},A_R] =& \frac{-1}{2q_2} g_{(1,2),R}\{\boldsymbol{A}^{-1}(D_{y_1})\cdot\nabla V_{23}\}g_{(1,2),R} \\ &+\frac{1}{2q_3} g_{(1,3),R}\{\boldsymbol{A}^{-1}(D_{y_1})\cdot\nabla V_{23}\}g_{(1,3),R}, \end{aligned} \tag{2.34}$$

we obtain

$$(\hat{H}_0+1)^{-1}i[V_{23},\hat{A}_R](\hat{H}_0+1)^{-1}=O(R^{-(1+\mu)}) \tag{2.35}$$

by virtue of Proposition 2.2.

Next we consider neutral-charged pair interactions $V_{1j}$ with $j\in\{2,3\}$. It is sufficient to deal with $V_{12}$ only. By a straightforward computation, we have

$$\begin{aligned} i[V_{12},A_R] =& i[V_{12},A^{(1,2)}] \\ &+ g_{(1,3),R}\{(\gamma_{cc}^{\{3\}}-\gamma_{cc}^{\{1,2\}})\cdot\nabla V_{12}\}g_{(1,3),R} \\ &+ g_{a_{\min},R}\{(\gamma_{cc}^{\{1,2,3\}}-\gamma_{cc}^{\{1,2\}})\cdot\nabla V_{12}\}g_{a_{\min},R} \\ &+ g_{a_{\max},R}\{(\gamma_{cc}^{\{1,2,3\}}-\gamma_{cc}^{\{1,2\}})\cdot\nabla V_{12}\}g_{a_{\max},R} \end{aligned} \tag{2.36}$$

by virtue of $\sum_{a\in\mathcal{A}\setminus\{(2,3)\}} g_{a,R}^2\equiv 1$. By virtue of Proposition 2.2 and $(\mathrm{V})_2$, we have

$$\begin{aligned} |(\nabla V_{12})(y_1-y_2)g_{(1,3),R}(y^{a_{\max}})| &\le CR^{-(1+\mu)}\langle y_2-y_3\rangle^{-(1+\mu)}, \\ |(\nabla V_{12})(y_1-y_2)g_{a_{\min},R}(y^{a_{\max}})| &\le CR^{-(1+\mu)}\langle y_2-y_3\rangle^{-(1+\mu)}. \end{aligned} \tag{2.37}$$

Then we obtain

$$\begin{aligned} &(\hat{H}_0+1)^{-1}g_{(1,3),R}\{U(\gamma_{cc}^{\{3\}}-\gamma_{cc}^{\{1,2\}})U^*\cdot\nabla V_{12}\}g_{(1,3),R}(\hat{H}_0+1)^{-1} \\ &\quad = O(R^{-(1+\mu)}), \\ &(\hat{H}_0+1)^{-1}g_{a_{\min},R}\{U(\gamma_{cc}^{\{1,2,3\}}-\gamma_{cc}^{\{1,2\}})U^*\cdot\nabla V_{12}\}g_{a_{\min},R}(\hat{H}_0+1)^{-1} \\ &\quad = O(R^{-(1+\mu)}). \end{aligned} \tag{2.38}$$

On the other hand, for $y^{a_{\max}}\in\operatorname{supp} g_{a_{\max},R}$,

$$|y_2-y_3|^2\le 2\rho^2R^2 \tag{2.39}$$

holds by virtue of Proposition 2.2. Using (2.39) and

$$\nabla V_{12}\mathbf{1}_{B_R(0)}+\nabla V_{12}\mathbf{1}_{B_R(0)^c}=\nabla V_{12}$$

with $B_R(0)=\{r\in\boldsymbol{R}^2 \mid |r|\le R\}$, we see that

$$\begin{aligned} &(\hat{H}_0+1)^{-1}g_{a_{\max},R}\{U(\gamma_{cc}^{\{1,2,3\}}-\gamma_{cc}^{\{1,2\}})U^*\cdot\nabla V_{12}\}g_{a_{\max},R}(\hat{H}_0+1)^{-1} \\ &\quad = K_R+O(R^{-\mu}), \end{aligned} \tag{2.40}$$

where $K_R$ is compact on $\mathcal{H}$, because $1_{B_{C_0}(0;Y^{a_{\max}})}(\hat{H}_0+1)^{-1}$ is compact on $\mathcal{H}$ for $C_0>0$ (see e.g. [AHS2]), where $B_{C_0}(0;Y^{a_{\max}})=\{y^{a_{\max}}\in Y^{a_{\max}}\mid |y^{a_{\max}}|_1\le C_0\}$. Here we used the simplicity of the geometric structure of three body systems in order to get the compactness of $K_R$. Therefore we obtain

$$\begin{aligned}(\hat{H}_0+1)^{-1}i[V_{12},\hat{A}_R](\hat{H}_0+1)^{-1}\\ =(\hat{H}_0+1)^{-1}i[V_{12},UA^{(1,2)}U^*](\hat{H}_0+1)^{-1}+O(R^{-\mu})+K_R.\end{aligned}\tag{2.41}$$

This completes the proof. □

By virtue of this Lemma 2.3, one can prove that $\hat{A}_R$ is a conjugate operator for $\hat{H}$ at $\lambda\ge\inf\Theta$ for sufficiently large $R>0$, by following e.g. the argument of [FH]. For details, see [A3].

# 3 The case where $d=3$

In this section, we state a construction of a conjugate operator for $\hat{H}$ only, because the proof of the Mourre estimate is quite similar to the one for the case where $d=2$. Throughout this section, we assume the condition $(\mathrm{V})_3$.

Let $C_k=\{c_k(1),\ldots,c_k(\#(C_k))\}$ for $a=\{C_1,C_2\}\in\mathcal{A}$, where $\#(C_k)$ is the number of the elements in the cluster $C_k$. The configuration space $Z^{C_k}$ is defined by

$$Z^{C_k}=\left\{(z_{c_k(1)},\ldots,z_{c_k(\#(C_k))})\in\boldsymbol{R}^{\#(C_k)}\;\middle|\;\sum_{l=1}^{\#(C_k)}m_{c_k(l)}z_{c_k(l)}=0\right\},$$

which is equipped with the metric defined by

$$\langle\zeta,\tilde{\zeta}\rangle=\sum_{l=1}^{\#(C_k)}m_{c_k(l)}z_{c_k(l)}\tilde{z}_{c_k(l)},\quad |\zeta|_1=\sqrt{\langle\zeta,\zeta\rangle}$$

for $\zeta=(z_{c_k(1)},\ldots,z_{c_k(\#(C_k))})\in\boldsymbol{R}^{\#(C_k)}$ and $\tilde{\zeta}=(\tilde{z}_{c_k(1)},\ldots,\tilde{z}_{c_k(\#(C_k))})\in\boldsymbol{R}^{\#(C_k)}$. We also define two subspaces $Z^a$ and $Z_a$ of $Z^{a_{\max}}$ by

$$Z^a=\left\{z\in Z^{a_{\max}}\;\middle|\;\sum_{l\in C_k}m_lz_l=0\text{ for each cluster }C_k\in a\right\},\quad Z_a=Z^{a_{\max}}\ominus Z^a,$$

and write $z^a=\pi_{\|}^a z$ and $z_a=\pi_{\|,a}z$ for $z\in Z^{a_{\max}}$, where $\pi_{\|}^a$ and $\pi_{\|,a}$ are the orthogonal projections of $Z^{a_{\max}}$ onto $Z^a$ and $Z_a$, respectively. One can identify $Z^a$ with $Z^{C_1}\oplus Z^{C_2}$.

Let $\lambda\ge\inf\Theta$. We will define the original operator $\tilde{A}$ of a conjugate operator $\hat{A}$ for the reduced Hamiltonian $\hat{H}$ at $\lambda$: We first introduce a Graf partition of unity $\{\zeta_a\}_{a\in\mathcal{A}}$ on $Z^{a_{\max}}$

such that $\zeta_a(z^{a_{\max}}) \in C^\infty(Z^{a_{\max}})$ with bounded derivatives, $0 \le \zeta_a(z^{a_{\max}}) \le 1$, on $\operatorname{supp}\zeta_a$ $|z_j - z_k| \ge \delta_1$ holds for any pair $(j,k) \not\subset a$ with some $\delta_1 > 0$, on $\operatorname{supp}\zeta_a$ $|z^a|_1 \le \delta_2$ holds with some $\delta_2 > 0$, and $\sum_{a\in\mathcal{A}} \zeta_a^2 \equiv 1$. Then we introduce an operator $\tilde{A}_R$ as follows: We put

$$\begin{aligned}\tilde{A}_R = &\frac{1}{2}(\langle z^{a_{\max}}, D_{z^{a_{\max}}}\rangle + \langle D_{z^{a_{\max}}}, z^{a_{\max}}\rangle)\\ &+ \sum_{a\in\mathcal{A}\setminus\{a_{\max}\}} \zeta_a\left(\frac{z^{a_{\max}}}{R\langle y_2 - y_3\rangle}\right) A^a \zeta_a\left(\frac{z^{a_{\max}}}{R\langle y_2 - y_3\rangle}\right)\\ &+ \zeta_{a_{\max}}\left(\frac{z^{a_{\max}}}{R\langle y_2 - y_3\rangle}\right) A_R \zeta_{a_{\max}}\left(\frac{z^{a_{\max}}}{R\langle y_2 - y_3\rangle}\right),\end{aligned} \tag{3.1}$$

where $D_{z^{a_{\max}}} = -i\nabla_{z^{a_{\max}}}$, $A^a$ and $A_R$ are the same as the one defined in §2. This definition is an extension of that of conjugate operator in the case where $N - L = L = 1$. One can check easily the fact that $\tilde{A}_R$ does commute with $k_{\text{total}}$. Then we denote by $\hat{A}_R$ the reduced operator of $U\tilde{A}_R U^*$ which acts on $\mathcal{H}$. Then we see that $(\hat{H}_0 + 1)^{-1} i[\hat{H}_0, \hat{A}_R](\hat{H}_0 + 1)^{-1}$ is bounded on $\mathcal{H}$ and

$$\begin{aligned}&(\hat{H}_0 + 1)^{-1} i[\hat{H}_0, \hat{A}_R](\hat{H}_0 + 1)^{-1}\\ &= (\hat{H}_0 + 1)^{-1}\left\{2\left(\hat{H}_0 - U\left(\sum_{j=2}^{3} H^{\{j\}}\right)U^*\right)\right\}(\hat{H}_0 + 1)^{-1}\\ &\quad + O(R^{-1}),\end{aligned} \tag{3.2}$$

which is an important estimate in order to prove the Mourre estimate for $\hat{H}$.

Then we obtain the following lemma concerned with $i[V, \hat{A}_R]$ as in §2, which is the key in order to obtain the Mourre estimate (1.13). We here omit the proof, because it is quite similar to the one of Lemma 2.3.

**Lemma 3.1.** *$(\hat{H}_0 + 1)^{-1} i[V, \hat{A}_R](\hat{H}_0 + 1)^{-1}$ is bounded on $\mathcal{H}$.*

As in §2, one can prove that $\hat{A}_R$ is a conjugate operator for $\hat{H}$ at $\lambda \ge \inf\Theta$ for sufficiently large $R > 0$, by virtue of this Lemma 3.1 and the HVZ theorem

$$\sigma_{\text{ess}}(\hat{H}) = [\inf\Theta, \infty). \tag{3.3}$$

*Remark.* The difference in the construction of a conjugate operator for $\hat{H}$ between the two cases where $d = 2$ and where $d = 3$ seems to be caused by the difference in the quantum scattering picture with a constant magnetic field between them, by virtue of $L = 2$, as mentioned in [A2]: In terms of the sets of indices of wave operators $\mathcal{A}_d$ which should be expected in the quantum scattering theory, one has

$$\mathcal{A}_2 = \{(2,3)\} \subsetneq \mathcal{A} \setminus \{a_{\max}\} = \mathcal{A}_3,$$

since charged particles and clusters are bound in the plane perpendicular to the constant magnetic field $\boldsymbol{B}$ as mentioned in §1.

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