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On the arithmetic distributions of simultaneous approximation convergents arising from Jacobi-Perron algorithm

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We fix a positive integer $d \geq 2$. Let $X = [0,1)^d$ with the Borel σ-algebra $\mathbb{B}$. Define a map $T : X \to X$ by

$$T((x_1, x_2, \ldots, x_d)) = \left(\frac{x_2}{x_1} - \left[\frac{x_2}{x_1}\right], \ldots, \frac{x_d}{x_1} - \left[\frac{x_d}{x_1}\right], \frac{1}{x_1} - \left[\frac{1}{x_1}\right]\right)$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in X$. Then there exists a unique absolutely continuous invariant probability measure μ. (X, T) is called the d-dimensional Jacobi-Perron algorithm. We put

$$\mathbf{k}(\mathbf{x}) = \mathbf{k}^{(0)}(\mathbf{x}) = (k_1, k_2, \ldots, k_d) = \left(\left[\frac{x_2}{x_1}\right], \left[\frac{x_3}{x_1}\right], \ldots, \left[\frac{x_d}{x_1}\right], \left[\frac{1}{x_1}\right]\right) \quad \text{for } \mathbf{x} \in X$$

and

$$\mathbf{k}^{(s)}(\mathbf{x}) = \left(k_1^{(s)}, k_2^{(s)}, \ldots, k_d^{(s)}\right) = \mathbf{k}\left(T^{s-1}(\mathbf{x})\right) \quad \text{for} \quad s \geq 1.$$

We first define $Q^{(0)}$ as the $(d+1) \times (d+1)$ identity matrix I_{d+1}; then recursively $Q^{(n)}$ for $n \geq 1$ as

$$Q^{(n)} = Q^{(n-1)} \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & k_1^{(n)} \\ 0 & 1 & \ldots & 0 & k_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & k_d^{(n)} \end{pmatrix}.$$

We set for $n \geq 1$

$$Q^{(n)} := \begin{pmatrix} p_1^{(n-d)} & p_1^{(n-d+1)} & \ldots & p_1^{(n-1)} & p_1^{(n)} \\ p_2^{(n-d)} & p_2^{(n-d+1)} & \ldots & p_2^{(n-1)} & p_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_d^{(n-d)} & p_d^{(n-d+1)} & \ldots & p_d^{(n-1)} & p_d^{(n)} \\ q^{(n-d)} & q^{(n-d+1)} & \ldots & q^{(n-1)} & q^{(n)} \end{pmatrix}.$$

Then the sequence

$$\left\{\left(\frac{p_1^{(k)}}{q^{(k)}}, \ldots, \frac{p_d^{(k)}}{q^{(k)}}\right) : k \geq 1 - d\right\}$$

*The main part of this note is based on a joint work with V. Berthé and H. Nakada [1] is called the simultaneous approximation convergents of $\mathbf{x}$ from the d-dimensional Jacobi-Perron algorithm. It is well-known that for any $\mathbf{x} = (x_1, x_2, \dots, x_d) \in X$

$$\lim_{n\to\infty} \frac{p_i^{(n)}}{q^{(n)}} = x_i \quad \text{for} \quad 1 \le i \le d$$

holds.

Our result is that for almost every $\mathbf{x} \in X$ the sequences of vectors $\{(q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) : n \ge 1\}$ and $\{(p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) : n \ge 1\}$ are both equidistributed modulo m for any integer $m \ge 2$. More precisely we put

$$\widetilde{\mathbb{Z}}_m^{d+1} = \{(\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1} : (\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \text{ generates } \mathbb{Z}_m\}$$

and

$$c_m = \sharp\, \widetilde{\mathbb{Z}}_m^{d+1} \quad \text{(the cardinality of } \widetilde{\mathbb{Z}}_m^{d+1}).$$

One easily sees that

$$\begin{aligned} c_m &= \varphi_{d+1}(m) \\ &= \sharp\{(a_1, a_2 \dots, a_{d+1}) \in \{1, \dots, m\}^{d+1} : \gcd(a_1, \dots, a_{d+1}, m) = 1\}, \end{aligned} \tag{1}$$

where φ_{d+1} denotes the Jordan totient function of order $d+1$; we thus have

$$c_m = m^{d+1} \prod_{p|m} (1 - p^{-(d+1)}),$$

where the notation $\prod_{p|m}$ stands for the product over the prime numbers p that divide m. Then we have the following:

Main Theorem. *For almost every $\mathbf{x} \in X$, we have*

$$\begin{aligned} &\lim_{N\to\infty} \frac{\sharp\{1 \le n \le N : (q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) \equiv (\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \ (\textit{mod. } m)\}}{N} \\ &= \lim_{N\to\infty} \frac{\sharp\{1 \le n \le N : (p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) \equiv (\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \ (\textit{mod. } m)\}}{N} \\ &= \frac{1}{c_m} = \frac{1}{\varphi_{d+1}(m)} = \frac{1}{m^{d+1} \prod_{p|m} (1 - p^{-(d+1)})} \end{aligned}$$

for any $(\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \in \widetilde{\mathbb{Z}}_m^{d+1}$ with any integer $m \ge 2$.

To prove main theorem, we consider for a given integer $m \ge 2$, the group $G(m)$ defined in a similar way as in [3]:

$$G(m) = \begin{cases} SL(d+1, \mathbb{Z}_m) & \text{if } d \text{ is even,} \\ SL_\pm(d+1, \mathbb{Z}_m) & \text{if } d \text{ is odd,} \end{cases}$$

where $SL(d+1, \mathbb{Z}_m)$ stands for the matrices with entries in $\mathbb{Z}_m$ with determinant 1, whereas $SL_\pm(d+1, \mathbb{Z}_m)$ stands for the matrices with entries in $\mathbb{Z}_m$ with determinant ± 1. Let us recall that (see for instance [6] or [5]) that

$$\sharp\, SL(d+1, \mathbb{Z}_m) = m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-i}) = m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m).$$

Let C_m denote the cardinality of $G(m)$. Since $SL(d+1, \mathbb{Z}_m)$ is a subgroup of $SL_\pm(d+1, \mathbb{Z}_m)$ of index 2 if d is odd and $m \neq 2$, one thus gets

$$C_m = \begin{cases} m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n}(1-p^{-i}) & \\ = m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m) & \text{if } d \text{ is even or } m = 2 \\ 2m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n}(1-p^{-i}) & \\ = 2m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m) & \text{if } d \text{ is odd and } m \neq 2. \end{cases} \tag{2}$$

We identify $Q^{(1)}$ with the $(d+1) \times (d+1)$ matrix with coefficients in $\mathbb{Z}_m$ obtained by reducing modulo m its entries, which we call J-P matrix. Here we note that $\det Q^{(1)} = 1$ or -1 if d is respectively even or odd, which implies that $Q^{(1)}$ belongs to the group $G(m)$, whatever may be the parity of d.

We define the map T_m on $X \times G(m)$ by

$$T_m(\mathbf{x}, A) = (T(\mathbf{x}),\ AQ^{(1)}).$$

T_m is said to be a $G(m)$-extension of the map T. We also define the probability measure δ_m on $G(m)$ by $(\frac{1}{C_m}, \dots, \frac{1}{C_m})$. Then it is easy to see that $\mu \times \delta_m$ is an invariant probability measure for T_m. Our question is whether $(T_m, \mu \times \delta_m)$ is ergodic or not. First, we show that the set of J-P matrices with $\mathbb{Z}_m$-entries (reduced modulo m) generates $G(m)$.

Theorem 1. *For any $B \in G(m)$, there exist J-P matrices $A_1, A_2, \dots, A_s$ such that*

$$B = A_1 A_2 \cdots A_s.$$

By using Theorem 1 we have the ergodicity of T_m.

Theorem 2. *The skew product $(X \times G(m), T_m, \mu \times \delta_m)$ is ergodic.*

From Theorem 2 and the individual ergodic theorem, we have the following proposition.

Proposition 1. *For a.e. $\mathbf{x} \in X$ and any $A \in G(m)$,*

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{1 \leq n \leq N : Q^{(n)} \equiv A \ (mod\ m)\} = \frac{1}{C_m}.$$

We are now able to give proof of main theorem.

Proof of Main Theorem

For any $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$, we denote by $N_{(\alpha_1,\alpha_2,\dots,\alpha_{d+1})}$ the number of elements in $G(m)$ such that the $(d+1)$th row is $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$. We will show that

$$N_{(\alpha_1,\alpha_2,\dots,\alpha_{d+1})} = \mathcal{C}_m \cdot m^d, \tag{3}$$

where $\mathcal{C}_m$ denotes the cardinality of $SL(d, \mathbb{Z}_m)$ or $SL_\pm(d, \mathbb{Z}_m)$ if d is even or odd, respectively. It is easy to see that

$$N_{(0,\dots,0,1)} = \mathcal{C}_m \cdot m^d. \tag{4}$$

Now we need the following lemma.

Lemma 1. *For any $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$, there exist J-P matrices A_1, A_2, $\dots$, A_s with $\mathbb{Z}_m$-entries such that*

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) = (0, \dots, 0, 1) A_1 A_2 \dots A_s,$$

where s depends on $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$.

From Lemma 1, we note that there always exists $D \in G(m)$ such that the $(d+1)$th row is $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$ for any $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1}$.

For any matrix E of the form

$$\begin{pmatrix} & * & & \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

ED is of the form

$$\begin{pmatrix} & * & & \\ \alpha_1 & \dots & \alpha_d & \alpha_{d+1} \end{pmatrix}.$$

This implies

$$N_{(\alpha_1,\alpha_2,\dots,\alpha_{d+1})} \geq N_{(0,\dots,0,1)}.$$

On the other hand, for any matrix D' of the form

$$\begin{pmatrix} & * & & \\ \alpha_1 & \dots & \alpha_d & \alpha_{d+1} \end{pmatrix},$$

$D' \cdot D^{-1}$ is of the form

$$\begin{pmatrix} & * & & \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

which implies

$$N_{(\alpha_1,\alpha_2,\dots,\alpha_{d+1})} \leq N_{(0,\dots,0,1)}.$$

Thus we have (3).

From Proposition 1 together with (3), we have

$$\lim_{N\to\infty} \frac{\sharp\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) \equiv (\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \pmod{m}\}}{N}$$
$$= \frac{C_m \cdot m^d}{C_m} = \frac{1}{c_m} \quad \text{for } \mu\text{-a.e. } \mathbf{x}.$$

Indeed one easily checks according to (1) and (2) that $\frac{C_m \cdot m^d}{C_m} = \frac{1}{c_m}$ holds. Since μ is equivalent to the Lebesgue measure, this holds for a.e. $\mathbf{x}$ with respect to the Lebesgue measure. If we consider the $(d+1)$th column, then the same argument shows the other equality. This completes the proof of Main Theorem. □

Finally we have the following corollary.

Corollary 1. *For a.e.* $\mathbf{x} \in X$ *and any* $a \in \mathbb{Z}_m$

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{1 \leq n \leq N : q^{(n)} \equiv a \pmod{m}\} = \frac{m^d \cdot \varphi_d(\gcd(a,m))}{\gcd(a,m) \cdot \varphi_{d+1}(m)}.$$

Remark 1. Let $\mathbb{F}_q$ denote the finite field of cardinality q and let $\mathbb{F}_q[X]$ be the set of polynomials with $\mathbb{F}_q$-coefficients. We denote by $\mathbb{L}$ the set of formal Laurent power series with negative degree. Since $\mathbb{L}$ is a compact Abelian group, there exists a unique normalized Haar measure m. We can define the Jacobi-Perron algorithm on $\mathbb{L}^d$ for any $d \geq 1$. In this case, m^d is invariant under this algorithm. Suppose that $\left(\frac{P_1^{(n)}}{Q^{(n)}}, \dots, \frac{P_d^{(n)}}{Q^{(n)}}\right)$ is the n-th convergent of $(f_1, \dots, f_d) \in \mathbb{L}^d$. For any $R \in \mathbb{F}_q[X]$, it is possible to prove the following : for any $A_1, \ldots, A_d, A_{d+1} \in \mathbb{F}_q[X]$ such that $A_1, \ldots, A_d, A_{d+1}, R$ are relatively prime,

$$\lim_{N\to\infty} \frac{\sharp\{1 \le n \le N : (P_1^{(n)}, \ldots, P_d^{(n)}, Q^{(n)}) \equiv (A_1, \ldots, A_d, A_{d+1}) \text{ (mod. } R)\}}{N} = c_R \quad \text{for } m^d\text{-a.e. } (f_1, \ldots, f_d) \in \mathbb{L}^d,$$

where c_R is a constant depending only on d and R. The proof is essentially the same as that of Main Theorem of this paper. We refer to K. Inoue and H. Nakada [2] for the study of the rates of convergence for Jacobi-Perron algorithm over $\mathbb{L}^d$ and to R. Natsui [4] for the $\mathbb{L}$-version of Jager-Liardet's result in the case of continued fractions.

Remark 2 (Accelerated Brun algorithm).
We put

$$Y = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in X : x_1 > x_2 > \cdots > x_d\}$$

and define a map $S : Y \to Y$ by

$$S((x_1, x_2, \ldots, x_d)) = \left(\frac{x_2}{x_1}, \ldots, \frac{x_{\varepsilon(\mathbf{x})}}{x_1}, \frac{1}{x_1} - a(\mathbf{x}), \frac{x_{\varepsilon(\mathbf{x})+1}}{x_1}, \ldots, \frac{x_d}{x_1}\right)$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in Y$, where

$$a(\mathbf{x}) = \left[\frac{1}{x_1}\right]$$

$$\varepsilon(\mathbf{x}) = \begin{cases} 1 & \text{if } \frac{1}{x_1} - \left[\frac{1}{x_1}\right] > \frac{x_2}{x_1} \\ i & \text{if } \frac{x_i}{x_1} > \frac{1}{x_1} - \left[\frac{1}{x_1}\right] > \frac{x_{i+1}}{x_1} \quad \text{for } 2 \le i \le d-1 \\ d & \text{if } \frac{x_d}{x_1} > \frac{1}{x_1} - \left[\frac{1}{x_1}\right] \end{cases}$$

Then we can define another simultaneous approximation convergents of $(x_1, x_2, \ldots, x_d)$ by a similar way (see [7]). In this case, it is not so hard to see that the associated matrices also generate $GL(d+1, \mathbb{Z}_m)$ and get the same results.

Remark 3 (Skew product map).
Let us consider the map U on $[0,1)^2$ defined by

$$U(x_1, x_2) = \left(\frac{1}{x_1} - \left[\frac{1}{x_1}\right], \frac{x_2}{x_1} - \left[\frac{x_2}{x_1}\right]\right).$$

Then it is easy to see that the associated matrices do not generate $SL(3, \mathbb{Z}_m)$.

References

[1] V. Berthé, H. Nakada and R. Natsui, *Arithmetic distribution of convergents arising from Jacobi-Perron algorithm*, preprint.

[2] K. Inoue and H. Nakada, *The modified Jacobi-Perron algorithm over $F_q(X)^d$*, Tokyo J. Math **26** (2003) 447–470.

[3] H. Jager and P. Liardet, *Distributions arithmétiques des dénominateurs de convergents de fractions continues*, Indag. Math **50** (1988) 181–197.

[4] R. Natsui, *On the group extension of the transformation associated to non-archimedean continued fractions*, to appear in Acta Math Hung.

[5] M. Newman, *Integral matrices*, Acadamic Press, New York London, 1972.

[6] J. Schulte, *Über die Jordansche Verallgemeinerung der Eulerschen Funktion*, Results Math. **36** (1999) 354–364.

[7] F. Schweiger, *Ergodic theory of fibred systems and metric number theory,* Oxford: Oxford University Press, 1995.