Transcendence of certain infinite products

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1 Introduction and the results

Duverney [1] introduced an inductive method to prove the transcendence of the number

$$\sum_{k=1}^{\infty} \frac{1}{a^{2^k} + b_k},$$

where $a (|a| \geq 2)$ is an integer, $\{b_k\}_{k \geq 1}$ is a sequence of integers satisfying $\log |b_k| = o(2^k)$, and $a^{2^k} + b_k \neq 0$ for every $k \geq 1$. Recently, Duverney and Nishioka [2] developed this method and gave a transcendence criterion for general series. As applications, they established necessary and sufficient conditions for transcendence of the following numbers

$$\sum_{k=0}^{\infty} \frac{a_k}{F_{r^k} + b_k}, \quad \sum_{k=0}^{\infty} \frac{a_k}{L_{r^k} + b_k},$$

where $\{a_k\}_{k \geq 0}$ and $\{b_k\}_{k \geq 0}$ are suitable sequences of algebraic numbers, and $F_n$ and $L_n$ are Fibonacci numbers and Lucas numbers defined by $F_{n+2} = F_{n+1} + F_n (n \geq 0)$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n (n \geq 0)$, $L_0 = 2$, $L_1 = 1$, respectively. The purpose of this article is to prove the transcendence of the values of infinite products of the form (1) by modifying the method in [2].

For an algebraic number $\alpha$, we denote by $|\alpha|$ the maximum of the absolute values of its conjugates and by $\text{den}(\alpha)$ the least positive integer such that $\text{den}(\alpha)\alpha$ is an algebraic integer, and define $||\alpha|| = \max\{|\alpha|, \text{den}(\alpha)\}$. Then we have the fundamental inequalities

$$|\alpha| \geq ||\alpha||^{-2[\mathbb{Q}(\alpha) : \mathbb{Q}]} \quad \text{and} \quad ||\alpha^{-1}|| \leq ||\alpha||^{2[\mathbb{Q}(\alpha) : \mathbb{Q}]}$$

for nonzero algebraic $\alpha$ (cf. Lemma 2.10.2 in [5]).

Let $K$ be an algebraic number field, $O_K$ be the ring of integer in $K$. Let $r \geq 2$ and $L \geq 1$ be integers, and let

$$\Phi_0(x) = \prod_{k=0}^{\infty} \frac{E_k(x^k)}{F_k(x^k)}$$

(1)
with

\[ E_k(x) = 1 + a_{k1}x + a_{k2}x^2 + \cdots + a_{kL}x^L \in K[x], \]

\[ F_k(x) = 1 + b_{k1}x + b_{k2}x^2 + \cdots + b_{kL}x^L \in K[x], \]

where \( \log ||a_{kl}||, \log ||b_{k1}|| = o(r^k), 1 \leq l \leq L \). We suppose that there exists a positive integer \( D \) such that \( DF_k(x) \in O_K[x] (k \geq 0) \).

Then for algebraic number \( \alpha \) satisfying \( 0 < |\alpha| < 1 \) and \( E_k(\alpha^{t^k})F_k(\alpha^{r^k}) \neq 0 (k \geq 0) \), we prove the following:

**Theorem 1.** \( \Phi_0(\alpha) \) is algebraic if and only if \( \Phi_0(x) \) is a rational function with coefficients in \( K \).

It should be noticed that in [2] they proved a similar result for infinite sums. The tools to prove Theorem 1 are also similar to those in [2], however we need some different techniques.

As applications, we have the following results.

**Theorem 2.** Let \( K \) be an algebraic number field, \( r \geq 2 \) be an integer, and

\[ \Phi(x) = \prod_{k=0}^{\infty} \left( 1 + a_k x^r \right), \]

where \( a_k \in K \) (\( k \geq 0 \)) and \( \log ||a_k|| = o(r^k) \). Let \( \alpha \in K \) with \( 0 < |\alpha| < 1 \) and \( 1 + a_k \alpha^r \neq 0 \) (\( k \geq 0 \)). Then \( \Phi(\alpha) \) is algebraic if and only if at least one of the following conditions holds:

(i) \( a_n = 0 \) for every large \( n \).

(ii) \( r = 2 \) and there exists a root of unity \( \omega \) such that \( a_n = \omega^{2^n} \) for every large \( n \).

Nishioka [4] proved that the numbers \( \prod_{k=0}^{\infty} (1 - \alpha^{rk}) \), \( r = 2, 3, 4, \ldots \) are algebraically independent for any fixed algebraic number \( \alpha \) with \( 0 < |\alpha| < 1 \). Furthermore, Tanaka [6] proved the algebraic independence of the numbers \( \prod_{k=0}^{\infty} (1 - \alpha_i^{rk}) \), \( i = 1, 2, \ldots, n \), for a linear recurrence \( \{a_k\}_{k \geq 0} \) and algebraic numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) under some suitable conditions.

In the following, we consider the binary recurrences \( \{R_n\}_{n \geq 0} \) defined by

\[ R_{n+2} = A_1 R_{n+1} + A_2 R_n, \quad A_1, A_2 \in \mathbb{Z} \setminus \{0\}, \quad R_0, R_1 \in \mathbb{Z}. \]

We suppose that \( |A_2| = 1 \) and \( \Delta = A_1^2 + 4A_2 > 0 \). Then \( R_n \) is written as

\[ R_n = g_1 \rho_1^n + g_2 \rho_2^n, \quad g_1, g_2 \in \mathbb{Q}(\rho_1), \quad \rho_1, \rho_2 \text{ are the roots of } g(x) = x^2 - A_1 x - A_2. \]

We may assume \( \rho_2 > |\rho_1| \), since \( A_1 \neq 0 \) and \( \Delta > 0 \). For a negative integer \( n \), we define \( R_n \) by (2).
Theorem 3. Let $R_n$ be a binary recurrence given by (2) and $r$, $c$, and $d$ be integers such that $r \geq 2$ and $c \geq 1$. Let $K$ be an algebraic number field and $a_k \in K$ satisfy $a_k \neq -R_{cr^k+d}$ ($k \geq 0$) and $\log||a_k|| = o(r^k)$. Then

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{R_{cr^k+d}}\right)$$

is algebraic if and only if at least one of the following conditions is satisfied:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = R_d$ for every large $n$.

(iii) $r = 2$, $g_1 \rho_1^d = g_2 \rho_2^d$, and there exists a root of unity $\omega$ such that $a_n = g_1 \rho_1^d(\omega^{2^n} + \omega^{-2^n})$ for every large $n$.

In the following examples, let $\{a_k\}_{k\geq 0}$, $r$, $c$, and $d$ be as in Theorem 3.

Example 1. Let $F_n$ be Fibonacci numbers defined above. Then

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{F_{cr^k+d}}\right)$$

is algebraic if and only if at least one of the following conditions holds:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = F_d$ for every large $n$.

In particular,

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{F_{r^k}}\right)$$

is algebraic if and only if $a_n = 0$ for all large $n$.

Example 2. Let $L_n$ be Lucas numbers defined above. Then

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{L_{cr^k+d}}\right)$$

is algebraic if and only if at least one of the following conditions is satisfied:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = L_d$ for every large $n$.

(iii) $r = 2$, $d = 0$, and there exists a root of unity $\omega$ such that $a_n = \omega^{2^n} + \omega^{-2^n}$ for every large $n$.

In particular, for any integer $a \neq 0$ and $r \geq 2$ the number $\prod_{k=1}^{\infty} \left(1 + \frac{a}{L_{r^k}}\right)$ is transcendental, except for two algebraic cases

$$\prod_{k=1}^{\infty} \left(1 + \frac{-1}{L_{2^k}}\right) = \frac{\sqrt{5}}{4}, \quad \prod_{k=1}^{\infty} \left(1 + \frac{2}{L_{2^k}}\right) = \sqrt{5},$$
which are obtained from the case (iii) with \( \omega = \frac{1 \pm \sqrt{-3}}{2} \) and \( \omega = \pm 1 \), respectively. These examples of algebraic infinite products involving Lucas numbers seems to be new.

2 Transcendence of \( \Phi_0(\alpha) \)

For a formal power series \( f(x) \in K[[x]] \) such that \( f(x) = \sum_{l \leq n} a_n x^n \) with \( a_l \neq 0 \), we define \( \text{ord} f(x) = l \).

Lemma 1. Let \( \Phi_0(x) \) and \( \alpha \) be given in Section 1. For every positive integer \( m \), suppose that there is a positive constant \( c(m) \) such that

\[
\text{ord} \left( A_0(x) + A_1(x) \Phi_0(x)^m \right) \leq c(m) M
\]

(3)

for any \( M \geq 1 \) and any polynomials \( A_0, A_1 \in K[x] \), not both zero, satisfying \( \deg A_0(x), \deg A_1(x) \leq M \). Then \( \Phi_0(\alpha) \) is transcendental.

Lemma 1 will be used in the proof of Theorem 1 in the next section. For the proof of Lemma 1, we apply the following criterion of Loxton and van der Poorten [3]. We put

\[
\Phi_n(x) = \prod_{k=0}^{\infty} \frac{E_{n+k}(x^r)}{F_{n+k}(x^r)}, \quad n \geq 0.
\]

Lemma 2 (cf. Theorem 2.9.1 in [5]). Let \( K \) be an algebraic number field, \( r \geq 2 \) be an integer, \( \{\Phi_n(x)\}_{n \geq 0} \) be a sequence in the ring of formal power series \( K[[x]] \), and \( \alpha \in K \) with \( 0 < |\alpha| < 1 \). If the following three properties are satisfied, then \( \Phi_0(\alpha) \) is transcendental.

(I) \( \Phi_n(\alpha^r^n) = a_n \Phi_0(\alpha) + b_n \) with \( a_n, b_n \in K \), and \( \log ||a_n||, \log ||b_n|| = O(r^n) \).

(II) If \( \Phi_n(x) = \sum_{i=0}^{\infty} a_i^{(n)} x^i \), then for any \( \varepsilon > 0 \) there is a positive integer \( n_0 \) such that

\[
\log ||a_i^{(n)}|| \leq \varepsilon r^n (1 + l)
\]

for any \( n \geq n_0 \) and \( l \geq 0 \).

(III) Let \( \{s_l\}_{l \geq 0} \) be variables and

\[
F(x; s) = F(x; \{s_l\}_{l \geq 0}) = \sum_{l=0}^{\infty} s_l x^l,
\]

in such a way that

\[
F(x; \sigma^{(n)}) = F(x; \{\sigma_i^{(n)}\}_{i \geq 0}) = \Phi_n(x).
\]
Then for any polynomials $P_0(x, s), \ldots, P_d(x, s) \in K[x, \{s_l\}_{l \geq 0}]$ and

$$E(x, s) = \sum_{j=0}^{d} P_j(x, s) F(x, s)^j,$$

there is positive integer $I$ with the following property: if $n$ is sufficiently large and $P_0(x, \sigma^{(n)}), \ldots, P_d(x, \sigma^{(n)})$ are not all zero, then $\text{ord} E(x, \sigma^{(n)}) \leq I$.

The property (I) follows from the functional equation

$$\Phi_n(x^{r^n}) = \Phi_0(x) \prod_{k=0}^{n-1} \frac{F_k(x^{r^k})}{E_k(x^{r^k})}. \quad (4)$$

It is not difficult to see that the property (II) is satisfied. The crucial point in applying Lemma 2 is to check property (III), which is done via Lemma 3.

**Lemma 3.** Suppose that $\Phi_0(x)$ satisfy the assumption 3. Then for every positive integer $d$, there exists a positive constant $c_d$ such that

$$\text{ord}(A_0(x) + A_1(x)\Phi_0(x) + \cdots + A_d(x)\Phi_0(x)^d) \leq c_d M$$

for any $M \geq 1$ and any polynomials $A_0(x), \ldots, A_d(x) \in K[x]$, not all zero, with $\deg A_i(x) \leq M \ (0 \leq i \leq d)$.

## 3 Proof of Theorem 1

We use the following lemma.

**Lemma 4 (Theorem 5 in [2]).** Let $r \geq 2$ be an integer, $K$ be a commutative field, and $f \in K[[x]]$. Let $\{m(n)\}_{n \geq 0}$ be an increasing sequence of nonnegative integers with $(m(n+1) - m(n)) \leq c_1$ for some $c_1 \geq 1$. Suppose that there exists a sequence $\{(P_n(x), Q_n(x))\}_{n \geq 0}$ in $K[x]^2$ such that

$$P_n(x)Q_{n+1}(x) - P_{n+1}(x)Q_n(x) \neq 0 \quad (5)$$

$$\deg Q_n(x), \deg P_n(x) \leq c_2 r^{m(n)} \quad (6)$$

$$\text{ord}(Q_n(x)f(x) - P_n(x)) \geq c_3 r^{m(n)} \quad (7)$$

for every $n \geq 0$, where $0 < c_2 < c_3$. Then we have

$$\text{ord}(A_0(x) + A_1(x)f(x)) \leq \left( c_2 r^{m(0)+2C} \left(1 + \frac{1}{c_3 - c_2}\right) + 1 \right) M$$

for any $M \geq 1$ and for any polynomials $A_0(x), A_1(x) \in K[x]$, not both zero, satisfying $\deg A_0(x), \deg A_1(x) \leq M$. 

For each \( f(x) = \Phi_0(x)^m \) \((m = 1, 2, \ldots)\), we construct a sequence \( \{(P_{m,n}, Q_{m,n})\}_{n \geq 0} \) satisfying the hypotheses of Lemma 4. Consider the \((mL, mL)\) Padé-approximants to \( \Phi_n(x)^m \), that is, polynomials \( A_{m,n}(x) \) and \( B_{m,n}(x) \) satisfying \( \deg A_{m,n}(x) \), \( \deg B_{m,n}(x) \leq mL \) and

\[
A_{m,n}(x)\Phi_n(x)^m - B_{m,n}(x) = O(x^{2mL+1}).
\]

By Siegel’s lemma (cf. Lemma 1.4.2 in [5]), we may assume that \( \log || \cdot || \) of the coefficients of \( A_{m,n}(x) \) and \( B_{m,n}(x) \) are \( o(r^n) \). Define

\[
D_{m,n}(x) = \begin{vmatrix}
A_{m,n}(x) & B_{m,n}(x) \\
A_{m,n+1}(x^r)F_n(x)^m & B_{m,n+1}(x^r)E_n(x)^m
\end{vmatrix}.
\]

Lemma 5. Suppose that \( D_{m,n}(x) \neq 0 \). Then

\[
\text{ord} \ (A_{m,n}(x)\Phi_n(x)^m - B_{m,n}(x)) < r(2mL + 1).
\]

Proof. This can be proved similarly as the proof of Lemma 4 in [2].

Replacing \( x \) by \( x^r \) in (8) and use the functional equation (4), we have

\[
Q_{m,n}^*(x)\Phi_0(x)^m - P_{m,n}^*(x) = O(x^{(2mL+1)r^n}),
\]

where

\[
Q_{m,n}^*(x) = A_{m,n}(x^r) \prod_{k=0}^{n-1} F_k(x^r)^m, \quad P_{m,n}^*(x) = B_{m,n}(x^r) \prod_{k=0}^{n-1} E_k(x^r)^m.
\]

Since \( \deg Q_{m,n}^*(x), \deg P_{m,n}^*(x) \leq 2mLr^n \), the sequence \( \{(P_{m,n}, Q_{m,n})\}_{n \geq 0} = \{(P_{m,l(m,n)}, Q_{m,l(m,n)})\}_{n \geq 0} \) satisfies hypotheses (6) and (7) of Lemma 4 for every increasing sequence \( \{l(m, n)\}_{n \geq 0} \). To study the condition (5) in Lemma 4, we need the following lemma. We put

\[
\triangle_{m,n}(x) = \begin{vmatrix}
Q_{m,n}^*(x) & P_{m,n}^*(x) \\
Q_{m,n+1}^*(x) & P_{m,n+1}^*(x)
\end{vmatrix}.
\]

Lemma 6. \( \triangle_{m,n}(x) = 0 \) if and only if \( D_{m,n}(x) = 0 \), that is,

\[
\left( \frac{E_n(x)}{F_n(x)} \right)^m = \frac{B_{m,n}(x)A_{m,n+1}(x^r)}{A_{m,n}(x)B_{m,n+1}(x^r)}.
\]

Proof. By definition, \( \triangle_{m,n}(x) = 0 \) if and only if

\[
\begin{vmatrix}
A_{m,n}(x^r) & B_{m,n}(x^r) \\
A_{m,n+1}(x^{r+1})F_n(x^r)^m & B_{m,n+1}(x^{r+1})E_n(x^r)^m
\end{vmatrix} = 0,
\]

which is equivalent to

\[
D_{m,n}(x) = \begin{vmatrix}
A_{m,n}(x) & B_{m,n}(x) \\
A_{m,n+1}(x^r)F_n(x)^m & B_{m,n+1}(x^r)E_n(x)^m
\end{vmatrix} = 0.
\]
Lemma 7. For each positive integer $m$, we define $f_{m,n}(x)$ by

$$f_{m,n}(x) = 1 - \frac{A_{m,n}(x)}{B_{m,n}(x)}\Phi_{n}(x)^{m}.$$  

Let $I$ be a positive integer and $\alpha$ be an algebraic number with $0 < |\alpha| < 1$. Then there exists a positive number $\eta_{m} < 1$ such that

$$0 < |f_{m,n}(\alpha^{n})| < \eta_{m}^{\text{ord} f_{m,n}(x)}$$

for every large $n$ satisfying $\text{ord} f_{m,n}(x) \leq I$.

Proof. We may assume $A_{m,n}(0) = B_{m,n}(0) = 1$ by (8). Let $\theta > 1$ and $A_{m,n}(x)/B_{m,n}(x) = \sum_{l=0}^{\infty} \tau_{l}^{(m,n)} x^{l}$. Then we obtain $||\tau_{l}^{(m,n)}|| \leq (\theta^{2mL})^{1}$ for any $n \geq n_{0}$ and $l \geq 0$. Let $\Phi_{n}(x)^{m} = (\sum_{l=0}^{\infty} \sigma_{l}^{(n)} x^{l})^{m} = \sum_{l=0}^{\infty} \mu_{l}^{(m,n)} x^{l}$, then we have

$$f_{m,n}(x) = 1 - \left( \sum_{l=0}^{\infty} \tau_{l}^{(m,n)} x^{l} \right) \left( \sum_{l=0}^{\infty} \sigma_{l}^{(n)} x^{l} \right) = \sum_{l=1}^{\infty} \left( \sum_{s+t=l} \tau_{s}^{(m,n)} \mu_{t}^{(m,n)} \right) x^{l},$$

where $||\sum_{s+t=l} \tau_{s}^{(m,n)} \mu_{t}^{(m,n)}|| \leq (\theta^{6mL})^{1}$ and den $(\sum_{s+t=l} \tau_{s}^{(m,n)} \mu_{t}^{(m,n)}) \leq (\theta^{5mL})^{1}$. We put

$$f_{m,n}(x) = a_{H}x^{H} + a_{H+1}x^{H+1} + \ldots, \quad a_{H} \neq 0,$$

where $1 \leq H \leq I$. Then

$$f_{m,n}(\alpha^{n}) = a_{H}\alpha^{Hr^{n}} + \frac{a_{H+1}}{a_{H}} \alpha^{(H+1)r^{n}} + \ldots,$$

Since $||a_{H}|| \leq (\theta^{6mL})^{Hr^{n}}$, we obtain

$$\left| \frac{a_{H+1}}{a_{H}} \alpha^{r^{n}} \right| \leq (\theta^{6mL})^{(1+2[K:Q])Ir^{n}} (\theta^{6mL})^{Ir^{n}}.$$

Choosing $\theta > 1$ with $\eta_{m} = (\theta^{6mL})^{(1+2[K:Q])Ir^{n}} |\theta^{6mL}| < 1$, we have

$$0 < |f_{m,n}(\alpha^{n})| < 2|\theta^{6mL}|^{r^{n}} < \eta_{m}^{r^{n}}$$

for sufficiently large $n$; which implies the lemma.

Lemma 8. $\Phi_{0}(\alpha)$ is algebraic if and only if $\Phi_{0}(x)^{m}$ is a rational function with coefficients in $K$ for some positive integer $m$.

Proof. We prove that if $\Phi_{0}(\alpha)$ is algebraic then there exists a positive integer $m$ such that $\Delta_{m,n}(x) = 0$ for every large $n$, which implies $\Phi_{0}(x)^{m}$ is a rational function.
by Lemma 6. For every integer \( m \), suppose that there exist infinitely many \( n \) satisfying \( \triangle_{m,n}(x) \neq 0 \). Denote by \( \{l(m,n)\}_{n \geq 0} \) the sequence satisfying
\[
\triangle_{m,l(m,n)}(x) \neq 0, \quad \triangle_{m,k}(x) = 0
\]
for every \( n \geq 0 \) and every \( k \) with \( l(m,n) < k < l(m,n+1) \). Then two cases occur:

(i) For every \( m \), \( l(m,n+1) - l(m,n) \leq C_m \) for some positive constant \( C_m \). Then it is clear that the determinant
\[
\begin{vmatrix}
Q^*_{m,l(m,n)}(x) & P^*_{m,l(m,n)}(x) \\
Q^*_{m,l(m,n+1)}(x) & P^*_{m,l(m,n+1)}(x)
\end{vmatrix} = \begin{vmatrix}
Q_{m,n}(x) & P_{m,n}(x) \\
Q_{m,n+1}(x) & P_{m,n+1}(x)
\end{vmatrix} \neq 0,
\]

namely, the condition (5) in Lemma 4 is satisfied. Hence we can apply Lemma 1 and find that \( \Phi_0(\alpha) \) is transcendental.

(ii) For some \( m \), \( \varlimsup_{n \to \infty} (l(m,n+1) - l(m,n)) = +\infty \). In this case, we have by using Lemma 6
\[
\left( \frac{E_k(x)}{F_k(x)} \right)^m = \frac{B_{m,k}(x)A_{m,k+1}(x^r)}{A_{m,k}(x)B_{m,k+1}(x^r)}
\]
for every \( k \) satisfying \( l(m,n) < k < l(m,n+1) \), so that
\[
\prod_{k=l(m,n)+1}^{l(m,n+1)-1} \left( \frac{E_k(x^r)}{F_k(x^r)} \right)^m = \frac{B_{m,l(m,n)+1}(x^{r(l(m,n)+1)})A_{m,l(m,n+1)}(x^{r(l(m,n)+1)})}{A_{m,l(m,n)+1}(x^{r(l(m,n)+1)})B_{m,l(m,n+1)}(x^{r(l(m,n)+1)})}.
\]

Let
\[
f_{m,l(m,n+1)}(x) = \frac{A_{m,l(m,n+1)}(x)}{B_{m,l(m,n+1)}(x)}\Phi_{l(m,n+1)}(x)^m - 1,
\]
where we may assume \( A_{m,l(m,n+1)}(0) = B_{m,l(m,n+1)}(0) = 1 \). Since \( \triangle_{m,l(m,n+1)}(x) \neq 0 \), we have \( D_{m,l(m,n+1)}(x) \neq 0 \) by Lemma 6. Therefore by Lemma 5
\[
\text{ord} f_{m,l(m,n+1)}(x) \leq \text{ord} (A_{m,l(m,n+1)}(x)\Phi_{l(m,n+1)}(x)^m - B_{m,l(m,n+1)}(x)) \leq r(2mL + 1).
\]

Applying Lemma 7, we see that there exists a positive number \( \eta_m < 1 \) such that
\[
0 < |f_{m,l(l(m,n+1))}(x)| < \eta_m^{r(l(m,n+1))}
\]
for every large \( n \). Since
\[
\Phi_0(x)^m = \prod_{k=0}^{l(m,n+1) - 1} \left( \frac{E_k(x)}{F_k(x)} \right)^m \prod_{k=l(m,n)+1}^{l(m,n+1)} \left( \frac{E_k(x)}{F_k(x)} \right)^m \Phi_{l(m,n+1)}(x)^m,
\]
we get by (9)

\[ f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}}) = \Phi_{0}(\alpha)^{m} - \prod_{k=0}^{l(m,n)} \left( \frac{E_{k}(\alpha^{r^{k}})}{F_{k}(\alpha^{r^{k}})} \right)^{m} \frac{B_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}})}{A_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}})}. \]

If \( \Phi_{0}(\alpha)^{m} \) is algebraic, then \( f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}}) \) is also algebraic and so there exists a constant \( C_{m} > 1 \) such that

\[ ||f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})|| \leq C_{m}^{r^{l(m,n)}}. \] (11)

These inequalities (10) and (11) contradict the fundamental inequality if \( n \) is large. Hence \( \Phi_{0}(\alpha) \) is transcendental, also in this case. The converse is trivial.

The next lemma together with Lemma 8 implies Theorem 1.

**Lemma 9.** \( \Phi_{0}(x) \) is a rational function with coefficients in \( K \) if and only if \( \Phi_{0}(x)^{m} \) is so for some positive integer \( m \).

**Proof.** Suppose that \( \Phi_{0}(x)^{m} \in K(x) \) for some integer \( m \geq 1 \), then \( \Phi_{0}(\alpha) \) is algebraic. By Lemma 8 there exists a positive integer \( m' \) such that \( \Delta_{m',n}(x) = 0 \) for every large \( n \), that is,

\[ \left( \frac{E_{n}(x)}{F_{n}(x)} \right)^{m'} = \frac{B_{m',n}(x)}{A_{m',n}(x)} \frac{A_{m',n+1}(x^{f})}{B_{m',n+1}(x^{f})}, \quad n \geq N. \]

Hence we have

\[ \Phi_{0}(x)^{mm'} = \left( \prod_{k=0}^{n-1} \frac{E_{k}(x^{r^{k}})}{F_{k}(x^{r^{k}})} \right)^{mm'} \left( \frac{B_{m',n}(x^{r^{n}})}{A_{m',n}(x^{r^{n}})} \right)^{m} \left( \frac{P(x)}{Q(x)} \right)^{m'}, \quad n \geq N. \]

for some \( P(x), Q(x) \in K[x] \). We can put

\[ \frac{B_{m',n}(x)}{A_{m',n}(x)} = C_{n}(x)p_{n}(x)^{m'}, \quad \frac{P(x)}{Q(x)} = R(x)q_{n}(x)^{m}, \]

where \( p_{n}(x), q_{n}(x) \in K(x)^{\times}, p_{n}(0) = 1 \) and \( C_{n}(x), R(x) \in K[x] \) with orders less than \( m' \) and \( m \) at each zero, respectively. Since \( B_{m',n}(x)/A_{m',n}(x) = 1 + O(x) \), we may assume \( C_{n}(0) = 1 \). If \( \deg C_{n}(x) \geq 1 \), there exists an \( \alpha \neq 0 \) with \( C_{n}(\alpha) = 0 \). Since \( C_{n}(x^{r^{n}})^{m} \in R(x)^{m'}(K(x)^{\times})^{mm'} \) and the order of \( C_{n}(x^{r^{n}}) \) at \( \alpha^{1/k} \) is less than \( m' \), we see that \( \alpha^{1/k} \) is a root of \( R(x) \). This implies \( m'r^{n} \leq m'\deg R(x) \). Hence \( C_{n}(x) = 1 \) for every large \( n \). Therefore we obtain

\[ \frac{B_{m',n}(x)}{A_{m',n}(x)} = \left( \frac{B_{n}(x)}{A_{n}(x)} \right)^{m'}, \quad n \geq M. \]
for some $A_n(x), B_n(x) \in K[x]$ satisfying $A_n(0) = B_n(0) = 1$, $(A_n(x), B_n(x)) = 1$, and $\deg A_n(x), \deg B_n(x) \leq L$. Then we have
\[
\frac{E_n(x)}{F_n(x)} = \frac{B_n(x)A_{n+1}(x^r)}{A_n(x)B_{n+1}(x^r)}, \quad n \geq M,
\]
that is, $\Phi_0(x)$ is a rational function with coefficients in $K$. The converse is trivial. Hence the proof is completed.

References


