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# SOME RIGOROUS RESULTS ON SCATTERING INDUCED DECOHERENCE

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ABSTRACT. We consider a non relativistic quantum system consisting of $K$ heavy and $N$ light particles in dimension three, where each heavy particle interacts with the light ones via a two-body potential $\alpha V$. No interaction is assumed among particles of the same kind.

Choosing an initial state in a product form and assuming $\alpha$ sufficiently small we characterize the asymptotic dynamics of the system in the limit of small mass ratio, with an explicit control of the error. In the case $K=1$ the result is extended to arbitrary $\alpha$.

The proof relies on a perturbative analysis and exploits a generalized version of the standard dispersive estimates for the Schrödinger group.

Exploiting the asymptotic formula, it is also outlined an application to the problem of the decoherence effect produced on a heavy particle by the interaction with the light ones.

## 1. INTRODUCTION

In this contribution we shall present some recent results about decoherence, proofs will be found in [AFFT2]. The study of the dynamics of a non relativistic quantum system composed by heavy and light particles is of interest in different contexts and, in particular, the search for asymptotic formulas for the wave function of the system in the small mass ratio limit is particularly relevant in many applications.

Here we consider the case of $K$ heavy and $N$ light particles in dimension three, where the heavy particles interact with the light ones via a two-body potential. To simplify the analysis we assume that light particles are not interacting among themselves and that the same is true for the heavy ones.

We are interested in the dynamics of the system when the initial state is in a product form, i.e. no correlation among the heavy and light particles is assumed at time zero. Moreover we consider the regime where only scattering processes between light and heavy particles can occur and no other reaction channel is possible.

We remark that the situation is qualitatively different from the usual case studied in molecular physics where the light particles, at time zero, are assumed to be in a bound state corresponding to some energy level $E_n(R_1,\ldots,R_K)$ produced by the interaction potential with the heavy ones considered in the fixed positions $R_1,\ldots,R_K$.

In that case it is well known that the standard Born-Oppenheimer approximation applies and one finds that, for small values of the mass ratio, the rapid motion of the light particles produces a persistent effect on the slow (semiclassical) motion of the heavy ones, described by the effective potential $E_n(R_1, \ldots, R_K)$ (see e.g. [H], [HJ] and references therein).
The main physical motivation at the root of our work is the attempt to understand in a quantitative way the loss of quantum coherence induced on a heavy particle by the interaction with the light ones. This problem has attracted much interest among physicists in the last years (see e.g. [JZ], [GF], [HS], [HUBHAZ], [GJKKSZ], [BGJKS] and references therein). In particular in ([HS], [HUBHAZ]) the authors performed a very accurate analysis of the possible sources of collisional decoherence in experiments of matter wave interpherometry. We consider the results presented in the final section of this contribution a rigorous version of some of their results.
At a qualitative level, the process has been clearly described in [JZ], where the starting point is the analysis of the two-body problem involving one heavy and one light particle. For a small value of the mass ratio, it is reasonable to expect a separation of two characteristic time scales, a slow one for the dynamics of the heavy particle and a fast one for the light particle. Therefore, for an initial state of the form $\phi(R)\chi(r)$, where $\phi$ and $\chi$ are the initial wave functions of the heavy and the light particle respectively, the evolution of the system is assumed to be given by the instantaneous transition

$$\phi(R)\chi(r) \rightarrow \phi(R)\,(S(R)\chi)\,(r) \tag{1.1}$$

where $S(R)$ is the scattering operator corresponding to the heavy particle fixed at the position $R$.
The transition (1.1) simply means that the final state is computed in a zero-th order adiabatic approximation, with the light particle instantaneously scattered far away by the heavy one considered as a fixed scattering center.
Notice that in (1.1) the evolution in time of the system is completely neglected, in the sense that time zero for the heavy particle corresponds to infinite time for the light one.
In [JZ] the authors start from formula (1.1) to investigate the effect of multiple scattering events. They assume the existence of collision times and a free dynamics of the heavy particle in between. In this way they restore, by hand, a time evolution of the system.
Our aim in this work is to give a mathematical analysis of this kind of process in the more general situation of many heavy and light particles.
Starting from the Schrödinger equation of the system we shall derive the asymptotic form of the wave function for small values of the mass ratio and give an estimate of the error. The result can be considered as a rigorous derivation of formula (1.1), generalized to the many particle case and modified taking into account the internal motion of the heavy particles.
Furthermore, we shall exploit the asymptotic form of the wave function to briefly outline how the decoherence effect produced on the heavy particles can be explicitly computed.
At this stage our analysis leaves untouched the question of the derivation of a master equation for the heavy particles in presence of an environment consisting of a rarefied gas of light particles (see e.g. [JZ], [HS]). Such derivation involves the more delicate question

of the control of the limit $N \to \infty$ and requires a non trivial extension of the techniques used here.
The analysis presented here generalizes previous results for the two-body case obtained in [DFT], where a one-dimensional system of two particles interacting via a zero-range potential was considered, and in [AFFT1], where the result is generalized to dimension three with a generic interaction potential (se also [CCF] for the case of a three-dimensional two-body system with zero-range interaction).

We now give a more precise formulation of the model. Let us consider the following Hamiltonian

$$H = \sum_{l=1}^{K} \left( -\frac{\hbar^2}{2M} \Delta_{R_l} + U_l(R_l) \right) + \sum_{j=1}^{N} \left( -\frac{\hbar^2}{2m} \Delta_{r_j} + \alpha_0 \sum_{l=1}^{K} V(r_j - R_l) \right) \tag{1.2}$$

acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{3(K+N)}) = L^2(\mathbb{R}^{3K}) \otimes L^2(\mathbb{R}^{3N})$.
The Hamiltonian (1.2) describes the dynamics of a quantum system composed by a sub-system of $K$ particles with position coordinates denoted by $R = (R_1, \dots, R_K) \in \mathbb{R}^{3K}$, each of mass $M$ and subject to the one-body interaction potential $U_l$, plus a sub-system of $N$ particles with position coordinates denoted by $r = (r_1, \dots, r_N) \in \mathbb{R}^{3N}$, each of mass $m$. The interaction among the particles of the two sub-systems is described by the two-body potential $\alpha_0 V$, where $\alpha_0 > 0$.
The potentials $U_l$, $V$ are assumed to be smooth and rapidly decreasing at infinity.
In order to simplify the notation we fix $\hbar = M = 1$ and denote $m = \varepsilon$; moreover the coupling constant will be rescaled according to $\alpha = \varepsilon \alpha_0$, with $\alpha$ fixed. Then the Hamiltonian takes the form

$$H(\varepsilon) = X + \frac{1}{\varepsilon} \sum_{j=1}^{N} \left( h_{0j} + \alpha \sum_{l=1}^{K} V(r_j - R_l) \right) \tag{1.3}$$

where

$$X = \sum_{l=1}^{K} \left( -\frac{1}{2} \Delta_{R_l} + U_l(R_l) \right) \tag{1.4}$$

$$h_{0j} = -\frac{1}{2} \Delta_{r_j} \tag{1.5}$$

We are interested in the following Cauchy problem

$$\begin{cases} i\frac{\partial}{\partial t}\Psi^\varepsilon(t) = H(\varepsilon)\Psi^\varepsilon(t) \\ \Psi^\varepsilon(0;R,r) = \phi(R)\prod_{j=1}^N \chi_j(r_j) \equiv \phi(R)\chi(r) \end{cases} \tag{1.6}$$

where $\phi$, $\chi_j$ are sufficiently smooth given elements of $L^2(\mathbb{R}^{3K})$ and $L^2(\mathbb{R}^3)$ respectively. Our aim is the characterization of the asymptotic behavior of the solution $\Psi^\varepsilon(t)$ for $\varepsilon \to 0$, with a control of the error.
Under suitable assumptions on the potentials and the initial state, we find that the asymptotic form $\Psi_a^\varepsilon(t)$ of the wave function $\Psi^\varepsilon(t)$ for $\varepsilon \to 0$ is explicitly given by

$$\Psi_a^\varepsilon(t;R,r) = \int dR' e^{-itX}(R,R')\phi(R')\prod_{j=1}^N \left(e^{-i\frac{t}{\varepsilon}h_{0j}}\Omega_+(R')^{-1}\chi_j\right)(r_j) \tag{1.7}$$

where, for any fixed $R \in \mathbb{R}^{3K}$, we have defined the following wave operator acting in the one-particle space $L^2(\mathbb{R}^3)$ of the $j$-th light particle

$$\Omega_+(R)\chi_j = \lim_{\tau\to+\infty} e^{i\tau h_j(R)}\, e^{-i\tau h_{0j}}\chi_j \tag{1.8}$$

and in (1.8) we have denoted $h_j(R) = h_{0j} + \alpha\sum_{l=1}^K V(r_j - R_l)$.
It should be remarked that (1.7) reduces to (1.1) if we formally set $t = 0$ and assume that $\Omega_+(R')^{-1}\chi_j$ can be replaced by $S(R')\chi_j$, which is approximately true for suitably chosen state $\chi_j$ (see e.g. [HS]).
It is important to notice that the asymptotic evolution defined by (1.7) is not factorized, due to the parametric dependence on the configuration of the heavy particles of the wave operator acting on each light particle state.
Then the asymptotic wave function describes an entangled state for the whole system of heavy and light particles. In turn this implies a loss of quantum coherence for the heavy particles as a consequence of the interaction with the light ones.
The precise formulation of the approximation result will be given in the next section. Here we only mention that in the case of an arbitrary number $K$ of heavy particles our result holds for $\alpha$ sufficiently small, while in the simpler case $K = 1$ we can prove the result for any $\alpha$.

## 2. Mian Result

Our main result is given in theorem 1 below and concerns the general case $K \geq 1$. In the special case $K = 1$ we find a stronger result, summarized in theorem 1'.
The reason is that for the first case we follow and adapt to our situation the approach to dispersive estimates valid for small potentials as given in [RS], while for the second one we can prove the result for any $\alpha$ exploiting the approach to dispersive estimates via wave operators developed in [Y].
As a consequence we shall introduce two sets of different assumptions on the potential $V$ and on the initial state $\chi$ of the light particles.

Let us denote by $W^{m,p}(\mathbb{R}^d)$, $H^m(\mathbb{R}^d)$ the standard Sobolev spaces and by $W^{m,p}_n(\mathbb{R}^d)$, $H^m_n(\mathbb{R}^d)$ the corresponding weighted Sobolev spaces, with $m, n, d \in \mathbb{N}$, $1 \leq p \leq \infty$.
Then we introduce the following assumptions

(A-1) $U_l \in W^{4,\infty}_2(\mathbb{R}^3)$, for $l = 1, \ldots, K$;
(A-2) $\phi \in H^4_2(\mathbb{R}^{3K})$ and $\|\phi\|_{L^2(\mathbb{R}^{3K})} = 1$;

and, moreover, for the case $K \geq 1$

(A-3) $V \in W^{4,1}(\mathbb{R}^3) \cap H^4(\mathbb{R}^3)$;
(A-4) $\chi \in L^1(\mathbb{R}^{3N}) \cap L^2(\mathbb{R}^{3N})$, $\chi(r) = \prod_{j=1}^N \chi_j(r_j)$, and $\|\chi_j\|_{L^2(\mathbb{R}^3)} = 1$ for $j = 1, \ldots N$.

while for the case $K = 1$

(A-5) $V \in W^{4,\infty}_\delta(\mathbb{R}^3)$, $\delta > 5$, and $V \geq 0$;
(A-6) $\chi \in W^{4,1}(\mathbb{R}^{3N}) \cap H^4(\mathbb{R}^{3N})$, $\chi(r) = \prod_{j=1}^N \chi_j(r_j)$, and $\|\chi_j\|_{L^2(\mathbb{R}^3)} = 1$ for $j = 1, \ldots N$.

We notice that, under the above assumptions, the Hamiltonian (1.3) is self-adjoint and bounded from below in $\mathcal{H}$, the wave operator introduced in (1.8) exists and moreover the expression for the asymptotic wave function (1.7) makes sense.

We now state our main result. Denoting by $\|\cdot\|$ the norm in $\mathcal{H}$, for the case $K \geq 1$ we have

**Theorem 1.** *Let $K \geq 1$ and let us assume that $U_l$, $\phi$, $V$, $\chi$ satisfy assumptions* (A-1),(A-2),(A-3),(A-4); *moreover let us fix $T$, $0 < T < \infty$, and define*

$$\alpha^* = \frac{\pi^{2/3}}{24K} \|V\|_{W^{4,1}}^{-1/3} \|V\|_{H^4}^{-2/3} \tag{2.1}$$

*Then for any $t \in (0, T]$ and $\alpha < \alpha^*$ we have*

$$\|\Psi^\varepsilon(t) - \Psi^\varepsilon_a(t)\| \leq C \sqrt{\frac{\varepsilon}{t}} \tag{2.2}$$

*where $C$ is a positive constant depending on the interaction, the initial state and $T$.*

On the other hand, for the case $K = 1$ we prove

**Theorem 1′.** *Let $K = 1$ and let us assume that $U$, $\phi$, $V$, $\chi$ satisfy assumptions* (A-1),(A-2),(A-5),(A-6); *moreover let us fix $T$, $0 < T < \infty$. Then for any $t \in (0, T]$ the estimate* (2.2) *holds, with a positive constant $C'$ depending on the interaction, the initial state and $T$.*

Let us briefly comment on the results stated in theorems 1, 1'.
The estimate (2.2) clearly fails for $t \to 0$ and this fact is intrinsic in the expression of $\Psi^\varepsilon_a(t)$, which doesn't approach $\Psi^\varepsilon(0)$ for $t \to 0$.
Another remark concerns the estimate of the error in (2.2), which is probably not optimal. Indeed in the simpler two-body case analysed in [DFT], where the explicit form of the unitary group is available, the error found is $O(\varepsilon)$.
We also notice that the knowledge of the explicit dependence of the constant $C$ on $T$ is clearly interesting and in our proof we have that $C$ grows with $T$, which is rather unnatural from the physical point of view and is a consequence of the specific method of the proof. In the two-body case studied in [AFFT1] it is shown that the constant $C$ is bounded for $T$ large.
Concerning the method of the proof, we observe that the approach is perturbative and it is essentially based on Duhamel's formula. The main technical ingredient for the estimates is a generalized version of the dispersive estimates for Schrödinger groups.
In fact, during the proof we shall consider the one-particle Hamiltonian for the $j$-th light particle $h_j(R)$, parametrically dependent on the positions $R \in \mathbb{R}^{3K}$ of the heavy ones.
In particular, we shall need estimates (uniform with respect to $R$) for the $L^\infty$-norm of derivatives with respect to $R$ of the unitary evolution $e^{-i\tau h_j(R)}\chi_j$.
Apparently, such kind of estimates haven't been considered in the literature (see e.g. [RS], [Sc], [Y]) and then we exhibit a proof (see section 4) for $K \geq 1$ and small potential, following the approach of [RS], and also for $K = 1$ and arbitrary potential, following [Y].

We list few notations which will be used in the following.
- For any fixed $R \in \mathbb{R}^{3K}$

$$h(R) = \sum_{j=1}^{N} h_j(R) = \sum_{j=1}^{N} \left( h_{0j} + \alpha \sum_{l=1}^{K} V(r_j - R_l) \right) \tag{2.3}$$

denotes an operator in the Hilbert space $L^2(\mathbb{R}^{3N})$, while $h_j(R)$ and $h_{0j}$ act in the one-particle space $L^2(\mathbb{R}^3)$ of the $j$-th light particle.
- For any $t > 0$

$$\xi(t; R, r) = \phi(R) \left(e^{-ith(R)}\chi\right)(r) = \phi(R) \prod_{j=1}^{N} \left(e^{-ith_j(R)}\chi_j\right)(r_j) \tag{2.4}$$

$$\zeta^\varepsilon(t; R, r) = \left[e^{-itX}\xi(\varepsilon^{-1}t)\right](R, r) \tag{2.5}$$

defines two vectors $\xi(t)$, $\zeta^\varepsilon(t) \in \mathcal{H}$.
- $V_{jl}$ denotes the multiplication operator by $V(r_j - R_l)$.
- The derivative of order $\gamma$ with respect to $s$-th component of $R_m$ is denoted by

$$D^\gamma_{m,s} = \frac{\partial^\gamma}{\partial R^\gamma_{m,s}}, \qquad \gamma \in \mathbb{N}, \quad m = 1, \dots, K, \quad s = 1, 2, 3 \tag{2.6}$$

with $D^1_{m,s} = D_{m,s}$.
- The operator norm of $A : E \to F$, where $E$, $F$ are Banach spaces, is denoted by $\|A\|_{\mathcal{L}(E,F)}$.

We give here the a sketch of proof of our main result and we briefly discuss the generalized dispersive estimates we have used.
We start with the proof of theorem 1 and then we assume $\alpha < \alpha^*$. This condition guarantees the validity of a key technical ingredient, i.e. the uniform dispersive estimate

$$\sup_R \left\| \left( \prod_{i=1}^{n} D^{\gamma_i}_{m_i,s_i} \right) e^{-ith_j(R)} \right\|_{\mathcal{L}(L^1,L^\infty)} \leq \frac{C_\gamma}{t^{3/2}} \tag{2.7}$$

The estimate (2.7) is valid for any string of integers $\gamma_i$ (including zero), $m = 1,\ldots,K$, $s = 1,2,3$ and $\alpha < \alpha^*$.
In the proof we also make use of the following uniform $L^2$ estimate

$$\sup_R \left\| \left( \prod_{i=1}^{n} D^{\gamma_i}_{m_i,s_i} \right) \prod_{k=1,k\neq j}^{N} e^{-ith_k(R)}\chi_k \right\|_{L^2(\mathbb{R}^{3(N-1)})} \leq \hat{C}_\gamma \tag{2.8}$$

The first step is to show that $\zeta^\varepsilon(t)$ is a good approximation of $\Psi^\varepsilon_a(t)$ and this is a direct consequence of the existence of the wave operator (1.8).
Indeed, from (1.7) and (2.5) we have

$$\begin{aligned}
&\|\Psi^\varepsilon_a(t) - \zeta^\varepsilon(t)\| \\
&= \left( \int dR |\phi(R)|^2 \left\| \prod_{j=1}^{N} e^{-i\frac{t}{\varepsilon}h_{0j}} \Omega_+(R)^{-1}\chi_j - \prod_{j=1}^{N} e^{-i\frac{t}{\varepsilon}h_j(R)}\chi_j \right\|^2_{L^2(\mathbb{R}^{3N})} \right)^{1/2} \\
&\leq \sup_R \left\| \prod_{j=1}^{N} e^{-i\frac{t}{\varepsilon}h_{0j}} \Omega_+(R)^{-1}\chi_j - \prod_{j=1}^{N} e^{-i\frac{t}{\varepsilon}h_j(R)}\chi_j \right\|_{L^2(\mathbb{R}^{3N})} \\
&\leq \sup_R \sum_{n=1}^{N} \Big\| e^{-i\frac{t}{\varepsilon}h_1(R)}\chi_1 \cdots e^{-i\frac{t}{\varepsilon}h_{n-1}(R)}\chi_{n-1} \left( e^{-i\frac{t}{\varepsilon}h_{0n}}\Omega_+(R)^{-1}\chi_n - e^{-i\frac{t}{\varepsilon}h_n(R)}\chi_n \right) \\
&\quad e^{-i\frac{t}{\varepsilon}h_{0n+1}}\Omega_+(R)^{-1}\chi_{n+1} \cdots e^{-i\frac{t}{\varepsilon}h_{0N}}\Omega_+(R)^{-1}\chi_N \Big\|_{L^2(\mathbb{R}^{3N})} \\
&\leq \sup_R \sum_{n=1}^{N} \left\| e^{i\frac{t}{\varepsilon}h_{0n}} e^{-i\frac{t}{\varepsilon}h_n(R)}\chi_n - \Omega_+(R)^{-1}\chi_n \right\|_{L^2}
\end{aligned} \tag{2.9}$$

Let us recall that for any $\tau > 0$

$$e^{i\tau h_{0n}} e^{-i\tau h_n(R)}\chi_n - \Omega_+(R)^{-1}\chi_n = i\alpha \int_\tau^\infty ds \;\, e^{ish_{0n}} V_R e^{-ish_n(R)}\chi_n \tag{2.10}$$

Then using the dispersive estimate (2.7) we conclude

$$\|\Psi_a^\varepsilon(t) - \zeta^\varepsilon(t)\| \leq \alpha \sup_R \left( \|V_R\|_{L^2} \sum_{n=1}^N \int_{t/\varepsilon}^\infty ds \left\| e^{-ish_n(R)} \chi_n \right\|_{L^\infty} \right)$$
$$\leq \frac{\sqrt{\varepsilon}}{\sqrt{t}} C_0 \alpha K \|V\|_{L^2} \left( \sum_{n=1}^N \|\chi_n\|_{L^1} \right) \tag{2.11}$$

The next and more delicate step is to show that $\zeta^\varepsilon(t)$ approximates the solution $\Psi^\varepsilon(t)$ . By a direct computation one has

$$i\frac{\partial}{\partial t}\zeta^\varepsilon(t) = H(\varepsilon)\zeta^\varepsilon(t) + \mathcal{R}^\varepsilon(t) \tag{2.12}$$

where

$$\mathcal{R}^\varepsilon(t) = \frac{\alpha}{\varepsilon} \sum_{j=1}^N \sum_{l=1}^K [e^{-itX}, V_{jl}]\xi(\varepsilon^{-1}t) \tag{2.13}$$

Using Duhamel's formula and writing

$$[e^{-itX}, V_{jl}] = (e^{-itX} - I)V_{jl} - V_{jl}(e^{-itX} - I) \tag{2.14}$$

we have

$$\|\Psi^\varepsilon(t) - \zeta^\varepsilon(t)\| \leq \int_0^t ds \|\mathcal{R}^\varepsilon(s)\|$$
$$\leq \frac{\alpha}{\varepsilon} \sum_{l=1}^K \sum_{j=1}^N \int_0^t ds \left[ \|(e^{-isX} - I)V_{jl}\xi(\varepsilon^{-1}s)\| + \|V_{jl}(e^{-isX} - I)\xi(\varepsilon^{-1}s)\| \right]$$
$$= \alpha \sum_{l=1}^K \sum_{j=1}^N \int_0^{\varepsilon^{-1}t} d\sigma \left( \mathcal{A}_{jl}^\varepsilon(\sigma) + \mathcal{B}_{jl}^\varepsilon(\sigma) \right) \tag{2.15}$$

where we have defined

$$\mathcal{A}_{jl}^\varepsilon(\sigma) = \|(e^{-i\varepsilon\sigma X} - I)V_{jl}\xi(\sigma)\| \tag{2.16}$$

$$\mathcal{B}_{jl}^\varepsilon(\sigma) = \|V_{jl}(e^{-i\varepsilon\sigma X} - I)\xi(\sigma)\| \tag{2.17}$$

The problem is then reduced to the estimate of the two terms (2.16) and (2.17). The basic idea is that both terms are controlled by $e^{-i\varepsilon\sigma X} - I$ for $\sigma$ small with respect to $\varepsilon^{-1}t$ and by the dispersive character of the unitary group $e^{-i\sigma h(R)}$ for $\sigma$ of the order $\varepsilon^{-1}t$. It turns out that such strategy is easily implemented for (2.16) while for (2.17) the estimate is a bit more involved.

The proof of theorem 1′ is obtained following exactly the same line of the previous one with only slight modifications, the main difference lies in the use of other generalized dispersive estimates.

Indeed we fix $K = 1$ and assume (A-1), (A-2), (A-5), (A-6); moreover we make use of the following uniform estimates which hold for any value of $\alpha$

$$\sup_R \left\| D_{s_1}^{\gamma_1} D_{s_2}^{\gamma_2} D_{s_3}^{\gamma_3} e^{-ith_k(R)} \chi_k \right\|_{L^\infty} \leq \frac{B_\gamma}{t^{3/2}} \|\chi_k\|_{W^{\gamma,1}} \tag{2.18}$$

$$\sup_R \left\| D_{s_1}^{\gamma_1} D_{s_2}^{\gamma_2} D_{s_3}^{\gamma_3} \prod_{k=1}^{N} e^{-ith_k(R)} \chi_k \right\|_{L^2(\mathbb{R}^{3N})} \leq \hat{B}_\gamma \tag{2.19}$$

where $\gamma = \sum_{i=1}^{3} \gamma_i$ and $B_\gamma$ and $\hat{B}_\gamma$ are positive constants, increasing with $\gamma$.
The estimates (2.18), (2.19) replace, in the case $K = 1$, the uniform estimates (2.7), (2.8), which hold for $\alpha < \alpha^*$ in the general case $K \geq 1$.
Now we shall briefly comment these dispersive estimates. Estimates (2.7) and (2.8) are pertubative results and they rely on the spectral representation of the unitary group of a light particle and on the Born expansion of the resolvent. It is then possible to show that in this case the Born series is absolutely convergent and that it is possible to derive with respect to $R$, which play here the role of a parameter, term by term. We adapted to our situation the method exploited in [RS]
Estimates (2.18) and (2.19) have a different nature since they are non perturbative and they rely on the approach to dispersive estimates via wave operator of Yajima, see [Y]. Using the mapping properties of the wave operators between Sobolev spaces and extracting the dependence on $R$ of the unitary group by translation operators, it is straightforward to prove (2.18) and (2.19).

## 3. Application to decoherence

Some of the most peculiar aspects of Quantum Mechanics are direct consequences of the superposition principle, i.e. the fact that the normalized superposition of two quantum states is a possible state for a quantum system. Interference effects between the two states and their consequences on the statistics of the expected results of a measurement performed on the system do not have any explanation within the realm of classical probability theory.
On the other hand this highly non-classical behavior is extremely sensitive to the interaction with the environment. The mechanism of irreversible diffusion of quantum correlations in the environment is generally referred to as decoherence. The analysis of this phenomenon within the frame of Quantum Theory is of great interest and, at the same time, of great difficulty inasmuch as results about the dynamics of large quantum systems are required in order to build up non-trivial models of environment.
In this section we consider the mechanism of decoherence on a heavy particles (the system) scattered by $N$ light particles (the environment). For this purpose we follow closely the line of reasoning of Joos and Zeh ([JZ]) and we exploit formula (1.7) for the asymptotic wave function in the simpler case $U = 0$.
(For other rigorous analysis of the mechanism of decoherence see e.g. [D], [DS], [CCF]).

All the information concerning the dynamical behavior of observables associated with the heavy particle is contained in the reduced density matrix, which in our case is the positive, trace class operator $\rho^\varepsilon(t)$ in $L^2(\mathbb{R}^3)$ with $\mathrm{Tr}\,\rho^\varepsilon(t) = 1$ and integral kernel given by

$$\rho^\varepsilon(t; R, R') = \int_{\mathbb{R}^{3N}} dr\, \Psi^\varepsilon(t; R, r)\overline{\Psi^\varepsilon}(t; R', r) \tag{3.1}$$

An immediate consequence of theorems 1, 1′ is that for $\varepsilon \to 0$ the operator $\rho^\varepsilon(t)$ converges in the trace class norm to the asymptotic reduced density matrix

$$\rho^a(t) = e^{-itX_0}\rho_0^a e^{itX_0} \tag{3.2}$$

where $\rho_0^a$ is a density matrix whose integral kernel is

$$\rho_0^a(R, R') = \phi(R)\overline{\phi}(R')\mathcal{I}(R, R') \tag{3.3}$$

$$\mathcal{I}(R, R') = \prod_{j=1}^{N} \left(\Omega_+(R')^{-1}\chi_j, \Omega_+(R)^{-1}\chi_j\right)_{L^2} \tag{3.4}$$

and $(\cdot,\cdot)_{L^2}$ denotes the scalar product in $L^2(\mathbb{R}^3)$.

Notice that the asymptotic dynamics of the heavy particle described by $\rho^a(t)$ is generated by $X_0$, i.e. the Hamiltonian of the heavy particle when the light particles are absent. The effect of the interaction with the light particles is expressed in the change of the initial state from $\phi(R)\overline{\phi}(R')$ to $\phi(R)\overline{\phi}(R')\mathcal{I}(R, R')$. Significantly the new initial state is not in product form, meaning that entanglement between the system and the environment has taken place. Yet, at this level of approximation, entanglement is instantaneous and no result about the dynamics of the decoherence process can be extracted from the approximate reduced density matrix.

Moreover notice that $\mathcal{I}(R, R) = 1$ , $\mathcal{I}(R, R') = \overline{\mathcal{I}}(R', R)$ and $|\mathcal{I}(R, R')| \leq 1$. For $N$ large $\mathcal{I}(R, R')$ tends to be exponentially close to zero for $R \neq R'$.

In ([AFFT1]) a concrete example was considered in the case $N = 1$. The initial condition for the heavy particle were chosen as a superposition of two identical wave packets heading one against the other. The wave packet of an isolated heavy particle would have shown interference fringes typical of a two slit experiment. The decrease in the interference pattern, induced by the interaction with a light particle, was computed and taken as a measure of the decoherence effect.

We want to give here a brief summary of the same analysis for any number of light particles where the enhancement of the decoherence effect due to multiple scattering is easily verified.

Let the initial state be the coherent superposition of two wave packets in the following form

$$\phi(R) = b^{-1}\left(f_\sigma^+(R) + f_\sigma^-(R)\right), \quad b \equiv \|f_\sigma^+ + f_\sigma^-\|_{L^2} \tag{3.5}$$

$$f_\sigma^\pm(R) = \frac{1}{\sigma^{3/2}} f\left(\frac{R \pm R_0}{\sigma}\right) e^{\pm i P_0 \cdot R}, \quad R_0, P_0 \in \mathbb{R}^3 \tag{3.6}$$

where $f$ is a real valued function in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ with $\|f\|_{L^2} = 1$, $R_0 = (0,0,|R_0|)$, $P_0 = (0,0,-|P_0|)$.
It is clear that under the free evolution the two wave packets (3.5) exhibit a significant overlap and the typical interference effect is observed.
On the other hand, if we take into account the interaction with the light particles and introduce the further assumption $\sigma\alpha\|\nabla V\|_{L^2} \ll 1$, it can be easily seen that $\rho^a(t)$ is approximated by

$$\rho^e(t) \;=\; e^{-itX_0}\rho_0^e\, e^{itX_0} \tag{3.7}$$

where $\rho_0^e$ has integral kernel

$$\rho_0^e(R,R') = \frac{1}{b^2}\left(\left|f_\sigma^+(R)\right|^2 + \left|f_\sigma^-(R)\right|^2 + \Lambda f_\sigma^+(R)\overline{f}_\sigma^-(R') + \overline{\Lambda} f_\sigma^-(R)\overline{f}_\sigma^+(R')\right) \tag{3.8}$$

$$\Lambda \equiv \prod_{j=1}^{N} \left(\Omega_+(R_0)^{-1}\chi_j, \Omega_+(-R_0)^{-1}\chi_j\right)_{L^2} \tag{3.9}$$

The proof is easily obtained adapting the proof given in ([AFFT1]) for the case $N = 1$.
It is clear from (3.9) that, if the interaction is absent, then $\Lambda = 1$ and (3.8) describes the pure state corresponding to the coherent superposition of $f_\sigma^+$ and $f_\sigma^-$ evolving according to the free Hamiltonian.
If the interaction with the light particles is present then $\Omega_+(R_0)^{-1} \neq I$ and $|\Lambda| \ll 1$ for $N$ large. For specific model interaction the factor $\Lambda$ can also be explicitly computed (see e.g. the one dimensional case treated in [DFT]).
This means that the only effect of the interaction on the heavy particle is to reduce the non diagonal terms in (3.8) by the factor $\Lambda$ and this means that the interference effects for the heavy particle are correspondingly reduced.
In this sense we can say that a (partial) decoherence effect on the heavy particle has been induced and, moreover, the effect is completely characterized by the parameter $\Lambda$.

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