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Transcendence of certain infinite products

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1 Introduction and the results

Duverney [1] introduced an inductive method to prove the transcendence of the number

\[ \sum_{k=1}^{\infty} \frac{1}{a^{2^k} + b_k}, \]

where \( a (|a| \geq 2) \) is an integer, \( \{b_k\}_{k \geq 1} \) is a sequence of integers satisfying \( \log|b_k| = o(2^k) \), and \( a^{2^k} + b_k \neq 0 \) for every \( k \geq 1 \). Recently, Duverney and Nishioka [2] developed this method and gave a transcendence criterion for general series. As applications, they established necessary and sufficient conditions for transcendence of the following numbers

\[ \sum_{k=0}^{\infty} \frac{a_k}{F_{r^k} + b_k}, \quad \sum_{k=0}^{\infty} \frac{a_k}{L_{r^k} + b_k}, \]

where \( \{a_k\}_{k \geq 0} \) and \( \{b_k\}_{k \geq 0} \) are suitable sequences of algebraic numbers, and \( F_n \) and \( L_n \) are Fibonacci numbers and Lucas numbers defined by \( F_{n+2} = F_{n+1} + F_n \) (\( n \geq 0 \)), \( F_0 = 0, F_1 = 1 \) and \( L_{n+2} = L_{n+1} + L_n \) (\( n \geq 0 \)), \( L_0 = 2, L_1 = 1 \), respectively. The purpose of this article is to prove the transcendence of the values of infinite products of the form (1) by modifying the method in [2].

For an algebraic number \( \alpha \), we denote by \( |\alpha| \) the maximum of the absolute values of its conjugates and by \( \text{den}(\alpha) \) the least positive integer such that \( \text{den}(\alpha) \alpha \) is an algebraic integer, and define \( ||\alpha|| = \max\{|\alpha|, \text{den}(\alpha)\} \). Then we have the fundamental inequalities

\[ |\alpha| \geq ||\alpha||^{-2[Q(\alpha):Q]} \quad \text{and} \quad ||\alpha^{-1}|| \leq ||\alpha||^{2[Q(\alpha):Q]} \]

for nonzero algebraic \( \alpha \) (cf. Lemma 2.10.2 in [5]).

Let \( K \) be an algebraic number field, \( O_K \) be the ring of integer in \( K \). Let \( r \geq 2 \) and \( L \geq 1 \) be integers, and let

\[ \Phi_0(x) = \prod_{k=0}^{\infty} \frac{E_k(x^{r^k})}{F_k(x^{r^k})} \]

(1)
with
\[ E_k(x) = 1 + a_{k1}x + a_{k2}x^2 + \cdots + a_{kL}x^L \in K[x], \]
\[ F_k(x) = 1 + b_{k1}x + b_{k2}x^2 + \cdots + b_{kL}x^L \in K[x], \]
where \( \log ||a_{kl}||, \log ||b_{k1}|| = o(r^k) \), \( 1 \leq l \leq L \).

We suppose that there exists a positive integer \( D \) such that \( DF_k(x) \in O_K[x] \) \( (k \geq 0) \).

Then for algebraic number \( \alpha \) satisfying \( 0 < |\alpha| < 1 \) and \( E_k(\alpha^{t^k})F_k(\alpha^{r^k}) \neq 0 \) \( (k \geq 0) \), we prove the following:

**Theorem 1.** \( \Phi_0(\alpha) \) is algebraic if and only if \( \Phi_0(x) \) is a rational function with coefficients in \( K \).

It should be noticed that in [2] they proved a similar result for infinite sums. The tools to prove Theorem 1 are also similar to those in [2], however we need some different techniques.

As applications, we have the following results.

**Theorem 2.** Let \( K \) be an algebraic number field, \( r \geq 2 \) be an integer, and
\[ \Phi(x) = \prod_{k=0}^{\infty} \left( 1 + a_k x^r \right), \]
where \( a_k \in K \) \( (k \geq 0) \) and \( \log ||a_k|| = o(r^k) \). Let \( \alpha \in K \) with \( 0 < |\alpha| < 1 \) and \( 1 + a_k \alpha^r \neq 0 \) \( (k \geq 0) \). Then \( \Phi(\alpha) \) is algebraic if and only if at least one of the following conditions holds:
(i) \( a_n = 0 \) for every large \( n \).
(ii) \( r = 2 \) and there exists a root of unity \( \omega \) such that \( a_n = \omega^{2^n} \) for every large \( n \).

Nishioka [4] proved that the numbers \( \prod_{k=0}^{\infty} (1 - \alpha^r) \), \( r = 2, 3, 4, \ldots \) are algebraically independent for any fixed algebraic number \( \alpha \) with \( 0 < |\alpha| < 1 \). Furthermore, Tanaka [6] proved the algebraic independence of the numbers \( \prod_{k=0}^{\infty} (1 - \alpha_i^r) \), \( i = 1, 2, \ldots, n \), for a linear recurrence \( \{a_k\}_{k \geq 0} \) and algebraic numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) under some suitable conditions.

In the following, we consider the binary recurrences \( \{R_n\}_{n \geq 0} \) defined by
\[ R_{n+2} = A_1 R_{n+1} + A_2 R_n, \quad A_1, A_2 \in \mathbb{Z} \setminus \{0\}, \quad R_0, R_1 \in \mathbb{Z}. \]
We suppose that \( |A_2| = 1 \) and \( \Delta = A_1^2 + 4A_2 > 0 \). Then \( R_n \) is written as
\[ R_n = g_1 \rho_1^n + g_2 \rho_2^n, \quad g_1, g_2 \in \mathbb{Q}(\rho_1)^{x}, \] (2)
where \( \rho_1 \) and \( \rho_2 \) are the roots of \( g(x) = x^2 - A_1 x - A_2 \). By the assumption, \( \rho_1 \rho_2 = \pm 1 \).

We may assume \( |\rho_1| > |\rho_2| \), since \( A_1 \neq 0 \) and \( \Delta > 0 \). For a negative integer \( n \), we define \( R_n \) by (2).
Theorem 3. Let $R_n$ be a binary recurrence given by (2) and $r$, $c$, and $d$ be integers such that $r \geq 2$ and $c \geq 1$. Let $K$ be an algebraic number field and $a_k \in K$ satisfy $a_k \neq -R_{cr^k+d}$ ($k \geq 0$) and $\log||a_k|| = o(r^k)$. Then

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{R_{cr^k+d}}\right)$$

is algebraic if and only if at least one of the following conditions is satisfied:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = R_d$ for every large $n$.

(iii) $r = 2$, $g_1 \beta_1^d = g_2 \beta_2^d$, and there exists a root of unity $\omega$ such that $a_n = g_1 \beta_1^d (\omega^{2^n} + \omega^{-2^n})$ for every large $n$.

In the following examples, let $\{a_k\}_{k \geq 0}$, $r$, $c$, and $d$ be as in Theorem 3.

Example 1. Let $F_n$ be Fibonacci numbers defined above. Then

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{F_{cr^k+d}}\right)$$

is algebraic if and only if at least one of the following conditions holds:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = F_d$ for every large $n$.

In particular,

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{F_{r^k}}\right)$$

is algebraic if and only if $a_n = 0$ for all large $n$.

Example 2. Let $L_n$ be Lucas numbers defined above. Then

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{L_{cr^k+d}}\right)$$

is algebraic if and only if at least one of the following conditions is satisfied:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = L_d$ for every large $n$.

(iii) $r = 2$, $d = 0$, and there exists a root of unity $\omega$ such that $a_n = \omega^{2^n} + \omega^{-2^n}$ for every large $n$.

In particular, for any integer $a \neq 0$ and $r \geq 2$ the number $\prod_{k=1}^{\infty} \left(1 + \frac{a}{L_{r^k}}\right)$ is transcendental, except for two algebraic cases

$$\prod_{k=1}^{\infty} \left(1 + \frac{-1}{L_{2^k}}\right) = \frac{\sqrt{5}}{4}, \quad \prod_{k=1}^{\infty} \left(1 + \frac{2}{L_{2^k}}\right) = \sqrt{5}.$$
which are obtained from the case (iii) with \( \omega = \frac{1 \pm \sqrt{-3}}{2} \) and \( \omega = \pm 1 \), respectively. These examples of algebraic infinite products involving Lucas numbers seems to be new.

2 Transcendence of \( \Phi_0(\alpha) \)

For a formal power series \( f(x) \in K[[x]] \) such that \( f(x) = \sum_{l \leq n} a_n x^n \) with \( a_l \neq 0 \), we define \( \text{ord} f(x) = l \).

**Lemma 1.** Let \( \Phi_0(x) \) and \( \alpha \) be given in Section 1. For every positive integer \( m \), suppose that there is a positive constant \( c(m) \) such that

\[
\text{ord} \left( A_0(x) + A_1(x) \Phi_0(x)^m \right) \leq c(m)M
\]

for any \( M \geq 1 \) and any polynomials \( A_0, A_1 \in K[x] \), not both zero, satisfying \( \deg A_0(x), \deg A_1(x) \leq M \). Then \( \Phi_0(\alpha) \) is transcendental.

Lemma 1 will be used in the proof of Theorem 1 in the next section. For the proof of Lemma 1, we apply the following criterion of Loxton and van der Poorten [3]. We put

\[
\Phi_n(x) = \prod_{k=0}^\infty \frac{E_{n+k}(x^{r^k})}{F_{n+k}(x^{r^k})}, \quad n \geq 0.
\]

**Lemma 2** (cf. Theorem 2.9.1 in [5]). Let \( K \) be an algebraic number field, \( r \geq 2 \) be an integer, \( \{\Phi_n(x)\}_{n \geq 0} \) be a sequence in the ring of formal power series \( K[[x]] \), and \( \alpha \in K \) with \( 0 < |\alpha| < 1 \). If the following three properties are satisfied, then \( \Phi_0(\alpha) \) is transcendental.

(I) \( \Phi_n(\alpha^{r^n}) = a_n \Phi_0(\alpha) + b_n \) with \( a_n, b_n \in K \), and \( \log ||a_n||, \log ||b_n|| = O(r^n) \).

(II) If \( \Phi_n(x) = \sum_{i=0}^\infty a_i^{(n)} x^i \), then for any \( \epsilon > 0 \) there is a positive integer \( n_0 \) such that

\[
\log ||a_i^{(n)}|| \leq \epsilon r^n (1 + l)
\]

for any \( n \geq n_0 \) and \( l \geq 0 \).

(III) Let \( \{s_i\}_{i \geq 0} \) be variables and

\[
F(x; s) = F(x; \{s_i\}_{i \geq 0}) = \sum_{l=0}^\infty s_l x^l,
\]

in such a way that

\[
F(x; \sigma^{(n)}) = F(x; \{\sigma_i^{(n)}\}_{i \geq 0}) = \Phi_n(x).
\]
Then for any polynomials $P_0(x, s), \ldots, P_d(x, s) \in K[x, \{s_i\}_{i \geq 0}]$ and 

$$E(x, s) = \sum_{j=0}^{d} P_j(x, s) F(x, s)^j,$$

there is positive integer $I$ with the following property: if $n$ is sufficiently large and $P_0(x, \sigma^{(n)}), \ldots, P_d(x, \sigma^{(n)})$ are not all zero, then $\text{ord} E(x, \sigma^{(n)}) \leq I$.

The property (I) follows from the functional equation

$$\Phi_n(x^{r^n}) = \Phi_0(x) \prod_{k=0}^{n-1} \frac{F_k(x^{x^k})}{E_k(x^{x^k})}.$$  

(4)

It is not difficult to see that the property (II) is satisfied. The crucial point in applying Lemma 2 is to check property (III), which is done via Lemma 3.

**Lemma 3.** Suppose that $\Phi_0(x)$ satisfy the assumption 3. Then for every positive integer $d$, there exists a positive constant $c_d$ such that

$$\text{ord}(A_0(x) + A_1(x)\Phi_0(x) + \cdots + A_d(x)\Phi_0(x)^d) \leq c_d M$$

for any $M \geq 1$ and any polynomials $A_0(x), \ldots, A_d(x) \in K[x]$, not all zero, with $\deg A_i(x) \leq M (0 \leq i \leq d)$.

### 3 Proof of Theorem 1

We use the following lemma.

**Lemma 4 (Theorem 5 in [2]).** Let $r \geq 2$ be an integer, $K$ be a commutative field, and $f \in K[[x]]$. Let $\{m(n)\}_{n \geq 0}$ be an increasing sequence of nonnegative integers with $(m(n+1) - m(n)) \leq c_1$ for some $c_1 \geq 1$. Suppose that there exists a sequence $\{(P_n(x), Q_n(x))\}_{n \geq 0}$ in $K[x]^2$ such that

$$P_n(x)Q_{n+1}(x) - P_{n+1}(x)Q_n(x) \neq 0$$  

(5)

$$\deg Q_n(x), \deg P_n(x) \leq c_2 r^{m(n)}$$  

(6)

$$\text{ord}(Q_n(x)f(x) - P_n(x)) \geq c_3 r^{m(n)}$$  

(7)

for every $n \geq 0$, where $0 < c_2 < c_3$. Then we have

$$\text{ord}(A_0(x) + A_1(x)f(x)) \leq \left(c_2 r^{m(0)+2C} \left(1 + \frac{1}{c_3-c_2}\right) + 1\right) M$$

for any $M \geq 1$ and for any polynomials $A_0(x), A_1(x) \in K[x]$, not both zero, satisfying $\deg A_0(x), \deg A_1(x) \leq M$.
For each $f(x) = \Phi_0(x)^m$ ($m = 1, 2, \ldots$), we construct a sequence $\{(P_{m,n}, Q_{m,n})\}_{n \geq 0}$ satisfying the hypotheses of Lemma 4. Consider the $(mL, mL)$ Padé-approximants to $\Phi_n(x)^m$, that is, polynomials $A_{m,n}(x)$ and $B_{m,n}(x)$ satisfying $\deg A_{m,n}(x), \deg B_{m,n}(x) \leq mL$ and

$$A_{m,n}(x)\Phi_n(x)^m - B_{m,n}(x) = O(x^{2mL+1}).$$

By Siegel’s lemma (cf. Lemma 1.4.2 in [5]), we may assume that $\log ||$ of the coefficients of $A_{m,n}(x)$ and $B_{m,n}(x)$ are $o(r^n)$. Define

$$D_{m,n}(x) = \left| \begin{array}{cc} A_{m,n}(x) & B_{m,n}(x) \\ A_{m,n+1}(x^r)F_n(x)^m & B_{m,n+1}(x^r)E_n(x)^m \end{array} \right|.$$

**Lemma 5.** Suppose that $D_{m,n}(x) \neq 0$. Then

$$\text{ord} (A_{m,n}(x)\Phi_n(x)^m - B_{m,n}(x)) < r(2mL + 1).$$

**Proof.** This can be proved similarly as the proof of Lemma 4 in [2].

Replacing $x$ by $x^n$ in (8) and use the functional equation (4), we have

$$Q_{m,n}^*(x)\Phi_0(x)^m - P_{m,n}^*(x) = O(x^{(2mL+1)r^n}),$$

where

$$Q_{m,n}^*(x) = A_{m,n}(x^n) \prod_{k=0}^{n-1} F_k(x^r)^m, \quad P_{m,n}^*(x) = B_{m,n}(x^n) \prod_{k=0}^{n-1} E_k(x^r)^m.$$ Since $\deg Q_{m,n}^*(x), \deg P_{m,n}^*(x) \leq 2mLr^n$, the sequence $\{(P_{m,n}, Q_{m,n})\}_{n \geq 0} = \{(P_{m,l(m,n)}, Q_{m,l(m,n)})\}_{n \geq 0}$ satisfies hypotheses (6) and (7) of Lemma 4 for every increasing sequence $\{l(m, n)\}_{n \geq 0}$. To study the condition (5) in Lemma 4, we need the following lemma. We put

$$\triangle_{m,n}(x) = \left| \begin{array}{cc} Q_{m,n}(x) & P_{m,n}(x) \\ Q_{m,n+1}(x) & P_{m,n+1}(x) \end{array} \right|.$$ **Lemma 6.** $\triangle_{m,n}(x) = 0$ if and only if $D_{m,n}(x) = 0$, that is,

$$\left( \frac{E_n(x)}{F_n(x)} \right)^m = \frac{B_{m,n}(x)A_{m,n+1}(x^r)}{A_{m,n}(x)B_{m,n+1}(x^r)}.$$ **Proof.** By definition, $\triangle_{m,n}(x) = 0$ if and only if

$$\left| \begin{array}{cc} A_{m,n}(x^n) & B_{m,n}(x^n) \\ A_{m,n+1}(x^{n+1})F_n(x^{r^n})^m & B_{m,n+1}(x^{n+1})E_n(x^{r^n})^m \end{array} \right| = 0,$$

which is equivalent to

$$D_{m,n}(x) = \left| \begin{array}{cc} A_{m,n}(x) & B_{m,n}(x) \\ A_{m,n+1}(x^r)F_n(x)^m & B_{m,n+1}(x^r)E_n(x)^m \end{array} \right| = 0.$$
Lemma 7. For each positive integer $m$, we define $f_{m,n}(x)$ by

$$f_{m,n}(x) = 1 - \frac{A_{m,n}(x)}{B_{m,n}(x)} \Phi_{n}(x)^m.$$  

Let $I$ be a positive integer and $\alpha$ be an algebraic number with $0 < |\alpha| < 1$. Then there exists a positive number $\eta_m < 1$ such that

$$0 < |f_{m,n}(\alpha^r)| < \eta_m^{\text{ord} f_{m,n}(x)}$$

for every large $n$ satisfying $\text{ord} f_{m,n}(x) \leq I$.

**Proof.** We may assume $A_{m,n}(0) = B_{m,n}(0) = 1$ by (8). Let $\theta > 1$ and $A_{m,n}(x)/B_{m,n}(x) = \sum_{i=0}^{\infty} \tau_i^{(m,n)} x^i$. Then we obtain $||\tau_i^{(m,n)}|| \leq (\theta^{2mL})^{1r}$ for any $n \geq n_0$ and $l \geq 0$. Let $\Phi_{n}(x)^m = (\sum_{i=0}^{\infty} \sigma_i^{(n)} x^i)^m = \sum_{i=0}^{\infty} \mu_i^{(m,n)} x^i$, then we have

$$f_{m,n}(x) = 1 - \left( \sum_{i=0}^{\infty} \tau_i^{(m,n)} x^i \right) \left( \sum_{i=0}^{\infty} \mu_i^{(m,n)} x^i \right) = \sum_{i=1}^{\infty} \left( \sum_{l+i=l} \tau_s^{(m,n)} \mu_t^{(m,n)} \right) x^i,$$

where $|\sum_{s+t=l} \tau_s^{(m,n)} \mu_t^{(m,n)}| \leq (\theta^{6mL})^{1r}$ and $\text{den} \left( \sum_{s+t=l} \tau_s^{(m,n)} \mu_t^{(m,n)} \right) \leq (\theta^{5mL})^{1r}$. We put

$$f_{m,n}(x) = a_H x^H + a_{H+1} x^{H+1} + \ldots,$$

where $1 \leq H \leq I$. Then

$$f_{m,n}(\alpha^r) = a_H \alpha^{Hr} + \frac{a_{H+1}}{a_H} \alpha^{(H+1)r} + \frac{a_{H+2}}{a_H} \alpha^{2r} + \cdots.$$  

Since $||a_H|| \leq (\theta^{6mL})^{Hr}$, we obtain

$$\left| \frac{a_{H+1}}{a_H} \alpha^{(H+1)r} \right| \leq (\theta^{6mL})^{(1+2[K:Q])Ir} |\theta^{6mL} \alpha|^{Ir}.$$  

Choosing $\theta > 1$ with $\eta_m = (\theta^{6mL})^{(1+2[K:Q])I}|\theta^{6mL} \alpha| < 1$, we have

$$0 < |f_{m,n}(\alpha^r)| < 2 |\theta^{6mL} \alpha|^{Hr} < \eta_m^{Hr}$$

for sufficiently large $n$; which implies the lemma.

Lemma 8. $\Phi_0(\alpha)$ is algebraic if and only if $\Phi_0(x)^m$ is a rational function with coefficients in $K$ for some positive integer $m$.

**Proof.** We prove that if $\Phi_0(\alpha)$ is algebraic then there exists a positive integer $m$ such that $\Delta_{m,n}(x) = 0$ for every large $n$, which implies $\Phi_0(x)^m$ is a rational function.
by Lemma 6. For every integer $m$, suppose that there exist infinitely many $n$ satisfying $\Delta_{m,n}(x) \neq 0$. Denote by $\{l(m,n)\}_{n \geq 0}$ the sequence satisfying

$$\Delta_{m,l(m,n)}(x) \neq 0, \quad \Delta_{m,k}(x) = 0$$

for every $n \geq 0$ and every $k$ with $l(m,n) < k < l(m,n+1)$. Then two cases occur:

(i) For every $m$, $l(m,n+1) - l(m,n) \leq C_m$ for some positive constant $C_m$. Then it is clear that the determinant

$$\begin{vmatrix} Q_{m,l(m,n)}^*(x) & P_{m,l(m,n)}^*(x) \\ Q_{m,l(m,n+1)}^*(x) & P_{m,l(m,n+1)}^*(x) \end{vmatrix} = \begin{vmatrix} Q_{m,n}(x) & P_{m,n}(x) \\ Q_{m,n+1}(x) & P_{m,n+1}(x) \end{vmatrix} \neq 0,$$

namely, the condition (5) in Lemma 4 is satisfied. Hence we can apply Lemma 1 and find that $\Phi_0(\alpha)$ is transcendental.

(ii) For some $m$, $\varlimsup_{n \to \infty} (l(m,n+1) - l(m,n)) = +\infty$. In this case, we have by using Lemma 6

$$\left( \frac{E_k(x)}{F_k(x)} \right)^m = \frac{B_{m,k}(x)A_{m,k+1}(x^r)}{A_{m,k}(x)B_{m,k+1}(x^r)}$$

for every $k$ satisfying $l(m,n) < k < l(m,n+1)$, so that

$$(9) \prod_{k=l(m,n)+1}^{l(m,n+1)-1} \left( \frac{E_k(x^r)}{F_k(x^r)} \right)^m = \frac{B_{m,l(m,n)+1}(x^r)}{A_{m,l(m,n)+1}(x^r)}A_{m,l(m,n+1)}(x^r)B_{m,l(m,n+1)}(x^r).$$

Let

$$f_{m,l(m,n+1)}(x) = \frac{A_{m,l(m,n+1)}(x)}{B_{m,l(m,n+1)}(x)}\Phi_{l(m,n+1)}(x)^m - 1,$$

where we may assume $A_{m,l(m,n+1)}(0) = 1$. Since $\Delta_{m,l(m,n+1)}(x) \neq 0$, we have $D_{m,l(m,n+1)}(x) \neq 0$ by Lemma 6. Therefore by Lemma 5

$$\text{ord} f_{m,l(m,n+1)}(x) \leq \text{ord} (A_{m,l(m,n+1)}(x)\Phi_{l(m,n+1)}(x)^m - B_{m,l(m,n+1)}(x))$$

$$\leq r(2mL + 1).$$

Applying Lemma 7, we see that there exists a positive number $\eta_m < 1$ such that

$$0 < |f_{m,l(m,n+1)}(\alpha^{r(m,n+1)})| < \eta_m^{r(m,n+1)}$$

(10) for every large $n$. Since

$$\Phi_0(x)^m = \prod_{k=0}^{l(m,n)} \left( \frac{E_k(x^r)}{F_k(x^r)} \right)^m \prod_{k=l(m,n)+1}^{l(m,n+1)-1} \left( \frac{E_k(x^r)}{F_k(x^r)} \right)^m \Phi_{l(m,n+1)}(x^{r(m,n+1)})^m,$$
we get by (9)

\[
\begin{align*}
    f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}}) & = \Phi_0(\alpha)^m - \prod_{k=0}^{l(m,n)} \left( \frac{E_k(\alpha^{r^k})}{F_k(\alpha^{r^k})} \right)^m \\
    & \geq \prod_{k=0}^{l(m,n)} \left( \frac{E_k(\alpha^{r^k})}{F_k(\alpha^{r^k})} \right)^m.
\end{align*}
\]

If $\Phi_0(\alpha)^m$ is algebraic, then $f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})$ is also algebraic and so there exists a constant $C_m > 1$ such that

\[
||f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})|| \leq C_m^{r^{l(m,n)}}.
\]

These inequalities (10) and (11) contradict the fundamental inequality if $n$ is large. Hence $\Phi_0(\alpha)$ is transcendental, also in this case. The converse is trivial.

The next lemma together with Lemma 8 implies Theorem 1.

**Lemma 9.** $\Phi_0(x)$ is a rational function with coefficients in $K$ if and only if $\Phi_0(x)^m$ is so for some positive integer $m$.

**Proof.** Suppose that $\Phi_0(x)^m \in K(x)$ for some integer $m \geq 1$, then $\Phi_0(\alpha)$ is algebraic. By Lemma 8 there exists a positive integer $m'$ such that $\Delta_{m',n}(x) = 0$ for every large $n$, that is,

\[
\left( \frac{E_n(x)}{F_n(x)} \right)^{m'} = \frac{B_{m',n}(x)A_{m',n+1}(x^{r^n})}{A_{m',n}(x)B_{m',n+1}(x)}.
\]

Hence we have

\[
\Phi_0(x)^{mm'} = \left( \prod_{k=0}^{n-1} \frac{E_k(x^{r^k})}{F_k(x^{r^k})} \right)^{mm'} \left( \frac{B_{m',n}(x^{r^n})}{A_{m',n}(x^{r^n})} \right)^m = \left( \frac{P(x)}{Q(x)} \right)^{m'},
\]

for some $P(x), Q(x) \in K[x]$. We can put

\[
\frac{B_{m',n}(x)}{A_{m',n}(x)} = C_n(x)p_n(x)^{m'}, \quad \frac{P(x)}{Q(x)} = R(x)q_n(x)^m,
\]

where $p_n(x), q_n(x) \in K(x)^\times$, $p_n(0) = 1$ and $C_n(x), R(x) \in K[x]$ with orders less than $m'$ and $m$ at each zero, respectively. Since $B_{m',n}(x)/A_{m',n}(x) = 1 + O(x)$, we may assume $C_n(0) = 1$. If $\deg C_n(x) \geq 1$, there exists an $\alpha \neq 0$ with $C_\alpha(\alpha) = 0$. Since $C_n(x^{r^n})^m \in R(x)^{m'}(K(x)^\times)^{mm'}$ and the order of $C_n(x^{r^n})$ at $\alpha^{\frac{1}{r^n}}$ is less than $m'$, we see that $\alpha^{\frac{1}{r^n}}$ is a root of $R(x)$. This implies $m'r^n \leq m'\deg R(x)$. Hence $C_n(x) = 1$ for every large $n$. Therefore we obtain

\[
\frac{B_{m',n}(x)}{A_{m',n}(x)} = \left( \frac{B_n(x)}{A_n(x)} \right)^{m'}, \quad n \geq M
\]
for some \( A_n(x), B_n(x) \in K[x] \) satisfying \( A_n(0) = B_n(0) = 1 \), \( (A_n(x), B_n(x)) = 1 \), and \( \deg A_n(x), \deg B_n(x) \leq L \). Then we have
\[
\frac{E_n(x)}{F_n(x)} = \frac{B_n(x)A_{n+1}(x^r)}{A_n(x)B_{n+1}(x^r)}, \quad n \geq M,
\] (12)
that is, \( \Phi_0(x) \) is a rational function with coefficients in \( K \). The converse is trivial. Hence the proof is completed.

References


