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Hankel determinants and substitutions – some results and problems

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1. Introduction. Let $A^*$ be a free monoid generated by a non-empty set $A$, i.e., $A^*$ is the set of finite words over $A$ with the empty word $\lambda$. We put $A^*: = A^* \cup A^*$, where $N$ is the set of non-negative integers, so that $A^*$ is the set of infinite words over $A$. Any monoid morphism $\sigma : A^* \rightarrow A^*$ can be extended to a map $\sigma : A^* \rightarrow A^*$ by $\sigma(a_0a_1a_2 \cdots ) = \sigma(a_0)\sigma(a_1)\sigma(a_2) \cdots (a_i \in A)$, which is a so-called substitution (over $A$). We say that $\sigma$ is of constant length $k$ iff $\sigma(x)$ is a finite word of length $k$ for all $x \in A$. A fixed point of a substitution $\sigma$ is an infinite word $\nu \in A^*$ satisfying $\sigma(\nu) = \nu$.

The fixed point of a substitution $\sigma$ over $\{a, b\}$ defined by

$\sigma(a) = ab, \sigma(b) = ba$ (resp., $\sigma(a) = ab, \sigma(b) = a$)

prefixed by $a$ is referred to as the Thue-Morse word (resp., the Fibonacci word).

Let $q \geq 1$ be an integer. We denote by $\text{ord}_q(n)$ the largest integer $e \geq 0$ such that $n$ is divisible by $q^e$. We say a word $w = w_1w_2w_3 \cdots$ is a $q$-adic Toeplitz word iff $w_m = w_n$ holds for any positive integers $m, n$ satisfying $\text{ord}_q(m) = \text{ord}_q(n)$. Let $\sigma$ be a substitution over an infinite alphabet $A_\infty = \{a_0, a_1, a_2, \ldots \}$ defined by

$\sigma(a_n) = a_0a_{n+1}$ ($n = 0, 1, 2, \ldots$).

For some of the symbols $a_0, a_1, a_2, \ldots$, we also write $a_0 = a, a_1 = b, a_2 = c$, etc. The substitution $\sigma$ has a unique fixed point

$\omega = abacabacabacaba \ldots$, which is a 2-adic Toeplitz word. Any 2-adic Toeplitz word over a finite or an infinite alphabet $B$ can be written by

$\tau(\omega) = \tau(\omega_1)\tau(\omega_2)\tau(\omega_3) \ldots$,

where $\tau$ is a map from $A_\infty$ to $B$, and $\omega = \omega_1\omega_2\omega_3 \ldots (\omega_i \in A_\infty)$. In this sense the word $\omega$
is a universal 2-adic Toeplitz word.

Our objective is to get something interesting related to determinants

\[ H_n^{(m)} = H_n^{(m)}[\varphi] := \det(\varphi_{m+i+1}^n)_{0 \leq i \leq n-1, 0 \leq j \leq n-1}, \]

\[ H_n = H_n[\varphi] := H_n^{(0)}[\varphi] \]

for a given infinite word \( \varphi = \varphi_0 \varphi_1 \varphi_2 \ldots \) \((\varphi \in A)\) over a finite, or an infinite alphabet \( A \), where \( H_n^{(m)} \) is considered to be an element of \( Z[A] \), i.e., a polynomial in independent variables \( \in A \) with integer coefficients. \( H_n^{(m)}[\varphi] \) can be extended to \((m, n) \in Z \times N\) by setting \( H_n^{(0)} := 1, \varphi_n := 0 \) \((m < 0)\), where \( N \) denotes the set of non-negative integers. In the following two sections, we give a very rough survey on the results related to \( H_n^{(m)} \). In Section 2, we give some results on \( H_n^{(m)}[\varphi] \) for a general word \( \varphi \), cf. [K-T-W], [T2]. In Section 3, we give some of the results related to \( H_n^{(m)}[\varphi] \) when \( \varphi \) is the Thue-Morse word, the Fibonacci word, and a Fibonacci-type word, cf. [A-P-ZXW-ZYW], [K-T-W], [T2]. In Section 4, we give a new characterization of the 2-adic Toeplitz words \( \varphi \) by an algebraic property (completely reducibility) of \( H_n[\varphi] \), cf. [M-T-Tn]. We shall give no proofs, but state only results with minimum definition.

2. General properties of \( H_n[\varphi] \). The set \( A^\ast \) becomes a complete metric space with respect to the metric defined by

\[ d(\xi, \eta) := \exp(-\inf(n; \xi_n \neq \eta_n)) \((\xi = \xi_0 \xi_1 \xi_2 \ldots, \eta = \eta_0 \eta_1 \eta_2 \ldots \in A^\ast \)(\xi_n, \eta_n \in A))\).

As usual, \( K((Z)) \) denotes the set of formal Laurent series of one variable \( Z \) over a field \( K \). We put

\[ K := Q(A) (\supset A) . \]

The set \( K((z^{-1})) \) becomes a metric space induced by a non-Archimedean norm defined by

\[ |\varphi^{(h)}| := \exp(-n_0 + h), \varphi_0 := \inf(n \in N; \varphi_n \neq 0) \(101 := 0\) \]

for

\[ \varphi^{(h)} = \sum_{n \geq 0} \varphi_n z^{-n+h} \in K((z^{-1})) \] (1)

with \( h \in Z := \{0, \pm 1, \pm 2, \ldots\} \). Note that \( |\varphi^{(h)}| = \exp h \) holds if \( \varphi = \varphi_0 \varphi_1 \varphi_2 \ldots \in A^\ast \subset A^\ast \).
\(K^*\). If \(\varphi\) is a finite word of length \(k\), then \(\varphi_n := 0\) for \(n \geq k\). For any given \(\varphi = \varphi_0 \varphi_1 \varphi_2 \cdots \in K^*\), we say that \((P, Q) \in K[z]^2\) is an \(h\)-Padé pair of order \(m\) for \(\varphi\) iff

\[||Q\varphi^{(h)}-P|| < \exp(-m), \quad Q \neq 0, \quad \deg Q := \deg Q \leq m\]  \hspace{1cm} (2)

holds. The usual Padé pair (for a formal Laurent series) agrees with the \(h\)-Padé pair with \(h = -1\) (for a word), cf. [N-S]. It is known that an \(h\)-Padé pair \((P, Q)\) of order \(m\) for \(\varphi\) always exists for any \(h \in \mathbb{Z}, m \geq 0, \varphi \in K^*\), cf. Lemma 1. [T2]. For \(h\)-Padé pairs \((P, Q)\) of order \(m\) for \(\varphi\), a rational function \(P/Q \in K(z)\) is uniquely determined for any given \(h \in \mathbb{Z}, m \geq 0, \varphi \in K^*\). The element \(P/Q \in K(z)\) for an \(h\)-Padé pair \((P, Q)\) of order \(m\) for \(\varphi \in K^*\) is referred to as the \(h\)-Padé approximant of order \(m\) for \(\varphi\). A number \(m \in \mathbb{N}\) is called a normal \(h\)-index for \(\varphi \in K^*\) if (2) implies \(\deg Q = m\). A normal \(h\)-Padé pair, i.e., \(\deg Q\) is a normal \(h\)-index, is said to be normalized if the leading coefficient of \(Q\) equals one. Normal \((-1)\)-indices (resp. \((-1)\)-Padé pairs, \((-1)\)-Padé approximants) will be simply referred to as normal indices (resp. Padé pairs, Padé approximants). The set of all the normal \(h\)-indices for \(\varphi\) will be denoted by

\[\Lambda_n(\varphi) := \{m \in \mathbb{N}; m \text{ is normal } h\text{-indices for } \varphi\}\]

\[\Lambda(\varphi) := \Lambda_{-1}(\varphi)\].

We can consider the series (1) over \(K = \mathbb{Q}(a, b, \ldots)\) with \(a, b, \ldots \in \mathbb{C}\). In such a case, \(\varphi^{(h)}\) defined by (1) turns out to be not only an element of \(C((z^{-1}))\), but also an analytic function on \(\{z \in \mathbb{C}; |z| > 1\}\), and the \(h\)-Padé approximant of order \(m\) for \(\varphi^{(h)}\) pointwise converges to \(\varphi^{(h)}\) with respect to the usual topology on \(\mathbb{C}\) for each \(z \in \mathbb{C}, |z| > 1\) as \(m\) tends to infinity.

**Proposition 1** (cf. [T1]). Let \(\varphi \in K^*\) be a word over \(K = \mathbb{Q}(A)\) with an alphabet \(A\) possibly consisting of infinite letters.

\[H_{n+1}(m)[\varphi] = (-1)^{m/2} \prod_{Q(\varphi) = 0} P(z) \quad (h, m \in \mathbb{Z}, m \geq 0),\]

where \((P, Q)\) is a normalized \(h\)-Padé pair of degree \(m\) for \(\varphi\), \(|x|\) denotes the largest integer not exceeding a real number \(x\), and \(\prod_{Q(\varphi) = 0}\) indicates a product.
taken over all the zeros of $Q$ with their multiplicity in any field $\widetilde{K}$ containing an algebraic closure of $K$.

**Remark 1.** We can take $\widetilde{K}=\mathbb{C}$ in Proposition 1 in the case where $A$ is a subset (possibly empty) of $\mathbb{C}$.

**Remark 2.** If $m$ is not a normal $h$-index of $\varphi$, then $P, Q \in K[z]$ have common zeros. Hence, it follows from Proposition 1 that $m \notin \Lambda_h(\varphi)$ implies $H_{n,m}(\varphi)=0$. The converse of this fact is valid, cf. Lemma 2, [T2].

In particular, Proposition 1 holds for all the fixed point $\varphi \in A^w(CK^*)$ of a substitution over any alphabet $A$. The following remark is useful, while it is valid only for a word $\varphi$ consisting of at most two symbols.

**Remark 3.** Let $M$ be a matrix of size $n \times n$ with entries consisting of two variables $a, b$ (symbols). Then

$$\det M = (a-b)^{n-1} (pa+qb) \in \mathbb{Z}[a,b],$$

where $p, q$ are integers defined by

$$p = \det M \mid (a, b) = (1, 0), \quad q = \det M \mid (a, b) = (0, 1).$$

3. **Thue-Morse, and Fibonacci cases.**

J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen considered $H_{n,m}(\zeta)$ for the Thue-Morse sequence $\zeta=ababab\cdots$ with $(a,b)=(1,0)$, and showed that the 2-dimensional word $H_1(\zeta) \mod 2$ of $(n,m) \in \mathbb{N}^2$ is 2-dimensionally automatic; it is remarkable that $A(\zeta)=\mathbb{N}$ is known, cf. [A-P-ZXW-ZYW].

In general, it is very difficult to give an explicit formula of $H_n(\eta(n))$ for a given infinite word $\varphi$ that is not periodic, while explicit formulae of $H_n(\eta)$ are completely given for the Fibonacci word $\eta=abaab\ldots$, cf. Theorems 1-5 in [K-T-W]. By $f_n$ we denote the $n$-th Fibonacci number ($f_0=1, f_1=f_0+f_1=2$). Let
\[ n = \sum_{i \geq 0} \delta_i(n)f_i \quad (\delta_i(n) \in \{0,1\}, \delta_{i+1}(n)\delta_i(n) = 0 \text{ for all } i \geq 0) \]

be the representation of \( n \) in the Fibonacci base due to Zeckendorf. We write

\[ m \equiv_k n \]

iff \( \delta_i(m) = \delta_i(n) \) holds for all \( 0 \leq i < k \). We put

\[ \tau(k,S) := \begin{cases} 1 & \text{if } k = s \pmod{6} \text{ for some } s \in S, \\ 0 & \text{otherwise}, \end{cases} \]

for a subset \( S \) of \( \{0,1,2,3,4,5\} \). Then we have, for instance,

**Proposition 2** (Theorem 3, [K-T-W]). For any \( k, m, i \geq 0 \) integers satisfying

\[ m \equiv_{k+1} n \quad 0 \leq i < f_{k-1} \]

the following formulae hold:

\[ H_{f_k}^{(m)}(n) \left| (a,b) = \left( 1, 0 \right) \right. = \tau(k;2)\tau(k;1,4)^i f_{k-1}, \]

if either \( \delta_{k+1}(m) = 0 \) and \( 0 \leq i < f_{k-1} \),

or \( \delta_{k+1}(m) = 1 \) and \( 0 \leq i < f_k \),

\[ = \tau(k;1,2,4) f_{k-2}, \]

if either \( \delta_{k+1}(m) = 0 \) and \( i = f_{k-1} \),

or \( i = f_{k+1}-1 \),

\[ = 0 \text{ otherwise,} \]

\[ H_{f_k}^{(m)}(n) \left| (a,b) = \left( 0, 1 \right) \right. = \tau(k;1,2,4)\tau(k;1,4)^i f_{k-2}, \]

if either \( \delta_{k+1}(m) = 0 \) and \( 0 \leq i < f_{k-1} \),

or \( \delta_{k+1}(m) = 1 \) and \( 0 \leq i < f_k \),

\[ = \tau(k;2) f_{k-3}, \]

if either \( \delta_{k+1}(m) = 0 \) and \( i = f_{k-1} \),

or \( i = f_{k+1}-1 \),

\[ = 0 \text{ otherwise.} \]

Notice that Proposition 2 together with Remark 3 gives a part of the explicit formulae for \( H_n^{(*)}( \eta ) \). In comparison with the automacity result for the 2-dimensional word \( (H_n^{(*)}( \eta ) (\text{mod } 2))_{(n,m) \in \mathbb{N}^2} \) for the Thue-Morse sequence.
given in [A-P-ZXW-ZYW], we gave an explicit expression of the
2-dimensional word \( H_n^{(m)}(\tau) \) \((n,m) \in \mathbb{N}^2\) for the Fibonacci word \( \tau \text{=abaabab...} \), which
is rather complicated, cf. Theorem 5 in [K-T-W].

In [T2], we developed a theory of analysis on words, especially for words of
Fibonacci type, i.e. the fixed points of substitutions of the form
\[
\sigma(a) := a \cdot b, \quad \sigma(b) := a \quad (k>0).
\tag{3}
\]
We denote by \( \tau = \tau(a,b;k) \) the fixed point of the substitution defined by (3), and
by \( |w|_* := \text{the number of occurrences of an identical symbol } x \text{ appearing in a word } w \). In Sections 2, 3 in [T2], we gave explicit formulae for the continued
fraction expansion with partial denominators \( K[z] \) \((K = \mathbb{Q}(a,b)) \) and normalized
Padé pairs for the Laurent series \( \tau^{-1}(z) = \tau(z,a,b;k) \) defined by (1) with \( \varrho = \tau \),
cf. Theorems 1-12, [T2]. For instance, we have

**Proposition 3 (Theorem 8, [T2]).** Let \( k \geq 2 \). The continued fraction expansion
of the Laurent series \( \tau^{-1}(z) \in K((z^{-1})) \) for \( \tau = \tau(a,b;k) \in \{a,b\}^* \) is given by
\[
\tau^{-1}(z) = [0; a^{-1}(z-1), (-1)^m(a-b)^{-1}h_m^*b_{m-1}^*, (-1)^n(a-b)h_m^{-1}h_{m+1}^{-1}(z-1)]_{m=0}^\infty,
\]
where
\[
\begin{align*}
h_n := |\sigma^*(a)|a^*+|\sigma^*(a)|b^*(=g.a+g_{n-1}b \in \mathbb{Z}[a,b]), & f_* = f_* := |\sigma^*(a)|, \\
b_* = b_*^*(z;k) := & z^f_* \sum_{0 \leq j < f_*+1} z^{-1} \sum_{1 \leq j \leq k-1} (k-j) z^{j-1} f_*+1+k \sum_{0 \leq j < f_*-1} z^j \in \mathbb{Z}[z].
\end{align*}
\]
If \((a,b) \in \mathbb{C}^2\), then Proposition 3 is valid under the condition
\[
a \neq b, \quad h_* (=g_*a+g_{*+1}b) \neq 0 \text{ for all } n \geq 0.
\tag{4}
\]
We can give explicit formulae for the continued fraction expansion of \( \tau^{-1}(z) =
\tau(z,a,b;k) \in C((z^{-1})) \) with \((a,b) \in \mathbb{C}^2\), which does not satisfy (4). For example,

**Proposition 4 (Theorem 9, [T2]).** Let \((a,b) \in \mathbb{C}^2\) with \( h_* = 0, a \neq 0, t \geq 0 \). Then
\[
\tau^{-1}(z) = [0; a_1, d_1, c_0, d_0, \ldots, c_{1-2}, d_{1-2}, e_1, e_2, i_m, j_m]_{m=0}^\infty
\]
holds with partial denominators \( K[z] \) given by
\[
\begin{align*}
a_1 &= a^{-1}(z-1) \\
c_m &= (-1)^* (a-b)h_m^{-1}h_{m+1}^{-1}(z-1)
\end{align*}
\]
\[ d_m = (-1)^{m+1} (a-b)^{-1} h_{m+1} z b_{m+1}^{*}, \]
\[ e_1 = (-1)^{t-1} (a-b) h_{t-1} z (z-1) b_{t-1}, \]
\[ e_2 = (-1)^{t} (a-b)^{-1} h_{t-1} z b_{t}, \]
\[ i_m = (-1)^{m+1} (a-b) h_{m+1} z g_{m+1}^{*} (z-1), \]
\[ j_m = (-1)^{m+1} (a-b)^{-1} h_{m+1} z b_{m+1}^{*}. \]

Concerning such continued fractions, we studied uniform convergence in Section 5, [T2]. Related to the product formula (Proposition 1), we studied the distribution, and the simplicity of the zero points of \( Q(z) \) for the Padé pairs \((P, Q)\) for \( \epsilon(z; a, b; k) \) in Section 4, [T2].

It is an interesting problem which asks whether we can do the same for the Thue-Morse word \( \zeta \). The fact \( \Lambda(\zeta) = N \) (cf. [A-P-ZXW-ZYW]), we mentioned, says that all the denominators of the continued fraction for \( \zeta^{(-1)}(z) \) are of degree 1. It is of special interest to find the continued fraction expansion for \( \zeta^{(-1)}(z) \) in a closed form.

We could not give a completely explicit formula for \( H_{m}^{(m)}[\epsilon(z; a, b; k)] \) when \( k \geq 2 \); we gave the following

Proposition 5 (cf. Corollary 6, [T2]).
\[ H_{f_n}^{(0)}[\epsilon(z; a, b; k)] = r_n (g_n a + g_n b) (a-b)^{f_n-1}, \]
\[ H_{g_{n+1}-1}^{(0)}[\epsilon(z; a, b; k)] = s_n (g_n a + g_n b) (a-b)^{f_{n+1}-1} (n \geq 0); \]
and \( H_{m}^{(0)} = 0 \) for all \( m \neq f_n \) and \( m \neq f_{n+1}-1 (n \geq 0) \), where \( r_n \neq 0, s_n \neq 0 \) are integers independent of \( a, b \).

4. Toeplitz cases. In this section, we consider Hankel determinants for 2-adic Toeplitz words

\[ w = w_1 w_2 w_3 \ldots \ (w_1 \in A). \]

Note that the numbering of the symbols starts from 1 (not from 0), cf. Sections 2, 3. Recall that a word \( w = w_1 w_2 w_3 \ldots \ (w_1 \in A) \) is a 2-adic Toeplitz word iff
\[ \text{ord}_z(m) = \text{ord}_z(n) \implies w_n = w_n. \]

Without loss of generality, we may suppose \( A = \{a_0, a_1, a_2, \ldots \} \). In some cases, we use symbols \( a, b, c, \ldots \) instead of \( a_0, a_1, a_2, \ldots \) as before. Recall also the universal 2-adic Toeplitz word

\[ \omega = abacabadabacabaeabacabadabacabaf \ldots \]

and the notation

\[ H_n[\omega] = H_n^{(w)}[\omega] \]

defined in Section 1. For example, by direct calculation, we have

\[
H_7[\omega] = \begin{bmatrix}
  a & b & a & c & a & b & a \\
  b & a & c & a & b & a & d \\
  a & c & a & b & a & d & a \\
  c & a & b & a & d & a & b \\
  a & b & d & a & b & a & b \\
  b & a & d & a & b & a & c \\
  a & d & a & b & a & c & a \\
\end{bmatrix}
\]

\[
= -4ab^2c^4+abc^5-ac^6+16ab^2c^3d-12abc^4d+2ac^5d-24ab^2c^2d^2 \\
+8abc^3d^2+ac^4d^2+16ab^2cd^3+8abc^2d^3-4ac^3d^3-4ab^4d^4-12abcd^4 \\
+ac^2d^4+4abd^5+2acd^5-ad^6 \\
= -a(2b-c-d)^2(c-d)^4.
\]

We say that a form (i.e., a homogeneous polynomial) \( P \in \mathbb{Z}[A] \) is completely reducible iff \( P = 0 \), or \( P \) can be factorized into linear forms \( \in \mathbb{Z}[A] \), i.e.,

\[
P = P_1P_2 \cdots P_k, \quad (\deg P_i = 1 \text{ for all } 1 \leq i \leq k).
\]

One can check that \( H_n[\omega] \) are non-zero completely reducible forms for small \( n \) (for \( n \leq 30 \) or so) by using the soft "Mathematica". This is a curious phenomenon, since, for instance, \( H_2[\omega] \) (resp., \( H_3[\omega] \), etc.) is not completely reducible for any word \( \omega \) having \( abc \) (resp., \( abacd \), etc.) as its prefix of \( w \). Related to such a phenomenon, we can show the following

**Proposition 6** (Main Theorem in [M-T-Tn]). Let \( \omega \) be a fixed point of a substitution of constant length 2. Suppose \( \omega \) is a word strictly over an alphabet
consisting of at least 3 symbols. Then $H_n[w]$ is completely reducible for all $n \geq 1$ if and only if $w$ is a 2-adic Toeplitz word.

The proof of this proposition together with something more interesting (probably) will appear in the forthcoming paper [M-T-Tn].

References


