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KURENAI
On the arithmetic distributions of simultaneous approximation convergents arising from Jacobi-Perron algorithm

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We fix a positive integer \( d \geq 2 \). Let \( X = [0,1)^d \) with the Borel \( \sigma \)-algebra \( B \). Define a map \( T : X \rightarrow X \) by

\[
T((x_1, x_2, \ldots, x_d)) = \left( \frac{x_2}{x_1} - \left\lfloor \frac{x_2}{x_1} \right\rfloor, \ldots, \frac{x_d}{x_1} - \left\lfloor \frac{x_d}{x_1} \right\rfloor, \frac{1}{x_1} - \left\lfloor \frac{1}{x_1} \right\rfloor \right)
\]

for \( x = (x_1, x_2, \ldots, x_d) \in X \). Then there exists a unique absolutely continuous invariant probability measure \( \mu \). \((X, T)\) is called the \( d \)-dimensional Jacobi-Perron algorithm. We put

\[
k(x) = k^{(0)}(x) = (k_1, k_2, \ldots, k_d) = \left( \left\lfloor \frac{x_2}{x_1} \right\rfloor, \left\lfloor \frac{x_3}{x_1} \right\rfloor, \ldots, \left\lfloor \frac{x_d}{x_1} \right\rfloor, \left\lfloor \frac{1}{x_1} \right\rfloor \right)
\]

for \( x \in X \) and

\[
k^{(s)}(x) = (k_1^{(s)}, k_2^{(s)}, \ldots, k_d^{(s)}) = k(T^{s-1}(x)) \text{ for } s \geq 1.
\]

We first define \( Q^{(0)} \) as the \((d+1) \times (d+1)\) identity matrix \( I_{d+1} \); then recursively \( Q^{(n)} \) for \( n \geq 1 \) as

\[
Q^{(n)} = Q^{(n-1)} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & k_1^{(n)} \\ 0 & 1 & \cdots & 0 & k_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & k_d^{(n)} \end{pmatrix}
\]

We set for \( n \geq 1 \)

\[
Q^{(n)} := \begin{pmatrix} p_1^{(n-d)} & p_1^{(n-d+1)} & \cdots & p_1^{(n-1)} & p_1^{(n)} \\ p_2^{(n-d)} & p_2^{(n-d+1)} & \cdots & p_2^{(n-1)} & p_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_d^{(n-d)} & p_d^{(n-d+1)} & \cdots & p_d^{(n-1)} & p_d^{(n)} \\ q^{(n-d)} & q^{(n-d+1)} & \cdots & q^{(n-1)} & q^{(n)} \end{pmatrix}
\]

Then the sequence

\[
\left\{ \begin{pmatrix} p_1^{(k)} \\ p_2^{(k)} \\ \vdots \\ p_d^{(k)} \\ q^{(k)} \end{pmatrix} : k \geq 1 - d \right\}
\]

*The main part of this note is based on a joint work with V. Berthé and H. Nakada [1]
is called the simultaneous approximation convergents of $x$ from the $d$-dimensional Jacobi-Perron algorithm. It is well-known that for any $x = (x_1, x_2, \ldots, x_d) \in X$

$$\lim_{n \to \infty} \frac{p_i^{(n)}}{q^{(n)}} = x_i \quad \text{for} \quad 1 \leq i \leq d$$

holds.

Our result is that for almost every $x \in X$ the sequences of vectors

$$\{(q^{(n-d)}, q^{(n-d+1)}, \ldots, q^{(n)} : n \geq 1\} \text{ and } \{(p_{1}^{(n)}, p_{2}^{(n)}, \ldots, p_{d}^{(n)}, q^{(n)} : n \geq 1\}$$

are both equidistributed modulo $m$ for any integer $m \geq 2$. More precisely we put

$$\hat{Z}_{m}^{d+1} = \{(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{d+1}) \in Z_{m}^{d+1} : (\alpha_{1}, \alpha_{2}, \ldots, \alpha_{d+1}) \text{ generates } Z_{m}\}$$

and

$$c_{m} = \# \hat{Z}_{m}^{d+1} \quad \text{(the cardinality of } \hat{Z}_{m}^{d+1}).$$

One easily sees that

$$c_{m} = \varphi_{d+1}(m)$$

$$= \#\{(a_{1}, a_{2}, \ldots, a_{d+1}) \in \{1, \ldots, m\}^{d+1} : \gcd(a_{1}, \ldots, a_{d+1}, m) = 1\} , \quad (1)$$

where $\varphi_{d+1}$ denotes the Jordan totient function of order $d + 1$; we thus have

$$c_{m} = m^{d+1} \prod_{p|m}(1 - p^{-(d+1)}),$$

where the notation $\prod_{p|m}$ stands for the product over the prime numbers $p$ that divide $m$. Then we have the following:

**Main Theorem.** For almost every $x \in X$, we have

$$\lim_{N \to \infty} \frac{\#\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \ldots, q^{(n)}) \equiv (\alpha_{1}, \alpha_{2}, \ldots, \alpha_{d+1}) \text{ (mod. } m)\} \cdot N}{N}$$

$$= \lim_{N \to \infty} \frac{\#\{1 \leq n \leq N : (p_{1}^{(n)}, p_{2}^{(n)}, \ldots, p_{d}^{(n)}, q^{(n)}) \equiv (\alpha_{1}, \alpha_{2}, \ldots, \alpha_{d+1}) \text{ (mod. } m)\} \cdot N}{N}$$

$$= \frac{1}{c_{m}} = \frac{1}{\varphi_{d+1}(m)} = \frac{1}{m^{d+1} \prod_{p|m}(1 - p^{-(d+1)})}$$

for any $(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{d+1}) \in \hat{Z}_{m}^{d+1}$ with any integer $m \geq 2$.

To prove main theorem, we consider for a given integer $m \geq 2$, the group $G(m)$ defined in a similar way as in [3]:

$$G(m) = \begin{cases} SL(d + 1, Z_{m}) & \text{if } d \text{ is even,} \\ SL_{\pm}(d + 1, Z_{m}) & \text{if } d \text{ is odd,} \end{cases}$$

where $SL(d + 1, Z_{m})$ stands for the matrices with entries in $Z_{m}$ with determinant 1, whereas $SL_{\pm}(d + 1, Z_{m})$ stands for the matrices with entries in $Z_{m}$ with determinant $\pm 1$. Let us recall that (see for instance [6] or [5]) that

$$\# SL(d + 1, Z_{m}) = m^{(d+1)^{2}-1} \prod_{i=2}^{d+1} \prod_{p|m}(1 - p^{-i}) = m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_{i}(m).$$
Let $C_m$ denote the cardinality of $G(m)$. Since $SL(d+1, \mathbb{Z}_m)$ is a subgroup of $SL_\pm(d+1, \mathbb{Z}_m)$ of index 2 if $d$ is odd and $m \neq 2$, one thus gets

$$C_m = \begin{cases} 
\frac{m^{(d+1)^2-1}}{(d+1)!} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-1}) & \text{if } d \text{ is even or } m = 2 \\
2m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-1}) & \text{if } d \text{ is odd and } m \neq 2.
\end{cases}$$

We identify $Q^{(1)}$ with the $(d + 1) \times (d + 1)$ matrix with coefficients in $\mathbb{Z}_m$ obtained by reducing modulo $m$ its entries, which we call J-P matrix. Here we note that $\det Q^{(1)} = 1$ or $-1$ if $d$ is respectively even or odd, which implies that $Q^{(1)}$ belongs to the group $G(m)$, whatever may be the parity of $d$.

We define the map $T_m$ on $X \times G(m)$ by

$$T_m(x, A) = (T(x), AQ^{(1)}).$$

$T_m$ is said to be a $G(m)$-extension of the map $T$. We also define the probability measure $\delta_m$ on $G(m)$ by $(\frac{1}{C_m^d}, \ldots, \frac{1}{C_m^d})$. Then it is easy to see that $\mu \times \delta_m$ is an invariant probability measure for $T_m$. Our question is whether $(T_m, \mu \times \delta_m)$ is ergodic or not. First, we show that the set of J-P matrices with $\mathbb{Z}_m$-entries (reduced modulo $m$) generates $G(m)$.

**Theorem 1.** For any $B \in G(m)$, there exist J-P matrices $A_1, A_2, \ldots, A_s$ such that $B = A_1A_2\cdots A_s$.

By using Theorem 1 we have the ergodicity of $T_m$.

**Theorem 2.** The skew product $(X \times G(m), T_m, \mu \times \delta_m)$ is ergodic.

From Theorem 2 and the individual ergodic theorem, we have the following proposition.

**Proposition 1.** For a.e. $x \in X$ and any $A \in G(m)$,

$$\lim_{N \to \infty} \frac{1}{N} \# \{1 \leq n \leq N : Q^{(n)} \equiv A \ (\text{mod } m)\} = \frac{1}{C_m}.$$

We are now able to give proof of main theorem.

**Proof of Main Theorem**

For any $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1}$, we denote by $N_{(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})}$ the number of elements in $G(m)$ such that the $(d+1)$th row is $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$. We will show that

$$N_{(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})} = C_m \cdot m^d,$$  

where $C_m$ denotes the cardinality of $SL(d, \mathbb{Z}_m)$ or $SL_\pm(d, \mathbb{Z}_m)$ if $d$ is even or odd, respectively. It is easy to see that

$$N_{(0, \ldots, 0, 1)} = C_m \cdot m^d.$$  

Now we need the following lemma.

**Lemma 1.** For any $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1}$, there exist J-P matrices $A_1, A_2, \ldots, A_s$ with $\mathbb{Z}_m$-entries such that

$$(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) = (0, \ldots, 0, 1)A_1A_2\cdots A_s,$$

where $s$ depends on $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$. 
From Lemma 1, we note that there always exists $D \in G(m)$ such that the $(d + 1)$th row is $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$ for any $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1}$.

For any matrix $E$ of the form
\[
\begin{pmatrix}
* \\
0 & \ldots & 0 & 1
\end{pmatrix},
\]
$ED$ is of the form
\[
\begin{pmatrix}
* \\
\alpha_1 & \ldots & \alpha_d & \alpha_{d+1}
\end{pmatrix}.
\]
This implies
\[
N_{(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})} \geq N_{(0, \ldots, 0, 1)}.
\]
On the other hand, for any matrix $D'$ of the form
\[
\begin{pmatrix}
* \\
\alpha_1 & \ldots & \alpha_d & \alpha_{d+1}
\end{pmatrix},
\]
$D' \cdot D^{-1}$ is of the form
\[
\begin{pmatrix}
* \\
0 & \ldots & 0 & 1
\end{pmatrix},
\]
which implies
\[
N_{(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})} \leq N_{(0, \ldots, 0, 1)}.
\]
Thus we have (3).

From Proposition 1 together with (3), we have
\[
\lim_{N \to \infty} \frac{\#\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \ldots, q^{(n)}) \equiv (\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \pmod{m}\}}{N} = \frac{c_m \cdot m^d}{c_m} = \frac{1}{c_m} \text{ for } \mu\text{-a.e. } x.
\]
Indeed one easily checks according to (1) and (2) that $\frac{c_m \cdot m^d}{c_m} = \frac{1}{c_m}$ holds. Since $\mu$ is equivalent to the Lebesgue measure, this holds for a.e. $x$ with respect to the Lebesgue measure. If we consider the $(d + 1)$th column, then the same argument shows the other equality. This completes the proof of Main Theorem.

Finally we have the following corollary.

**Corollary 1.** For a.e. $x \in X$ and any $a \in \mathbb{Z}_m$
\[
\lim_{N \to \infty} \frac{1}{N} \#\{1 \leq n \leq N : q^{(n)} \equiv a \pmod{m}\} = \frac{m^d \cdot \varphi_d(\gcd(a, m))}{\gcd(a, m) \cdot \varphi_{d+1}(m)}.
\]

**Remark 1.** Let $\mathbb{F}_q$ denote the finite field of cardinality $q$ and let $\mathbb{F}_q[X]$ be the set of polynomials with $\mathbb{F}_q$-coefficients. We denote by $L$ the set of formal Laurent power series with negative degree. Since $L$ is a compact Abelian group, there exists a unique normalized Haar measure $m$. We can define the Jacobi-Perron algorithm on $L^d$ for any $d \geq 1$. In this case, $m^d$ is invariant under this algorithm. Suppose that
\[
\left(\frac{P_1^{(n)}}{Q_1^{(n)}}, \ldots, \frac{P_d^{(n)}}{Q_d^{(n)}}\right)
\]
is the $n$-th convergent of $(f_1, \ldots, f_d) \in L^d$. For any $R \in \mathbb{F}_q[X]$,
it is possible to prove the following: for any \( A_1, \ldots, A_d, A_{d+1} \in \mathbb{F}_q[X] \) such that \( A_1, \ldots, A_d, A_{d+1}, R \) are relatively prime,

\[
\lim_{N \to \infty} \frac{\#\{1 \leq n \leq N : (P_1^{(n)}, \ldots, P_d^{(n)}, Q^{(n)}) \equiv (A_1, \ldots, A_d, A_{d+1}) \pmod{R}\}}{N} = c_R \quad \text{for m\textsuperscript{d}-a.e.} \quad (f_1, \ldots, f_d) \in \mathbb{L}^d,
\]

where \( c_R \) is a constant depending only on \( d \) and \( R \). The proof is essentially the same as that of Main Theorem of this paper. We refer to K. Inoue and H. Nakada [2] for the study of the rates of convergence for Jacobi-Perron algorithm over \( \mathbb{L}^d \) and to R. Natsui [4] for the \( L \)-version of Jager-Liardet's result in the case of continued fractions.

**Remark 2 (Accelerated Brun algorithm).**

We put

\[
Y = \{ x = (x_1, x_2, \ldots, x_d) \in X : x_1 > x_2 > \cdots > x_d \}
\]

and define a map \( S : Y \to Y \) by

\[
S((x_1, x_2, \ldots, x_d)) = \left( \frac{x_2}{x_1}, \frac{x_3(x)}{x_1}, \frac{1}{x_1} - a(x), \frac{x_{d(x)+1}}{x_1}, \ldots, \frac{x_d}{x_1} \right)
\]

for \( x = (x_1, x_2, \ldots, x_d) \in Y \), where

\[
a(x) = \left\lfloor \frac{1}{x_1} \right\rfloor
\]

\[
\epsilon(x) = \begin{cases} 1 & \text{if } \frac{1}{x_1} - \left\lfloor \frac{1}{x_1} \right\rfloor > \frac{x_2}{x_1} \\ i & \text{if } \frac{x_i}{x_1} > \frac{1}{x_1} - \left\lfloor \frac{1}{x_1} \right\rfloor > \frac{x_{i+1}}{x_1} & \text{for } 2 \leq i \leq d - 1 \\ d & \text{if } \frac{x_d}{x_1} > \frac{1}{x_1} - \left\lfloor \frac{1}{x_1} \right\rfloor 
\end{cases}
\]

Then we can define another simultaneous approximation convergents of \((x_1, x_2, \ldots, x_d)\) by a similar way (see [7]). In this case, it is not so hard to see that the associated matrices also generate \( GL(d + 1, \mathbb{Z}_m) \) and get the same results.

**Remark 3 (Skew product map).**

Let us consider the map \( U \) on \([0,1)^2 \) defined by

\[
U(x_1, x_2) = \left( \frac{1}{x_1} - \left\lfloor \frac{1}{x_1} \right\rfloor, \frac{x_2}{x_1} - \left\lfloor \frac{x_2}{x_1} \right\rfloor \right)
\]

Then it is easy to see that the associated matrices do not generate \( SL(3, \mathbb{Z}_m) \).

**References**


