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LONG-RANGE SCATTERING AT LOW ENERGIES

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We shall give an account on various recent results [DS2] on the quantum mechanical scattering theory in the low-energy regime for a class of negative slowly decaying potentials including the attractive Coulombic. This includes the construction of wave operators of Isozaki–Kitada type diagonalizing the *whole* continuous part of the Hamiltonian, and the study of corresponding generalized eigenfunctions and S–matrices. The S–matrices are strongly continuous at the zero-energy threshold. This result is used to derive (oscillatory) asymptotics of Dollard type S–matrices for the subclass of potentials where both families of operators are defined. For another subclass of potentials we have shown that the location of the singularities of the kernel of $S(\lambda)$ experiences an abrupt change from passing from positive energies λ to the limiting energy $\lambda = 0$ given by an explicit classical rule.

Our work may be seen as a continuation of [DS1] and [FS] in which the classical low-energy scattering theory was developed and a complete expansion of the resolvent at the zero-energy threshold was obtained, respectively.

It is well-known that one can extract some detailed information about the kernel of the S–matrix $S(\lambda)(\omega, \omega')$ by using the Isozaki–Kitada construction of the wave operators, see [IK1], [IK2], [Ya2] and [RY]. However there is one severe limitation of this approach namely that the energy needs to be bounded away from zero, viz. $\lambda \geq \lambda_0 > 0$. It turns out to be non-trivial to include $\lambda_0 = 0$, and one needs additional conditions on the potential. The conditions imposed in [DS1] and [DS2] are dictated by our desire to find nontrivial interesting behavior of scattering quantities at the zero-energy threshold; they count the virial condition and spherical symmetry of the leading term of the potential (both imposed at infinity only).

To simplify the presentation let us throughout this review assume that the potential takes the form (with $x \in \mathbb{R}^d$; $d \geq 2$)

$$V(x) = -\gamma|x|^{-\mu} + O(|x|^{-\mu-\epsilon}), \tag{1}$$

where $\mu \in (0, 2)$ and $\gamma, \epsilon > 0$. For derivatives, assume that $\partial^\beta\{V(x) + \gamma|x|^{-\mu}\} = O(|x|^{-\mu-\epsilon-|\beta|})$. We look at the Hamiltonian $H = H_0 + V$

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where $H_0 = 2^{-1}p^2$, $p = -\mathrm{i}\nabla_x$; compactly supported singularities may be included.

We proved in [DS1] that there exist *asymptotic normalized velocities* for any classical scattering orbit, i.e. a solution to Newton's equation with $|x(t)| \to \infty$,

$$\omega^{\pm} = \pm \lim_{t\to\pm\infty} x(t)/|x(t)|; \tag{2}$$

notice that this includes orbits with arbitrary energy $\lambda \geq 0$.

Taking a similar definition for asymptotic normalized velocities in quantum mechanics and virtually any preferred definition of wave operators we may simultaneous diagonalize H and either all components of ω^- or those of ω^+. We consider the corresponding S–matrices $S(\lambda)$ as operators on a related space, viz. $L^2(S^{d-1})$, and for its Schwartz kernel $S(\lambda)(\omega,\omega')$ the variables ω' and ω represent ω^- and ω^+, respectively. Our main result concerns a specific construction for which $S(\lambda)$ is regular at zero energy.

Theorem 1. *Suppose V obeys* (1). *There exists the strong limit*

$$S(+0) = \lim_{\lambda\to 0+} S(\lambda) \ \textit{in}\ \mathcal{B}(L^2(S^{d-1})).$$

The operator $S(+0)$ is unitary on $L^2(S^{d-1})$, and its kernel $S(+0)(\omega,\omega')$ is smooth outside the set $\{(\omega,\omega')\,|\, \omega\cdot\omega' = \cos\frac{\mu}{2-\mu}\pi\}$.

We remark that for the strictly homogeneous potential, $V(r) = -\gamma r^{-\mu}$, the (non-collision) zero-energy orbits are given by the implicit equation (in polar coordinates)

$$\frac{2}{1+\cos\left((2-\mu)(\theta(t)-\theta_{\mathrm{tp}})\right)} = \left(\frac{r(t)}{r_{\mathrm{tp}}}\right)^{2-\mu}. \tag{3}$$

Whence the angle betweeen the asymptotic direction and the angle θ_{tp} of the turning point is equal to $\frac{\pi}{2-\mu}$, which is in complete agreement with the relation $\omega\cdot\omega' = \cos\frac{\mu}{2-\mu}\pi$ of Theorem 1.

As indicated above our wave operators are modelled after the Isozaki–Kitada Fourier integral operator constructions

$$W^{\pm} f = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} J^{\pm} \mathrm{e}^{-\mathrm{i}tH_0} f; \tag{4}$$

$$(J^{\pm} f)(x) = (2\pi)^{-d/2} \int \mathrm{e}^{\mathrm{i}\phi^{\pm}(x,\xi)} a^{\pm}(x,\xi)\hat{f}(\xi)\mathrm{d}\xi. \tag{5}$$

Here the phases $\phi^{\pm}(x,\xi)$ are constructed in outgoing and incoming regions in terms of velocity fields $\nabla\phi^{\pm}(y(t)) = \dot{y}(t)$ for solutions to the problems

$$\begin{cases} \ddot{y}(t) = -\nabla V(y(t)), \\ \frac{1}{2}\xi^2 = \frac{1}{2}\dot{y}(t)^2 + V(y(t)), \\ y(\pm 1) = x, \\ \xi = \lim_{t\to\pm\infty} \dot{y}(t). \end{cases} \tag{6}$$

Now to incorporate the low-energy regime we change variables to "blow up" a discontinuity at $\lambda = 0$. This amounts to looking at $\xi = \sqrt{2\lambda}\omega$ as depending on two independent variables $\lambda \geq 0$ and $\omega \in S^{d-1}$. Whence we look instead at phases $\phi^{\pm}(x, \omega, \lambda)$ constructed in outgoing and incoming regions in terms of velocity fields $\nabla\phi^{\pm}(y(t)) = \dot{y}(t)$ for solutions to the problems

$$\begin{cases} \ddot{y}(t) = -\nabla V(y(t)), \\ \lambda = \frac{1}{2}\dot{y}(t)^2 + V(y(t)), \\ y(\pm 1) = x, \\ \omega = \pm \lim_{t \to \pm\infty} y(t)/|y(t)|. \end{cases} \tag{7}$$

We proved in [DS1] that indeed continuous phases $\phi^{\pm}(x, \omega, \lambda)$ can be constructed along this line. (We also studied smoothness and growth properties of these phase functions.) The S–matrices of Theorem 1 are given in terms of (4) and (5) with these phases.

We can relate our S–matrices to standard ones, for example the standard short-range S–matrices defined for $\mu > 1$. Let us here elaborate on another example provided by the Dollard S–matrices, $S_{\text{dol}}(\lambda)$, for $\mu \in (2^{-1}, 1]$. (We impose for simplicity an extra symmetry condition.) The Dollard wave operators are defined as

$$W^{\pm}_{\text{dol}} f = \lim_{t \to \pm\infty} e^{itH} U_{\text{dol}}(t) f;$$

$$U_{\text{dol}}(t) = \mathrm{e}^{-\mathrm{i}\int_0^t (p^2/2 + V(sp)1_{\{|sp| \geq R_0\}})\,\mathrm{d}s}, \ R_0 > 0.$$

Theorem 2. *Suppose V obeys* (1) *with $\mu \in (2^{-1}, 1]$, and that it is spherically symmetric. Then*

$$S_{\text{dol}}(\lambda) = \mathrm{e}^{-\mathrm{i}2\int_{R_0}^{\infty}(\sqrt{2\lambda} - \sqrt{2(\lambda - V(r))} - (2\lambda)^{-1/2}V(r))\,\mathrm{d}r} S(\lambda); \tag{8}$$

the phase factor is oscillatory as $\lambda \to 0$.

A particularly illuminating example is given by the purely Coulombic case $V = -\gamma r^{-1}$ in dimension $d \geq 3$. In this case one may compute (using special functions, see Yafaev [Ya3] for an explicit formula)

$$S_{\text{dol}}(\lambda) = \mathrm{e}^{\mathrm{i}\lambda^{-1/2}\{C_1 \ln\lambda + C_2 + o(\lambda^0)\}}(P + o(\lambda^0)), \tag{9}$$

where $(P\tau)(\omega) = \tau(-\omega)$ in agreement with Theorems 1 and 2. This formula is also in agreement with the rule, deflection angle $= -\pi$, valid for the classical zero energy Coulomb problem (see [Ne, p. 126] for example).

We remark that for short-range potentials there is also oscillatory behavior; this was proved in [Ya1] in the one-dimensional setting.

We have regularity results for generalized eigenfunctions associated with (4). Traditionally these functions are labelled $\Psi^{\pm}(\cdot, \xi)$ where ξ denotes the asymptotic momentum, cf. (6). There is a discontinuity at $\xi = 0$ which again may be "blown up" writing $\xi = \sqrt{2\lambda}\omega$ and looking instead at $\Psi^{\pm}(\cdot,\omega,\lambda) = \Psi^{\pm}(\cdot,\xi)$. Parametrized in this way we show that the generalized eigenfunctions are continuous at the energy $\lambda = 0$. We remark that they are not smooth in $\lambda \geq 0$ at the threshold (neither so are the S–matrices), which may seem somewhat surprising given the fact that the boundary value of the resolvent $(H - \lambda - i0)^{-1}$ (interpreted as acting between appropriate weighted spaces) indeed has this property; this operator appears in certain stationary formulas that we utilize. See [BGS] for explicit expansions in the purely Coulombic case.

Finally we remark that it is possible to do an analysis of the spacial asymptotics of averaged generalized eigenfunctions. Not surprisingly the qualitative behavior changes abruptly from passing from positive energies to zero energy. Such asymptotics is an integral part of the analysis of S–matrices done by Vasy, see [Va1] and [Va2].

Our proof of the classical rule, $\omega \cdot \omega' = \cos \frac{\mu}{2-\mu}\pi$, for the location of zero-energy singularities is based on a "propagation of scattering singularities result", see [Hö], [Me], [HMV] and [Va2]. Whence it is rather abstract and does not provide detailed information about the nature of the singularities. Presently it is not known how to obtain such information in the regime $\mu > 1$. We remark that the partial wave analysis of [Kv] on essentially the same problem in our opinion is incomplete.

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