

Ground State of the Polaron in the Relativistic Quantum Electrodynamics

北海道大学大学院理学研究科 佐々木 格 (Itaru Sasaki) *
Department of Mathematics, Hokkaido University

1 Introduction

We consider a system of one relativistic electron moving the Euclidean space \mathbb{R}^3 and interacting with the quantized electromagnetic field — We call it the quantized Dirac-Maxwell model. We assume that there is no external potential. We denote H by the Hamiltonian of the quantized Dirac-Maxwell model. Since there is no external potential, the total momentum of the system is conserved, namely the Hamiltonian H strongly commutes with the total momentum operator $\mathbf{P} = (P_1, P_2, P_3)$:

$$[H, P_j] = 0, \quad j = 1, 2, 3.$$

Hence the Hamiltonian H has the direct integral decomposition

$$H \cong \int_{\mathbb{R}^3}^{\oplus} H(\mathbf{p}) d\mathbf{p},$$
$$\mathbf{P} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbf{p} d\mathbf{p}.$$

Physically, the self-adjoint operator $H(\mathbf{p})$ is the Hamiltonian of the system which fixed total momentum \mathbf{p} . This model which fixed total momentum is called the polaron model in the relativistic quantum electrodynamics(QED). In this proceeding, we show some results about this polaron model. The most important fact of the polaron model is that the operator $H(\mathbf{p})$ is bounded from below for all values of fine-structure constant ([12]). We are interested

*This work was supported by JSPS Fellowship(DC2)

in the ground state energy $E(\mathbf{p})$ which is the minimum point of the spectrum $\sigma(H(\mathbf{p}))$. We give some properties of $E(\mathbf{p})$. In particular, we give the paramagnetic-type inequality

$$E(\mathbf{p}) \leq E(0), \quad \mathbf{p} \in \mathbb{R}^3. \quad (1)$$

To assume the existence of ground state of $H(\mathbf{p})$, we can show the strict paramagnetic-type inequality

$$E(\mathbf{p}) < E(0), \quad \mathbf{p} \in \mathbb{R}^3 \setminus \{0\}.$$

The polaron model of *non-relativistic* a charged particle was studied in several papers ([6, 7, 9, 10]), and a survey of results for the non-relativistic polaron is described in the book [14]. Let $H_{\text{NR}}(\mathbf{p})$ be the Hamiltonian of the Pauli-Fierz polaron model — which is a non-relativistic version of the quantized Dirac-Maxwell polaron model —, and let $E_{\text{NR}}(\mathbf{p}) := \inf \sigma(H_{\text{NR}}(\mathbf{p}))$ be the ground state energy of the Pauli-Fierz polaron. If the charged particle has no spin, $E_{\text{NR}}(\mathbf{p})$ satisfies the following diamagnetic-type inequality ([14, Section 15.2])

$$E_{\text{NR}}(0) \leq E_{\text{NR}}(\mathbf{p}), \quad \mathbf{p} \in \mathbb{R}^3. \quad (2)$$

This is a reverse inequality of (1). It is open problem whether the inequality (2) holds (or does not hold) for a non-relativistic electron with spin 1/2. Although we can show that the inequality (1) holds in the quantized Dirac-Maxwell polaron model.

Moreover we give some condition for $H(\mathbf{p})$ to have the ground state.

2 Definition of Models

In this proceeding, we take an units such that $c = \hbar = 1$, where “ c ” is the speed of light, \hbar is Planck’s constant/(2π). Let $\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$ be the Hilbert space for the relativistic electron. The Hilbert space for the photon is defined by

$$\mathcal{F}_{\text{rad}} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_{\text{sym}}^n L^2(\mathbb{R}^3 \times \{1, 2\}) \right],$$

which is the Boson Fock space over $L^2(\mathbb{R}^3 \times \{1, 2\})$. Note that we define $\bigotimes_{\text{sym}}^0 L^2(\mathbb{R}^3 \times \{1, 2\}) := \mathbb{C}$.

The Hilbert space for the quantized Dirac-Maxwell operator is given by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\text{rad}}.$$

Let $\omega(\mathbf{k}) := |\mathbf{k}|$, ($\mathbf{k} \in \mathbb{R}^3$) be the dispersion relation of the photon. The function ω defines a nonnegative self-adjoint operator on $L^2(\mathbb{R}^3 \times \{1, 2\})$. The self-adjoint operator ω is the Hamiltonian of 1-photon. n -photon Hamiltonian is defined by

$$\omega^{[n]} := \sum_{j=1}^n \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \overset{j\text{-th}}{\omega} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \quad n = 1, 2, 3, \dots$$

which is a nonnegative self-adjoint operator acting on the n -photon Hilbert space $\otimes_{\text{sym}}^n L^2(\mathbb{R}^3 \times \{1, 2\})$. We set $\omega^{[0]} := 0$ the 0-photon Hamiltonian, which is a self-adjoint operator on the vacuum \mathbb{C} . The total photon Hamiltonian is defined by

$$H_f := \bigoplus_{n=0}^{\infty} \omega^{[n]},$$

which is a nonnegative self-adjoint operator acting on \mathcal{F}_{rad} . It is easy to see that the vacuum $\Omega := (1, 0, 0, \dots) \in \mathcal{F}_{\text{rad}}$ satisfies $H_f \Omega = 0$ and the vector Ω is unique eigenvector of H_f .

For each vector $f \in L^2(\mathbb{R}^3 \times \{1, 2\})$ we define a closed operator $a(f)^*$ on \mathcal{F}_{rad} by

$$\text{Dom}(a(f)^*) := \left\{ \Psi \in \mathcal{F}_{\text{rad}} \mid \sum_{n=1}^{\infty} n \|S_n f \otimes \Psi^{(n-1)}\|^2 < \infty \right\},$$

$$(a(f)^* \Psi)^{(n)} := \sqrt{n} S_n f \otimes \Psi^{(n-1)}, \quad \Psi \in \text{Dom}(a(f)^*),$$

where S_n is the symmetrization operator on $\otimes_{\text{sym}}^n L^2(\mathbb{R}^3 \times \{1, 2\})$. We set $a(f) := (a(f)^*)^*$ the adjoint of $a(f)^*$. The operators $a(f)$, $a(f)^*$ are called an annihilation, creation operator, respectively. $a(f)$, $a(f)^*$ satisfy the following canonical commutation relations(CCR):

$$[a(f), a(g)^*] = \langle f, g \rangle,$$

$$[a(f), a(g)] = [a(f)^*, a(g)^*] = 0.$$

Let $\mathbf{e}^{(\lambda)} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\lambda = 1, 2$ be polarization vectors:

$$\mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{e}^{(\mu)}(\mathbf{k}) = \delta_{\lambda, \mu}, \quad \mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{k} = 0, \quad \mathbf{k} \in \mathbb{R}^3, \lambda, \mu \in \{1, 2\}.$$

In addition we assume that the each component of the polarization vectors is Borel measurable.

We choose and fix a function $\rho \in L^2(\mathbb{R}^3) \cap \text{Dom}(\omega^{-1/2})$. We set

$$g_j(\mathbf{k}, \lambda; \mathbf{x}) := |\mathbf{k}|^{-1/2} \rho(\mathbf{k}) e_j^{(\lambda)}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^3.$$

For each $\mathbf{x} \in \mathbb{R}^3$, $g_j(\mathbf{x}) := g_j(\cdot; \mathbf{x}) \in L^2(\mathbb{R}^3 \times \{1, 2\})$. The quantized vector potential at point $\mathbf{x} \in \mathbb{R}^3$ is defined by

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &:= (A_1(\mathbf{x}), A_2(\mathbf{x}), A_3(\mathbf{x})), \\ A_j(\mathbf{x}) &:= \frac{1}{\sqrt{2}} \overline{(a(g_j(\mathbf{x})) + a(g_j(\mathbf{x}))^*)}, \quad j = 1, 2, 3. \end{aligned}$$

For each $\mathbf{x} \in \mathbb{R}^3$, the operator $A_j(\mathbf{x})$ is a self-adjoint operator on \mathcal{F}_{rad} . The Hilbert space \mathcal{F} can be identified as follows:

$$\mathcal{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^4 \otimes \mathcal{F}_{\text{rad}} d^3 \mathbf{x}.$$

The quantized vector potential in the Dirac-Maxwell model is defined by

$$\mathbf{A}(\hat{\mathbf{x}}) := \int_{\mathbb{R}^3}^{\oplus} \mathbf{A}(\mathbf{x}) d^3 \mathbf{x},$$

which is a decomposable self-adjoint operator on \mathcal{F} .

The Hamiltonian of the quantized Dirac-Maxwell system is given by

$$\begin{aligned} \text{Dom}(H) &:= \text{Dom}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) \otimes_{\text{alg}} \text{Dom}(H_f), \\ H &:= (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + M\beta) \otimes \mathbb{1} + \mathbb{1} \otimes H_f - q\boldsymbol{\alpha} \cdot \mathbf{A}(\hat{\mathbf{x}}), \end{aligned}$$

where \otimes_{alg} means the algebraic tensor product and

$$\begin{aligned} \boldsymbol{\alpha} &= (\alpha_1, \alpha_2, \alpha_3), \quad \hat{\mathbf{p}} := -i\nabla \\ \alpha_j &:= \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix}, \quad \beta := \begin{bmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix}, \\ \sigma_1 &:= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned}$$

$M \in \mathbb{R}$ is the mass of the electron and $q \in \mathbb{R}$ is a constant which proportional to the fine-structure constant.

We omit the tensor product between \mathcal{H} and \mathcal{F}_{rad} through this proceeding.

The Hamiltonian of the quantum system must be self-adjoint (or essentially self-adjoint). At first the essential self-adjointness of H has proved by A. Arai [2]. In the following proposition, we show an improved result.

Proposition 2.1 (Essential self-adjointness). *Assume that $\rho \in \text{Dom}(\omega^{-1})$. Then, \bar{H} is self-adjoint.*

Proof. The proof is almost same as in the proof of [2]. But we do not assume the condition $\rho \in \text{Dom}(\omega^{1/2})$, and our comparison operator for the Nelson's commutator theorem differs from the operator used in [2]. Our comparison operator is the following

$$K_0 := \sqrt{-\Delta} + H_f + 1.$$

It is easy to see that $\text{Dom}(H)$ is a core for K_0 . By a standard estimates, we can obtain

$$\|H\Psi\| \leq C\|K_0\Psi\|, \quad \Psi \in \text{Dom}(H),$$

where C is a constant(see [2, Proof of Theorem 1.3]). For a constant $\Lambda > 0$, we denote by $\chi_\Lambda(\mathbf{k})$ the characteristic function of the ball $\{\mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| < \Lambda\}$. Let

$$\begin{aligned} g_{j,\Lambda}(\mathbf{k}, \lambda; \mathbf{x}) &:= \chi_\Lambda(\mathbf{k})g_j(\mathbf{k}, \lambda, \mathbf{x}), \\ A_{j,\Lambda}(\mathbf{x}) &:= \frac{1}{\sqrt{2}} \overline{a(g_{j,\Lambda}(\mathbf{x})) + a(g_{j,\Lambda}(\mathbf{x}))^*}, \\ \mathbf{A}_\Lambda(\mathbf{x}) &:= (A_{1,\Lambda}(\mathbf{x}), A_{2,\Lambda}(\mathbf{x}), A_{3,\Lambda}(\mathbf{x})). \end{aligned}$$

For each $\Psi \in \text{Dom}(H)$ with $\|\Psi\| = 1$, we have

$$\begin{aligned} &|\langle H\Psi, K_0\Psi \rangle - \langle K_0\Psi, H\Psi \rangle| \\ &= |q| \cdot |\langle \boldsymbol{\alpha} \cdot \mathbf{A}(\mathbf{x})\Psi, K_0\Psi \rangle - \langle K_0\Psi, \boldsymbol{\alpha} \cdot \mathbf{A}(\mathbf{x})\Psi \rangle| \\ &= |q| \lim_{\Lambda \rightarrow \infty} |\langle \boldsymbol{\alpha} \cdot \mathbf{A}_\Lambda(\mathbf{x})\Psi, K_0\Psi \rangle - \langle K_0\Psi, \boldsymbol{\alpha} \cdot \mathbf{A}_\Lambda(\mathbf{x})\Psi \rangle|. \end{aligned}$$

By using the CCR, we have

$$\begin{aligned} &|\langle H_f\Psi, \alpha_j A_{j,\Lambda}(\mathbf{x})\Psi \rangle - \langle \alpha_j A_{j,\Lambda}(\mathbf{x})\Psi, H_f\Psi \rangle| \\ &\leq \frac{1}{\sqrt{2}} |\langle \alpha_j \Psi, [a(\omega g_{j,\Lambda}(\mathbf{x}))^* - a(\omega g_{j,\Lambda})(\mathbf{x})]\Psi \rangle| \\ &\leq \sqrt{2} \|a(\omega g_{j,\Lambda}(\mathbf{x}))\Psi\| \\ &\leq \sqrt{2} \|\omega^{1/2} g_{j,\Lambda}(0)\| \cdot \|H_f^{1/2}\Psi\| \\ &\leq 2\|\rho\|_{L^2(\mathbb{R}^3)} \langle \Psi, K_0\Psi \rangle. \end{aligned}$$

Let $a_\lambda(\mathbf{k})$ be the operator valued distributional kernel of $a(f)$ (see [12]). We

can rigorously calculate as follows:

$$\begin{aligned}
& |\langle \sqrt{-\Delta} \Psi, \alpha_j A_{j,\Lambda}(\mathbf{x}) \Psi \rangle - \langle \alpha_j A_{j,\Lambda}(\mathbf{x}) \Psi, \sqrt{-\Delta} \Psi \rangle | \\
& \leq \sqrt{2} |\langle \alpha_j \Psi, [\sqrt{-\Delta}, a(g_{j,\Lambda}(\mathbf{x}))] \Psi \rangle | \\
& \leq \sqrt{2} \left| \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} \langle \alpha_j \Psi, |\mathbf{k}|^{-1/2} \chi_{\Lambda}(\mathbf{k}) \rho(\mathbf{k})^* a_{\lambda}(\mathbf{k}) e_j^{(\lambda)}(\mathbf{k}) [\sqrt{-\Delta}, e^{i\mathbf{k}\cdot\mathbf{x}}] \Psi \rangle \right| \\
& \leq \sqrt{2} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} |\mathbf{k}|^{-1/2} |\rho(\mathbf{k})| \cdot \| [\sqrt{-\Delta}, e^{-i\mathbf{k}\cdot\mathbf{x}}] \Psi \| \cdot \| a_{\lambda}(\mathbf{k}) \Psi \|. \quad (3)
\end{aligned}$$

On the other hand, we have

$$[\sqrt{-\Delta}, e^{-i\mathbf{k}\cdot\mathbf{x}}] = (\sqrt{-\Delta} - e^{-i\mathbf{k}\cdot\mathbf{x}} \sqrt{-\Delta} e^{i\mathbf{k}\cdot\mathbf{x}}) e^{-i\mathbf{k}\cdot\mathbf{x}} = (|\hat{\mathbf{p}}| - |\hat{\mathbf{p}} + \mathbf{k}|) e^{-i\mathbf{k}\cdot\mathbf{x}},$$

which implies that

$$\| [\sqrt{-\Delta}, e^{-i\mathbf{k}\cdot\mathbf{x}}] \| \leq |\mathbf{k}|.$$

Hence we have

$$\begin{aligned}
\text{the right hand side of (3)} & \leq \sqrt{2} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} |\rho(\mathbf{k})| \cdot \| |\mathbf{k}|^{1/2} a_{\lambda}(\mathbf{k}) \Psi \| \\
& \leq \| \rho \|_{L^2(\mathbb{R}^3)} \| H_f^{1/2} \Psi \|. \leq \| \rho \|_{L^2(\mathbb{R}^3)} \langle \Psi, K_0 \Psi \rangle.
\end{aligned}$$

Therefore we obtain

$$|\langle H \Psi, K_0 \Psi \rangle - \langle K_0 \Psi, H \Psi \rangle| \leq \text{const} \cdot \langle \Psi, K_0 \Psi \rangle.$$

By these facts and Nelson's commutator theorem, we conclude that the operator H is essentially self-adjoint. \blacksquare

Remark. In the above proposition, we show the essential self-adjointness of the quantized Dirac-Maxwell operator without external potential. It is important to prove that $H + V$ is essentially self-adjoint for an external scalar potential. A. Arai found a condition such that $H + V$ is essentially self-adjoint ([2]). However his condition for V does not include the most important Coulomb potential V_C . It is an interesting problem to prove $H + V_C$ is essentially self-adjoint.

Remark. There is few *rigorous* research on the quantized Dirac-Maxwell operator H . The fundamental results for H are written in [1, 2]. In the paper [4], A. Arai research the non-relativistic limit of the Hamiltonian H .

Let T be a closed operator on $L^2(\mathbb{R}^3 \times \{1, 2\})$. We set

$$T^{[n]} := \overline{\sum_{j=1}^n \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes T \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}},$$

$$d\Gamma(T) := 0 \oplus T \oplus T^{[2]} \oplus \cdots \oplus T^{[n]} \oplus \cdots .$$

The closed operator $d\Gamma(T)$ is called the second quantization operator of T . Note that $H_f = d\Gamma(\omega)$.

The electron momentum and photon momentum are defined by

$$\hat{\mathbf{p}} := -i\nabla = (-i\partial_{x_1}, -i\partial_{x_2}, -i\partial_{x_3}),$$

$$\mathbf{P}_{\text{rad}} := d\Gamma(\mathbf{k}) = (d\Gamma(k_1), d\Gamma(k_2), d\Gamma(k_3)),$$

respectively. The total momentum of the quantized Dirac-Maxwell system is given by

$$\mathbf{P} := (P_1, P_2, P_3) := \overline{\hat{\mathbf{p}} + \mathbf{P}_{\text{rad}}}$$

The operators P_1, P_2, P_3 are strongly commuting self-adjoint operators.

Since now we don't consider an external potential, the Hamiltonian H strongly commutes with \mathbf{P} :

Proposition 2.2 (Conservation of the total momentum). *The quantized Dirac-Maxwell Hamiltonian H strongly commutes with the total momentum operators $\mathbf{P} = (P_1, P_2, P_3)$.*

Proof. [1, Proposition 3.2] ■

Let

$$Q := \overline{\hat{\mathbf{x}} \cdot \mathbf{P}_{\text{rad}}} = \overline{\sum_{j=1}^3 \hat{x}_j \otimes d\Gamma(k_j)}$$

$$(U_F \psi)(\tilde{\mathbf{p}}) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \psi(\mathbf{x}) e^{-i\tilde{\mathbf{p}} \cdot \mathbf{x}} d^3 \mathbf{x}, \quad \psi \in L^2(\mathbb{R}^3; \mathbb{C}^4).$$

Note that U_F is the Fourier transformation on the electron Hilbert space and the variable after the transformation is $\tilde{\mathbf{p}}$. We define a unitary transformation:

$$U := (U_F \otimes \mathbb{1}) \exp(iQ).$$

By this unitary operator the total momentums are diagonalized as follows:

Proposition 2.3 (Diagonalization of the total momentum).

$$\begin{aligned} UPU^* &= \tilde{\mathbf{p}}, \\ U\bar{H}U^* &= \overline{\alpha \cdot \tilde{\mathbf{p}} + M\beta + H_f - \alpha \cdot d\Gamma(\hat{\mathbf{k}}) - q\alpha \cdot \mathbf{A}(0)} \end{aligned}$$

Proof. See [2]. ■

Remark. By the unitary operator U , the total momentum \mathbf{P} are transformed to $\tilde{\mathbf{p}}$, which is a multiplication operator by $\tilde{\mathbf{p}}$.

For each $\mathbf{p} \in \mathbb{R}^3$ we define

$$H(\mathbf{p}) := \alpha \cdot \mathbf{p} + M\beta + H_f - \alpha \cdot d\Gamma(\hat{\mathbf{k}}) - q\alpha \cdot \mathbf{A},$$

where $\mathbf{A} := \mathbf{A}(0)$. $H(\mathbf{p})$ is a symmetric operator on $\mathbb{C}^4 \otimes \mathcal{F}_{\text{rad}}$.

We need the other identification of the Hilbert space:

$$\mathcal{F} \cong L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\text{rad}} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^4 \otimes \mathcal{F}_{\text{rad}} d^3\tilde{\mathbf{p}}.$$

Under this identification, the following holds:

Proposition 2.4. *Assume that $\rho \in \text{Dom}(\omega^{-1})$. Then $\overline{H(\mathbf{p})}$ is self-adjoint and*

$$U\bar{H}U^* = \int_{\mathbb{R}^3}^{\oplus} \overline{H(\tilde{\mathbf{p}})} d^3\tilde{\mathbf{p}}.$$

Remark. $\overline{H(\mathbf{p})}$ is just the Hamiltonian with the fixed total momentum $\mathbf{p} \in \mathbb{R}^3$ — the Hamiltonian of the quantized Dirac-Maxwell polaron.

3 $H(\mathbf{p})$ is bounded from below

The most important fact for $H(\mathbf{p})$ is the following theorem:

Theorem 3.1. *Suppose that $\rho \in \text{Dom}(\omega^{-1})$. Then $\overline{H(\mathbf{p})}$ is bounded from below.*

Remark. Although the quantized Dirac-Maxwell operator H is not bounded from below, $H(\mathbf{p})$ is bounded from below.

Remark. In Theorem 3.1, the specialities of polarization vectors are essential. If the functions $\mathbf{e}^{(1)}(\mathbf{k})$, $\mathbf{e}^{(2)}(\mathbf{k})$ are not polarization vectors, the Hamiltonian $H(\mathbf{p})$ may not be bounded from below (see [1, Proposition 4.1]).

Lemma 3.2. *Let A be a positive self-adjoint operator on a separable Hilbert space. Let B be a symmetric operator with $\text{Dom}(A) \subset \text{Dom}(B)$ and*

$$\|B\Psi\| \leq \|A\Psi\|, \quad \Psi \in \text{Dom}(A).$$

Then the operator $A + B$ is positive symmetric.

Proof. By the Kato-Rellich theorem, for all $\epsilon \in (-1, 1)$, $A + \epsilon B$ is positive self-adjoint. Therefore $\langle \Psi, (A + B)\Psi \rangle \geq 0$ for all $\Psi \in \text{Dom}(A)$. ■

The first key of the proof of Theorem 3.1 is the next lemma.

Lemma 3.3. *If the following estimate holds*

$$\|(d\Gamma(\omega) + E)\Psi\|^2 \geq \|\alpha \cdot (d\Gamma(\mathbf{k}) + q\mathbf{A})\Psi\|^2, \quad \Psi \in \text{Dom}(H_f),$$

for a constant $E > 0$. Then $H(\mathbf{p})$ is bounded from below.

Proof. $\alpha \cdot \mathbf{p} + M\beta$ is bounded. Applying Lemma 3.2, the result follows. ■

We set $\mathbf{g}_\Lambda := (g_{1,\Lambda}, g_{2,\Lambda}, g_{3,\Lambda})$ and $g_{j,\Lambda}(\mathbf{k}, \lambda) = \chi_\Lambda(\mathbf{k})g_j(\mathbf{k}, \lambda; \mathbf{x} = 0)$, $j = 1, 2, 3$. One can easily show that:

Lemma 3.4. *For all $\Psi \in \text{Dom}(H_f)$, the equality*

$$\begin{aligned} \|\alpha \cdot (d\Gamma(\mathbf{k}) + q\mathbf{A})\Psi\|^2 &= \sum_{j=1}^3 \|(d\Gamma(k_j) + qA_j)\Psi\|^2 \\ &\quad - \frac{q}{\sqrt{2}} \lim_{\Lambda \rightarrow \infty} \langle \Psi, \mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}_\Lambda) + a(i\mathbf{k} \times \mathbf{g}_\Lambda)^*]\Psi \rangle \end{aligned}$$

holds, where $\mathbf{S} := (S_1, S_2, S_3)$ and $S_j := \sigma_j \oplus \sigma_j$.

We set $g_j(\mathbf{k}, \lambda) := g_j(\mathbf{k}, \lambda; \mathbf{x} = 0)$.

Lemma 3.5. *For all $\Psi \in \text{Dom}(H_f)$ and $\epsilon > 0$, the following inequality holds:*

$$\lim_{\Lambda \rightarrow \infty} |\langle \Psi, \mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}_\Lambda) + a(i\mathbf{k} \times \mathbf{g}_\Lambda)^*]\Psi \rangle| \leq \epsilon \langle \Psi, H_f \Psi \rangle + 2\epsilon^{-1} \|\rho\|_{L^2(\mathbb{R}^3)}^2 \|\Psi\|^2.$$

Proof.

$$\begin{aligned}
& |\langle \Psi, \mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}_\Lambda) + a(i\mathbf{k} \times \mathbf{g}_\Lambda)^*] \Psi \rangle| \\
& \leq 2 \left| \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} \langle \Psi, -i\mathbf{S} \cdot (\mathbf{k} \times \mathbf{g}_\Lambda(\mathbf{k}, \lambda))^* a_\lambda(\mathbf{k}) \Psi \rangle \right| \\
& \leq 2 \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} \|\mathbf{S} \cdot (\mathbf{k} \times \mathbf{g}_\Lambda(\mathbf{k}, \lambda)) \Psi\| \cdot \|a_\lambda(\mathbf{k}) \Psi\| \\
& = 2 \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} \chi_\Lambda(\mathbf{k}) \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} |\mathbf{k} \times \mathbf{e}^{(\lambda)}(\mathbf{k})|_{\mathbb{C}^3} \|\Psi\| \cdot \|a_\lambda(\mathbf{k}) \Psi\| \\
& = 2 \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} |\rho(\mathbf{k})| \cdot \|\omega(\mathbf{k})^{1/2} a_\lambda(\mathbf{k}) \Psi\| \cdot \|\Psi\| \\
& \leq 2 \left[\sum_{\lambda=1,2} \int_{\mathbb{R}^3} |\rho(\mathbf{k})|^2 d\mathbf{k} \right]^{1/2} \left[\sum_{\lambda=1,2} \int_{\mathbb{R}^3} \|\omega(\mathbf{k})^{1/2} a_\lambda(\mathbf{k}) \Psi\|^2 \right]^{1/2} \\
& = 2\sqrt{2} \|\rho\|_{L^2(\mathbb{R}^3)} \cdot \|H_f^{1/2} \Psi\| \cdot \|\Psi\|
\end{aligned}$$

Lemma 3.6. *For all $\Psi \in \text{Dom}(H_f)$ and $\epsilon > 0$, the following inequality holds:*

$$\langle \Psi, \mathbf{A}^2 \Psi \rangle \leq \left(4 + 2\epsilon + \frac{2}{\epsilon}\right) \|\rho\|_{L^2(\mathbb{R}^3)}^2 \langle \Psi, H_f \Psi \rangle + \left(2 + \frac{2}{\epsilon}\right) \|\omega^{-1/2} \rho\|_{L^2(\mathbb{R}^3)}^2 \|\Psi\|^2.$$

Proof.

$$\begin{aligned}
\langle \Psi, \mathbf{A}^2 \Psi \rangle & \leq \sum_{j=1}^3 \left[(1 + \epsilon) \|a(g_j) \Psi\|^2 + \left(1 + \frac{1}{\epsilon}\right) \|a(g_j)^* \Psi\|^2 \right] \\
& \leq \sum_{j=1}^3 \left[(1 + \epsilon) \|\omega^{-1/2} g_j\|^2 \cdot \|H_f^{1/2} \Psi\|^2 \right. \\
& \quad \left. + \left(1 + \frac{1}{\epsilon}\right) \|\omega^{-1/2} g_j\|^2 \cdot \|H_f^{1/2} \Psi\|^2 + \left(1 + \frac{1}{\epsilon}\right) \|g_j\|^2 \cdot \|\Psi\|^2 \right]
\end{aligned}$$

The main key in the proof of Theorem 3.1 is the following Lemma:

Lemma 3.7. *For all $\Psi \in \text{Dom}(H_f)$, the inequality*

$$\begin{aligned}
& \|H_f \Psi\|^2 - \sum_{j=1}^3 \|\text{d}\Gamma(k_j) \Psi\|^2 - q \langle \text{d}\Gamma(\mathbf{k}) \Psi, \mathbf{A} \Psi \rangle - q \langle \mathbf{A} \Psi, \text{d}\Gamma(\mathbf{k}) \Psi \rangle \\
& \geq -2q^2 G \langle \Psi, H_f \Psi \rangle.
\end{aligned}$$

holds, where

$$G := \int_{\mathbb{R}^3} \frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|^2} d\mathbf{k} + \sup_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \frac{1}{|\mathbf{k}|} \int_{\mathbb{R}^3} \frac{|\rho(\mathbf{k}')|^2}{|\mathbf{k}'|^2} \left(\mathbf{k} \cdot \frac{\mathbf{k}'}{|\mathbf{k}'|} \right) d\mathbf{k}'$$

is a finite constant.

Proof. We prove this lemma by a formal calculation. One, however, can verify this proof by a tedious calculation. We set

$$\begin{aligned} \mathcal{J} &:= \sum_{\lambda=1,2} \int, & \mathcal{J}' &:= \sum_{\mu=1,2} \int, \\ a &:= a_\lambda(\mathbf{k}), & b &:= a_\mu(\mathbf{k}'). \end{aligned}$$

Then, we have

$$\begin{aligned} &\|H_f \Psi\|^2 - \sum_{j=1}^3 \|\mathrm{d}\Gamma(k_j) \Psi\|^2 - q \langle \mathrm{d}\Gamma(\mathbf{k}) \Psi, \mathbf{A} \Psi \rangle - q \langle \mathbf{A} \Psi, \mathrm{d}\Gamma(\mathbf{k}) \Psi \rangle \\ &= \mathcal{J} d\mathbf{k} \mathcal{J}' d\mathbf{k}' \left[|\mathbf{k}| \cdot |\mathbf{k}'| \langle a^* a \Psi, b^* b \Psi \rangle - \mathbf{k} \cdot \mathbf{k}' \langle a^* a \Psi, b^* b \Psi \rangle \right] \\ &\quad - \frac{q}{\sqrt{2}} \mathcal{J} \mathcal{J}' (\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)^*) \langle a^* a \Psi, b \Psi \rangle d\mathbf{k} d\mathbf{k}' + \text{c.c.} \\ &\quad - \frac{q}{\sqrt{2}} \mathcal{J} \mathcal{J}' (\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)) \langle a^* a \Psi, b^* \Psi \rangle d\mathbf{k} d\mathbf{k}' + \text{c.c.} \end{aligned} \tag{4}$$

By the CCR, we have

$$\begin{aligned} \langle a^* a \Psi, b^* b \Psi \rangle &= \langle a \Psi, ab^* b \Psi \rangle = \langle ab \Psi, ab \Psi \rangle + \delta(\mathbf{k} - \mathbf{k}') \delta_{\lambda, \mu} \langle a \Psi, b \Psi \rangle, \\ \langle a^* a \Psi, b \Psi \rangle &= \langle a \Psi, ab \Psi \rangle, \\ \langle a^* a \Psi, b^* \Psi \rangle &= \langle ab \Psi, a \Psi \rangle + \delta(\mathbf{k} - \mathbf{k}') \delta_{\lambda, \mu} \langle a \Psi, \Psi \rangle. \end{aligned}$$

Since the polarization vectors are orthogonal to \mathbf{k} , we have $\mathbf{k} \cdot \mathbf{g}(\mathbf{k}, \lambda) = 0$.

We set

$$F := F_\mu(\mathbf{k}, \mathbf{k}') := -\sqrt{2}q \frac{\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)}{|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'}$$

Then we have

$$\begin{aligned}
(4) &= \int \int' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}') \langle ab\Psi, ab\Psi \rangle d\mathbf{k}d\mathbf{k}' \\
&\quad - \sqrt{2q} \int \int' (\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)^*) [\langle a\Psi, ab\Psi \rangle + \langle ab\Psi, a\Psi \rangle] d\mathbf{k}d\mathbf{k}' \\
&\quad - \sqrt{2q} \int \int' (\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)) \langle ab\Psi, a\Psi \rangle d\mathbf{k}d\mathbf{k}' \\
&= \int \int' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}') \left[\langle ab\Psi, ab\Psi \rangle + F \langle a\Psi, ab\Psi \rangle + F^* \langle ab\Psi, a\Psi \rangle \right] d\mathbf{k}d\mathbf{k}' \\
&= \int \int' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}') \|a(b + F)\Psi\|^2 d\mathbf{k}d\mathbf{k}' \\
&\quad - \int \int' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}') |F|^2 \|a\Psi\|^2 d\mathbf{k}d\mathbf{k}' \\
&\geq - \int \int' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}') |F|^2 \|a\Psi\|^2 d\mathbf{k}d\mathbf{k}' \\
&= -2q^2 \int \int' \frac{|\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)|^2}{|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'} \|a\Psi\|^2 d\mathbf{k}d\mathbf{k}' \\
&= -2q^2 \int d\mathbf{k} \left[\frac{1}{|\mathbf{k}|} \int' \frac{|\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)|^2}{|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'} d\mathbf{k}' \right] |\mathbf{k}| \cdot \|a\Psi\|^2 \\
&\geq -2q^2 \left[\sup_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \frac{1}{|\mathbf{k}|} \int' \frac{|\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)|^2}{|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'} d\mathbf{k}' \right] \|H_f^{1/2} \Psi\|^2.
\end{aligned}$$

By using the property of polarization vectors, we obtain

$$G = \left[\sup_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \frac{1}{|\mathbf{k}|} \int' \frac{|\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)|^2}{|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'} d\mathbf{k}' \right].$$

Proof of Theorem 3.1. By Lemma 3.4-3.7, Lemma 3.3 holds for a large constant $E > 0$. Therefore $H(\mathbf{p})$ is bounded from below. ■

4 Properties of the quantized Dirac-Maxwell Polaron

We define

$$H_m(\mathbf{p}) := H(\mathbf{p}) + mN_b,$$

where $N_b := d\Gamma(\mathbb{1})$ is the number operator. Throughout this proceeding, we assume that $\rho \in \text{Dom}(\omega^{-1})$ when we consider the case $m = 0$. The operator $H_m(\mathbf{p})$ is essentially self-adjoint and bounded from below. For a constant $m \geq 0$, we set

$$E_m(\mathbf{p}) := \inf \sigma(\overline{H_m(\mathbf{p})}), \quad E(\mathbf{p}) := E_0(\mathbf{p}).$$

$E_m(\mathbf{p})$ is called the ground state energy of $\overline{H_m(\mathbf{p})}$. The ground state energy $E_m(\mathbf{p})$ depends on all the constants in $H_m(\mathbf{p})$ — the total momentum $\mathbf{p} \in \mathbb{R}^3$, the electron mass $M \in \mathbb{R}$, the virtual photon mass $m \geq 0$ and the coupling constant $g \in \mathbb{R}$.

If $m > 0$, for all total momentum $\mathbf{p} \in \mathbb{R}^3$, the massive Hamiltonian $\overline{H_m(\mathbf{p})}$ has a ground state [1], i.e., $E_m(\mathbf{p})$ is an eigenvalue of $\overline{H_m(\mathbf{p})}$. In the *non-relativistic* polaron, the massive non-relativistic polaron Hamiltonian $H_{\text{NR}}(\mathbf{p})$ may not have a ground state for $|\mathbf{p}| > 1$ under a suitable units¹ (see [6]). This means that an dressed one electron state of total momentum \mathbf{p} with $|\mathbf{p}| > 1$ does not exist. This fact is interpreted as follows:

In the non-relativistic polaron model (in particular the Pauli-Fierz polaron model), the electron is described non-relativistically and the photon is described relativistically. The velocities of non-relativistic electron and relativistic photon are given by

$$\begin{aligned} \mathbf{v}_{\text{nr-electron}} &:= i \left[\frac{\hat{\mathbf{p}}^2}{2M}, \mathbf{x} \right] = \frac{\hat{\mathbf{p}}}{M}, \\ \mathbf{v}_{\text{photon}} &:= i \left[\sqrt{-\Delta}, \mathbf{x} \right] = \frac{-i\nabla}{\sqrt{-\Delta}}, \end{aligned}$$

which implies that the non-relativistic electron can have arbitrary large velocity and the velocity of photon does not exceed 1. Hence if the velocity of non-relativistic electron is larger than 1, the Cherenkov radiation occurs and the electron is unstable. In the case $|\mathbf{p}| > 1$, the non-relativistic polaron model $H_{\text{NR}}(\mathbf{p})$ includes an electron with $\mathbf{v}_{\text{nr-electron}} > 1$ and hence a stable dressed one electron state does not exist (see [6]). Moreover the condition about \mathbf{p} for $H_{\text{NR}}(\mathbf{p})$ to have a ground state has a strong restriction on the virtual photon mass $m > 0$ (see [14, Section 15.2]).

In our relativistic polaron model, the electron and photon are described relativistically and the velocity of the relativistic electron is given by

$$\mathbf{v}_{\text{electron}} := i[\boldsymbol{\alpha} \cdot \mathbf{p} + M\beta, \mathbf{x}] = \boldsymbol{\alpha}.$$

¹the electron mass $M = 1$.

Hence the velocity of the relativistic electron does not exceed 1, and the Cherenkov radiation does not occur. Therefore for any $\mathbf{p} \in \mathbb{R}^3$ the massive relativistic polaron $H_m(\mathbf{p})$ can have a ground state.

In the massless case $m = 0$, it is expect that the non-relativistic polaron $H_{\text{NR}}(\mathbf{p})$ has no ground state for all $\mathbf{p} \neq 0$ ([6]). Nevertheless the Dirac polaron Hamiltonian $H(\mathbf{p})$ has a ground state for a suitable condition including the infrared regularization condition.

To prove the existence of a ground state, it is important to study the ground state energy $E_m(\mathbf{p})$. To emphasis of a dependence of the variables, we sometimes write $E_m(\mathbf{p})$ as $E_m(\mathbf{p}, M)$ or $E_m(\mathbf{p}, M, q)$. In the following, we show the properties of $E_m(\mathbf{p})$ without proofs².

Proposition 4.1 (Concavity of the ground state energy). *$E_m(\mathbf{p})$ is a concave function in the variables $(\mathbf{p}, M, m, \mathbf{q}) \in \mathbb{R}^3 \times \mathbb{R} \times [0, \infty) \times \mathbb{R}$.*

Proposition 4.2 (Continuity of the ground state energy). *$E_m(\mathbf{p}, M)$ is a Lipschitz continuous function of (\mathbf{p}, M) , i.e.,*

$$|E_m(\mathbf{p}, M) - E_m(\mathbf{p}', M)| \leq \sqrt{|\mathbf{p} - \mathbf{p}'|^2 + (M - M')^2}, \quad \mathbf{p}, \mathbf{p}' \in \mathbb{R}^3, M, M' \in \mathbb{R}.$$

Proposition 4.3 (Reflective symmetry in the electron mass M). *$\overline{H_m(\mathbf{p}, M)}$ is unitarily equivalent to $\overline{H_m(\mathbf{p}, -M)}$. In particular $E_m(\mathbf{p}, M) = E_m(\mathbf{p}, -M)$, and $E_m(\mathbf{p}, M) \leq E_m(\mathbf{p}, 0)$.*

Proposition 4.4 (Symmetry in the total momentum). *Assume that for an orthogonal matrix $T \in O(3)$, $|\rho(\mathbf{k})| = |\rho(T\mathbf{k})|$ a.e. $\mathbf{k} \in \mathbb{R}^3$. Then $\overline{H_m(\mathbf{p})}$ is unitarily equivalent to $\overline{H_m(T\mathbf{p})}$, and $E_m(\mathbf{p}) = E_m(T\mathbf{p})$.*

If the cutoff function $|\rho(\mathbf{k})|$ has the reflective symmetry at the origin, the following inequality holds.

Theorem 4.5 (Paramagnetic type inequality). *Assume that $|\rho(\mathbf{k})| = |\rho(-\mathbf{k})|$, a.e. $\mathbf{k} \in \mathbb{R}^3$. Then, the paramagnetic type inequality*

$$E_m(\mathbf{p}) \leq E_m(0), \quad \mathbf{p} \in \mathbb{R}^3$$

holds.

Remark. In the Pauli-Fierz polaron model without spin, the ground state energy $E_{\text{NR}}(\mathbf{p})$ satisfies the diamagnetic type inequality (see [14]):

$$E_{\text{NR}}(0) \leq E_{\text{NR}}(\mathbf{p}), \quad \mathbf{p} \in \mathbb{R}.$$

²The parts of proofs are written in [13]

Assuming that $H_m(0)$ has a ground state, we can obtain the following strict paramagnetic type inequality:

Theorem 4.6 (Strict paramagnetic type inequality). *Assume that $|\rho(\mathbf{k})| = |\rho(-\mathbf{k})|$ a.e. $\mathbf{k} \in \mathbb{R}^3$. If $H_m(0)$ has a ground state, then*

$$E_m(\mathbf{p}) < E_m(0), \quad \text{for all } \mathbf{p} \in \mathbb{R}^3 \setminus \{0\}.$$

Remark. When $m > 0$, the massive Hamiltonian $H_m(0)$ has a ground state. In the massless case $m = 0$, $H(0)$ has a ground state if an infrared cutoff is imposed (see the conditions in the following Theorems).

Proposition 4.7 (Spherical symmetry in the total momentum). *Assume that $|\rho(\mathbf{k})|$ is a rotation invariant function. Then $\overline{H_m(\mathbf{p})}$ is unitarily equivalent to $H_m(\mathbf{p}')$ for all $\mathbf{p}' \in \mathbb{R}^3$ with $|\mathbf{p}| = |\mathbf{p}'|$. In particular $E_m(\mathbf{p})$ is rotation invariant in \mathbf{p} , and $E_m(\mathbf{p}) \geq E_m(\mathbf{p}')$ if $|\mathbf{p}| \leq |\mathbf{p}'|$.*

Proposition 4.8 (Massless limit of the ground state energy). *$E_m(\mathbf{p})$ is monotone non-decreasing in $m \geq 0$ and*

$$\lim_{m \rightarrow +0} E_m(\mathbf{p}) = E_0(\mathbf{p}).$$

Generally, by Proposition 4.2, the following inequality holds:

$$0 \leq E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}|, \quad \mathbf{p}, \mathbf{k} \in \mathbb{R}^3.$$

This quantity is important for the existence of a ground state. If the electron mass M is not zero, we can get a strict inequality:

Proposition 4.9. *In the case $m > 0$, the inequality $E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| > 0$ holds for all $M \neq 0$, $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$ and $\mathbf{p} \in \mathbb{R}^3$. In the massless case $m = 0$, for all $\mathbf{p} \in \mathbb{R}^3$, there exists a constant $M \geq 0$ such that the inequality $E(\mathbf{p} - \mathbf{k}) - E(\mathbf{p}) + |\mathbf{k}| > 0$ holds for all $|M| > M$ and $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$.*

The following two theorems are main results in this report:

Theorem 4.10. *Suppose that*

$$\liminf_{m \rightarrow +0} \int_{\mathbb{R}^3} \frac{q^2}{(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^2} \frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|} d\mathbf{k} < 1. \quad (5)$$

Then the polaron Hamiltonian $\overline{H(\mathbf{p})}$ has a ground state.

The condition (5) has a restriction in q , and $E_m(\mathbf{p})$ depends on q . Next we show that an another existence theorem of a ground state. In the following theorem, there is no restriction in the coupling constant q .

Theorem 4.11. *Suppose that ρ is rotation invariant and there is an open set $S \subset \mathbb{R}^3$ such that $\bar{S} := \text{supp } \rho$ and ρ is continuously differentiable function in S . Assume that for all R , the set $S_R := \{\mathbf{k} \in S \mid |\mathbf{k}| < R\}$ has the cone property, and*

$$\limsup_{m \rightarrow +0} \int_S \frac{q^2}{(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^2} \frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|} d\mathbf{k} < \infty,$$

and for all $p \in [1, 2)$ and $R > 0$, the following inequalities hold:

$$\begin{aligned} \sup_{0 < m < 1} \int_{S_R} \left[(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-2} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p d\mathbf{k} < \infty, \\ \sup_{0 < m < 1} \int_{S_R} \left[(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1} \frac{|\nabla \rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p d\mathbf{k} < \infty, \\ \sup_{0 < m < 1} \int_{S_R} \left[(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1} \frac{1}{\sqrt{k_1^2 + k_2^2}} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p d\mathbf{k} < \infty. \end{aligned}$$

Then $\overline{H(\mathbf{p})}$ has a ground state.

Remark (Examples of the cutoff function). We set

$$\begin{aligned} \rho_1 &:= \chi_D, \quad D := \{\mathbf{k} \in \mathbb{R}^3 \mid 0 < \epsilon < |\mathbf{k}| < \Lambda\}, \\ \rho_2 &:= |\mathbf{k}| \exp(-|\mathbf{k}|^2), \end{aligned}$$

where ϵ, Λ are constants such that $0 < \epsilon < \Lambda$. For the zero total momentum $\mathbf{p} = 0$ and a large electron mass $M > 0$, we can show that the conditions in Theorem 4.11 are true with $\rho = \rho_j$, ($j = 1, 2$).

References

- [1] A. Arai, Fundamental Properties of the Hamiltonian of a Dirac Particle Coupled to the Quantized Radiation Field, *Hokkaido Univ. Preprint Series in Math*, No. 447, 1999.
- [2] A. Arai, A particle-field Hamiltonian in relativistic quantum electrodynamics, *J. Math. Phys.* **41** (2000), 4271-4283.
- [3] A. Arai, Fock Spaces and Quantum Fields, Nippon-Hyouronsha, Tokyo, 2000(in Japanese).
- [4] A. Arai, Non-relativistic limit of a Dirac-Maxwell operator in relativistic quantum electrodynamics, *Rev. Math. Phys.* **15** (2003), 245-270.

- [5] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* 151 (1997) 455-503.
- [6] T. Chen, Operator-theoretic infrared renormalization and construction of dressed 1-particle states in non-relativistic QED, mp_arc 01-310.
- [7] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschr. Phys.*, **22**, 159-198, (1974).
- [8] M. Griesemer, E. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. math.* **145** (2001), 557-595.
- [9] F. Hiroshima and H. Spohn Ground state degeneracy of the Pauli-Fierz Hamiltonian with spin, *Adv. Theor. Math. Phys.* **5** (2001) 1091-1104.
- [10] F. Hiroshima, H. Spohn, Mass renormalization in nonrelativistic quantum electrodynamics, *J. Math. Phys.* **46**, 042302, (2005).
- [11] E.H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, American Mathematical Society, 1997.
- [12] I. Sasaki, Ground state energy of polaron in the relativistic quantum electrodynamics, *J. Math. Phys.* **46**, 1 (2005).
- [13] I. Sasaki, Ground state of a model in the relativistic quantum electrodynamics with a fixed total momentum, mp_arc 05-433, (2005).
- [14] H. Spohn, *Dynamics of Charged Particles and Their Radiation Field*, Cambridge University Press, Cambridge, (2004).
- [15] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol.I*, Academic Press, New York, 1972
- [16] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol. II*, Academic Press, New York, 1975.
- [17] B. Thaller, *The Dirac Equation*, Springer-Verlag, Berlin, Heidelberg, 1992.