<table>
<thead>
<tr>
<th>Title</th>
<th>Transcendence of certain infinite products (Analytic Number Theory and Surrounding Areas)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Tachiya, Yohei</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1511: 19-28</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58615">http://hdl.handle.net/2433/58615</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Transcendence of certain infinite products

Yohei Tachiya
Department of Mathematics, Keio University

1 Introduction and the results

Duverney [1] introduced an inductive method to prove the transcendence of the number

$$\sum_{k=1}^{\infty} \frac{1}{a^{2^k} + b_k},$$

where $a (|a| \geq 2)$ is an integer, $\{b_k\}_{k\geq 1}$ is a sequence of integers satisfying $\log|b_k| = o(2^k)$, and $a^{2^k} + b_k \neq 0$ for every $k \geq 1$. Recently, Duverney and Nishioka [2] developed this method and gave a transcendence criterion for general series. As applications, they established necessary and sufficient conditions for transcendence of the following numbers

$$\sum_{k=0}^{\infty} \frac{a_k}{F_{r^k} + b_k}, \quad \sum_{k=0}^{\infty} \frac{a_k}{L_{r^k} + b_k},$$

where $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ are suitable sequences of algebraic numbers, and $F_n$ and $L_n$ are Fibonacci numbers and Lucas numbers defined by $F_{n+2} = F_{n+1} + F_n (n \geq 0)$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n (n \geq 0)$, $L_0 = 2$, $L_1 = 1$, respectively. The purpose of this article is to prove the transcendence of the values of infinite products of the form (1) by modifying the method in [2].

For an algebraic number $\alpha$, we denote by $|\alpha|$ the maximum of the absolute values of its conjugates and by $\text{den}(\alpha)$ the least positive integer such that $\text{den}(\alpha)\alpha$ is an algebraic integer, and define $||\alpha|| = \max\{|\alpha|, \text{den}(\alpha)\}$. Then we have the fundamental inequalities

$$|\alpha| \geq ||\alpha||^{-2[Q(\alpha):\mathbb{Q}]} \quad \text{and} \quad ||\alpha^{-1}|| \leq ||\alpha||^{2[Q(\alpha):\mathbb{Q}]}$$

for nonzero algebraic $\alpha$ (cf. Lemma 2.10.2 in [5]).

Let $K$ be an algebraic number field, $O_K$ be the ring of integer in $K$. Let $r \geq 2$ and $L \geq 1$ be integers, and let

$$\Phi_0(x) = \prod_{k=0}^{\infty} \frac{E_k(x^{r^k})}{F_k(x^{r^k})}$$

(1)
with
\[ E_k(x) = 1 + a_{k1}x + a_{k2}x^2 + \cdots + a_{kL}x^L \in K[x], \]
\[ F_k(x) = 1 + b_{k1}x + b_{k2}x^2 + \cdots + b_{kL}x^L \in K[x], \]
where \( \log ||a_{kl}||, \log ||b_{kl}|| = o(r^k), 1 \leq l \leq L \). We suppose that there exists a positive integer \( D \) such that \( DF_k(x) \in O_K[x] (k \geq 0) \).

Then for algebraic number \( \alpha \) satisfying \( 0 < |\alpha| < 1 \) and \( E_k(\alpha^{t^k})F_k(\alpha^{r^k}) \neq 0 (k \geq 0) \), we prove the following:

**Theorem 1.** \( \Phi_0(\alpha) \) is algebraic if and only if \( \Phi_0(x) \) is a rational function with coefficients in \( K \).

It should be noticed that in [2] they proved a similar result for infinite sums. The tools to prove Theorem 1 are also similar to those in [2], however we need some different techniques.

As applications, we have the following results.

**Theorem 2.** Let \( K \) be an algebraic number field, \( r \geq 2 \) be an integer, and
\[ \Phi(x) = \prod_{k=0}^{\infty} \left( 1 + a_k x^r \right), \]
where \( a_k \in K (k \geq 0) \) and \( \log ||a_k|| = o(r^k) \). Let \( \alpha \in K \) with \( 0 < |\alpha| < 1 \) and \( 1 + a_k \alpha^r \neq 0 (k \geq 0) \). Then \( \Phi(\alpha) \) is algebraic if and only if at least one of the following conditions holds:
(i) \( a_n = 0 \) for every large \( n \).
(ii) \( r = 2 \) and there exists a root of unity \( \omega \) such that \( a_n = \omega^{2^n} \) for every large \( n \).

Nishioka [4] proved that the numbers \( \prod_{k=0}^{\infty} (1 - \alpha^{r^k}) \), \( r = 2, 3, 4, \ldots \) are algebraically independent for any fixed algebraic number \( \alpha \) with \( 0 < |\alpha| < 1 \). Furthermore, Tanaka [6] proved the algebraic independence of the numbers \( \prod_{k=0}^{\infty} (1 - \alpha_i^{r^k}) \), \( i = 1, 2, \ldots, n \), for a linear recurrence \( \{a_k\}_{k \geq 0} \) and algebraic numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) under some suitable conditions.

In the following, we consider the binary recurrences \( \{R_n\}_{n \geq 0} \) defined by
\[ R_{n+2} = A_1 R_{n+1} + A_2 R_n, \quad A_1, A_2 \in \mathbb{Z} \setminus \{0\}, \quad R_0, R_1 \in \mathbb{Z}. \]
We suppose that \( |A_2| = 1 \) and \( \Delta = A_1^2 + 4A_2 > 0 \). Then \( R_n \) is written as
\[ R_n = g_1 \rho_1^n + g_2 \rho_2^n, \quad g_1, g_2 \in \mathbb{Q}(\rho_1 \rho_2), \quad (2) \]
where \( \rho_1 \) and \( \rho_2 \) are the roots of \( g(x) = x^2 - A_1 x - A_2 \). By the assumption, \( \rho_1 \rho_2 = \pm 1 \). We may assume \( |\rho_1| > |\rho_2| \), since \( A_1 \neq 0 \) and \( \Delta > 0 \). For a negative integer \( n \), we define \( R_n \) by (2).
Theorem 3. Let $R_n$ be a binary recurrence given by (2) and $r$, $c$, and $d$ be integers such that $r \geq 2$ and $c \geq 1$. Let $K$ be an algebraic number field and $a_k \in K$ satisfy $a_k \neq -R_{cr^k+d} (k \geq 0)$ and $\log||a_k|| = o(r^k)$. Then

$$
\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{R_{cr^k+d}}\right)
$$

is algebraic if and only if at least one of the following conditions is satisfied:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = R_d$ for every large $n$.

(iii) $r = 2$, $g_1 \rho_1^d = g_2 \rho_2^d$, and there exists a root of unity $\omega$ such that $a_n = g_1 \rho_1^d(\omega^{2^n} + \omega^{-2^n})$ for every large $n$.

In the following examples, let \{a_k\}_{k \geq 0}, \ r, \ c, \ and \ d \ be \ as \ in \ Theorem \ 3.

Example 1. Let $F_n$ be Fibonacci numbers defined above. Then

$$
\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{F_{cr^k+d}}\right)
$$

is algebraic if and only if at least one of the following conditions holds:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = F_d$ for every large $n$.

In particular,

$$
\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{F_n}\right)
$$

is algebraic if and only if $a_n = 0$ for all large $n$.

Example 2. Let $L_n$ be Lucas numbers defined above. Then

$$
\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{L_{cr^k+d}}\right)
$$

is algebraic if and only if at least one of the following conditions is satisfied:

(i) $a_n = 0$ for every large $n$.

(ii) $r = 2$ and $a_n = L_d$ for every large $n$.

(iii) $r = 2, \ d = 0$, and there exists a root of unity $\omega$ such that $a_n = \omega^{2^n} + \omega^{-2^n}$ for every large $n$.

In particular, for any integer $a \neq 0$ and $r \geq 2$ the number $\prod_{k=1}^{\infty} \left(1 + \frac{a}{L_{r^k}}\right)$ is transcendental, except for two algebraic cases

$$
\prod_{k=1}^{\infty} \left(1 + \frac{-1}{L_{2^k}}\right) = \frac{\sqrt{5}}{4}, \quad \prod_{k=1}^{\infty} \left(1 + \frac{2}{L_{2^k}}\right) = \sqrt{5},
$$
which are obtained from the case (iii) with $\omega = \frac{1 \pm \sqrt{-3}}{2}$ and $\omega = \pm 1$, respectively. These examples of algebraic infinite products involving Lucas numbers seems to be new.

2 Transcendence of $\Phi_0(\alpha)$

For a formal power series $f(x) \in K[[x]]$ such that $f(x) = \sum_{l \leq n} a_n x^n$ with $a_l \neq 0$, we define $\text{ord} f(x) = l$.

**Lemma 1.** Let $\Phi_0(x)$ and $\alpha$ be given in Section 1. For every positive integer $m$, suppose that there is a positive constant $c(m)$ such that

$$\text{ord} \ (A_0(x) + A_1(x)\Phi_0(x)^m) \leq c(m)M$$

for any $M \geq 1$ and any polynomials $A_0, A_1 \in K[x]$, not both zero, satisfying $\deg A_0(x), \deg A_1(x) \leq M$. Then $\Phi_0(\alpha)$ is transcendental.

Lemma 1 will be used in the proof of Theorem 1 in the next section. For the proof of Lemma 1, we apply the following criterion of Loxton and van der Poorten [3]. We put

$$\Phi_n(x) = \prod_{k=0}^{\infty} \frac{E_{n+k}(x^{r^k})}{F_{n+k}(x^{r^k})}, n \geq 0.$$

**Lemma 2** (cf. Theorem 2.9.1 in [5]). Let $K$ be an algebraic number field, $r \geq 2$ be an integer, $\{\Phi_n(x)\}_{n \geq 0}$ be a sequence in the ring of formal power series $K[[x]]$, and $\alpha \in K$ with $0 < |\alpha| < 1$. If the following three properties are satisfied, then $\Phi_0(\alpha)$ is transcendental.

(I) $\Phi_n(\alpha^{r^n}) = a_n \Phi_0(\alpha) + b_n$ with $a_n, b_n \in K$, and $\log ||a_n||, \log ||b_n|| = O(r^n)$.

(II) If $\Phi_n(x) = \sum_{l=0}^{\infty} \sigma_l^{(n)} x^l$, then for any $\epsilon > 0$ there is a positive integer $n_0$ such that

$$\log ||\sigma_l^{(n)}|| \leq \epsilon r^n (1 + l)$$

for any $n \geq n_0$ and $l \geq 0$.

(III) Let $\{s_l\}_{l \geq 0}$ be variables and

$$F(x; s) = F(x; \{s_l\}_{l \geq 0}) = \sum_{l=0}^{\infty} s_l x^l,$$

in such a way that

$$F(x; \sigma^{(n)}) = F(x; \{\sigma_l^{(n)}\}_{l \geq 0}) = \Phi_n(x).$$
Then for any polynomials \(P_0(x, s), \ldots, P_d(x, s) \in K[x, \{s_i\}_{i \geq 0}]\) and
\[
E(x, s) = \sum_{j=0}^{d} P_j(x, s) F(x, s)^j,
\]
there is positive integer \(I\) with the following property: if \(n\) is sufficiently large and \(P_0(x, \sigma^{(n)}), \ldots, P_d(x, \sigma^{(n)})\) are not all zero, then \(\text{ord}E(x, \sigma^{(n)}) \leq I\).

The property (I) follows from the functional equation
\[
\Phi_n(x^{f^n}) = \Phi_0(x) \prod_{k=0}^{n-1} \frac{F_k(x^{\tau^k})}{E_k(x^{\tau^k})}.
\] (4)

It is not difficult to see that the property (II) is satisfied. The crucial point in applying Lemma 2 is to check property (III), which is done via Lemma 3.

**Lemma 3.** Suppose that \(\Phi_0(x)\) satisfy the assumption 3. Then for every positive integer \(d\), there exists a positive constant \(c_d\) such that
\[
\text{ord}(A_0(x) + A_1(x)\Phi_0(x) + \cdots + A_d(x)\Phi_0(x)^d) \leq c_d M
\]
for any \(M \geq 1\) and any polynomials \(A_0(x), \ldots, A_d(x) \in K[x]\), not all zero, with \(\deg A_i(x) \leq M\) \((0 \leq i \leq d)\).

## 3 Proof of Theorem 1

We use the following lemma.

**Lemma 4** (Theorem 5 in [2]). Let \(r \geq 2\) be an integer, \(K\) be a commutative field, and \(f \in K[[x]]\). Let \(\{m(n)\}_{n \geq 0}\) be an increasing sequence of nonnegative integers with \((m(n+1) - m(n)) \leq c_1\) for some \(c_1 \geq 1\). Suppose that there exists a sequence \(\{(P_n(x), Q_n(x))\}_{n \geq 0}\) in \(K[x]^2\) such that
\[
P_n(x)Q_{n+1}(x) - P_{n+1}(x)Q_n(x) \neq 0
\] (5)
\[
\deg Q_n(x), \deg P_n(x) \leq c_2 r^{m(n)}
\] (6)
\[
\text{ord}(Q_n(x)f(x) - P_n(x)) \geq c_3 r^{m(n)}
\] (7)
for every \(n \geq 0\), where \(0 < c_2 < c_3\). Then we have
\[
\text{ord}(A_0(x) + A_1(x)f(x)) \leq \left(c_2 r^{m(0) + 2c} \left(1 + \frac{1}{c_3 - c_2}\right) + 1\right) M
\]
for any \(M \geq 1\) and for any polynomials \(A_0(x), A_1(x) \in K[x]\), not both zero, satisfying \(\deg A_0(x), \deg A_1(x) \leq M\).
For each \( f(x) = \Phi_0(x)^m \) \((m = 1, 2, \ldots)\), we construct a sequence \( \{(P_{m,n}, Q_{m,n})\}_{n \geq 0} \) satisfying the hypotheses of Lemma 4. Consider the \((mL, mL)\) Padé-approximants to \( \Phi_n(x)^m \), that is, polynomials \( A_{m,n}(x) \) and \( B_{m,n}(x) \) satisfying \( \deg A_{m,n}(x), \deg B_{m,n}(x) \leq mL \) and
\[
A_{m,n}(x)\Phi_n(x)^m - B_{m,n}(x) = O(x^{2mL+1}).
\] (8)

By Siegel's lemma (cf. Lemma 1.4.2 in [5]), we may assume that \( \log || \) of the coefficients of \( A_{m,n}(x) \) and \( B_{m,n}(x) \) are \( o(r^n) \). Define
\[
D_{m,n}(x) = \left| \begin{array}{cc}
A_{m,n}(x) & B_{m,n}(x) \\
A_{m,n+1}(x^r)F_n(x)^m & B_{m,n+1}(x^r)E_n(x)^m
\end{array} \right|.
\]

**Lemma 5.** Suppose that \( D_{m,n}(x) \neq 0 \). Then
\[
\text{ord} \left( A_{m,n}(x)\Phi_n(x)^m - B_{m,n}(x) \right) < r(2mL + 1).
\]

**Proof.** This can be proved similarly as the proof of Lemma 4 in [2].

Replacing \( x \) by \( x^{r^n} \) in (8) and use the functional equation (4), we have
\[
Q_{m,n}^*(x)\Phi_0(x)^m - P_{m,n}^*(x) = O(x^{(2mL+1)r^n}),
\]
where
\[
Q_{m,n}^*(x) = A_{m,n}(x^{r^n}) \prod_{k=0}^{n-1} F_k(x)^m, \quad P_{m,n}^*(x) = B_{m,n}(x^{r^n}) \prod_{k=0}^{n-1} E_k(x)^m.
\]

Since \( \deg Q_{m,n}^*(x), \deg P_{m,n}^*(x) \leq 2mLr^n \), the sequence \( \{(P_{m,n}, Q_{m,n})\}_{n \geq 0} = \{ (P_{m,l(m,n)}, Q_{m,l(m,n)}) \}_{n \geq 0} \) satisfies hypotheses (6) and (7) of Lemma 4 for every increasing sequence \( \{l(m,n)\}_{n \geq 0} \). To study the condition (5) in Lemma 4, we need the following lemma. We put
\[
\Delta_{m,n}(x) = \left| \begin{array}{cc}
Q_{m,n}^*(x) & P_{m,n}^*(x) \\
Q_{m,n+1}^*(x) & P_{m,n+1}^*(x)
\end{array} \right|.
\]

**Lemma 6.** \( \Delta_{m,n}(x) = 0 \) if and only if \( D_{m,n}(x) = 0 \), that is,
\[
\left( \frac{E_n(x)}{F_n(x)} \right)^m = \frac{B_{m,n}(x)A_{m,n+1}(x^r)}{A_{m,n}(x)B_{m,n+1}(x^r)}.
\]

**Proof.** By definition, \( \Delta_{m,n}(x) = 0 \) if and only if
\[
\left| \begin{array}{cc}
A_{m,n}(x^{r^n}) & B_{m,n}(x^{r^n}) \\
A_{m,n+1}(x^{r^{n+1}})F_n(x)^m & B_{m,n+1}(x^{r^{n+1}})E_n(x)^m
\end{array} \right| = 0,
\]
which is equivalent to
\[
D_{m,n}(x) = \left| \begin{array}{cc}
A_{m,n}(x) & B_{m,n}(x) \\
A_{m,n+1}(x^r)F_n(x)^m & B_{m,n+1}(x^r)E_n(x)^m
\end{array} \right| = 0.
\]
Lemma 7. For each positive integer \( m \), we define \( f_{m,n}(x) \) by

\[
f_{m,n}(x) = 1 - \frac{A_{m,n}(x)}{B_{m,n}(x)} \Phi_{n}(x)^{m}.
\]

Let \( I \) be a positive integer and \( \alpha \) be an algebraic number with \( 0 < |\alpha| < 1 \). Then there exists a positive number \( \eta_{m} < 1 \) such that

\[
0 < |f_{m,n}(\alpha^{r^m})| < \eta_{m}^{ord f_{m,n}(x)}
\]

for every large \( n \) satisfying \( ord f_{m,n}(x) \leq I \).

**Proof.** We may assume \( A_{m,n}(0) = B_{m,n}(0) = 1 \) by (8). Let \( \theta > 1 \) and \( \frac{A_{m,n}(x)}{B_{m,n}(x)} = \sum_{i=0}^{\infty} \tau_{i}^{(m,n)} x^{i} \). Then we obtain \( ||\tau_{i}^{(m,n)}|| \leq (\theta^{2mL})^{i} \) for any \( n \geq n_{0} \) and \( l \geq 0 \). Let \( \Phi_{n}(x)^{m} = \left( \sum_{i=0}^{\infty} \sigma_{i}^{(n)} x^{i} \right)^{m} = \sum_{i=0}^{\infty} \mu_{i}^{(m,n)} x^{i} \), then we have

\[
f_{m,n}(x) = 1 - \left( \sum_{i=0}^{\infty} \tau_{i}^{(m,n)} x^{i} \right) \left( \sum_{i=0}^{\infty} \mu_{i}^{(m,n)} x^{i} \right) = \sum_{i=1}^{\infty} \left( \sum_{s+t=i} \tau_{s}^{(m,n)} \mu_{t}^{(m,n)} \right) x^{i},
\]

where \( \left| \sum_{s+t=i} \tau_{s}^{(m,n)} \mu_{t}^{(m,n)} \right| \leq (\theta^{6mL})^{i} \) and \( \left( \sum_{s+t=i} \tau_{s}^{(m,n)} \mu_{t}^{(m,n)} \right) \leq (\theta^{5mL})^{i} \). We put

\[
f_{m,n}(x) = a_{H} x^{H} + a_{H+1} x^{H+1} + \ldots, \quad a_{H} \neq 0,
\]

where \( 1 \leq H \leq I \). Then

\[
f_{m,n}(\alpha^{r^m}) = a_{H} \alpha^{H r^m} + a_{H+1} \alpha^{H+1} + \ldots.
\]

Since \( ||a_{H}|| \leq (\theta^{6mL})^{H r^m} \), we obtain

\[
\left| \frac{a_{H+1}}{a_{H}} \alpha^{r^m} \right| \leq (\theta^{6mL})^{(1+2[K:Q])I r^n} |\theta^{6mL} \alpha|^{r^m}.
\]

Choosing \( \theta > 1 \) with \( \eta_{m} = (\theta^{6mL})^{(1+2[K:Q])I |\theta^{6mL} \alpha|} < 1 \), we have

\[
0 < |f_{m,n}(\alpha^{r^m})| < 2 |\theta^{6mL} \alpha|^{H r^m} < \eta_{m}^{H r^n}
\]

for sufficiently large \( n \); which implies the lemma.

**Lemma 8.** \( \Phi_{0}(\alpha) \) is algebraic if and only if \( \Phi_{0}(x)^{m} \) is a rational function with coefficients in \( K \) for some positive integer \( m \).

**Proof.** We prove that if \( \Phi_{0}(\alpha) \) is algebraic then there exists a positive integer \( m \) such that \( \Delta_{m,n}(x) = 0 \) for every large \( n \), which implies \( \Phi_{0}(x)^{m} \) is a rational function
by Lemma 6. For every integer $m$, suppose that there exist infinitely many $n$ satisfying $\Delta_{m,n}(x) \neq 0$. Denote by \( \{l(m,n)\}_{n \geq 0} \) the sequence satisfying

$$\Delta_{m,l(m,n)}(x) \neq 0, \quad \Delta_{m,k}(x) = 0$$

for every $n \geq 0$ and every $k$ with $l(m,n) < k < l(m,n+1)$. Then two cases occur:

(i) For every $m$, $l(m,n+1) - l(m,n) \leq C_m$ for some positive constant $C_m$. Then it is clear that the determinant

$$\begin{vmatrix}
Q_{m,l(m,n)}^{*}(x) & P_{m,l(m,n)}^{*}(x) \\
Q_{m,l(m,n+1)}^{*}(x) & P_{m,l(m,n+1)}^{*}(x)
\end{vmatrix} = \begin{vmatrix}
Q_{m,n}(x) & P_{m,n}(x) \\
Q_{m,n+1}(x) & P_{m,n+1}(x)
\end{vmatrix} \neq 0,
$$

namely, the condition (5) in Lemma 4 is satisfied. Hence we can apply Lemma 1 and find that $\Phi_0(\alpha)$ is transcendental.

(ii) For some $m$, $\varlimsup_{n \to \infty} (l(m,n+1) - l(m,n)) = +\infty$. In this case, we have by using Lemma 6

$$\left( \frac{E_k(x)}{F_k(x)} \right)^m = \frac{B_{m,k}(x)A_{m,k+1}(x^r)}{A_{m,k}(x)B_{m,k+1}(x^r)}$$

for every $k$ satisfying $l(m,n) < k < l(m,n+1)$, so that

$$\prod_{k=l(m,n)+1}^{l(m,n+1)-1} \left( \frac{E_k(x^r)}{F_k(x^r)} \right)^m = \frac{B_{m,l(m,n)+1}(x^r(l(m,n+1)))A_{m,l(m,n+1)}(x^r(z)))}{A_{m,l(m,n)+1}(x^r(l(m,n+1)))B_{m,l(m,n+1)}(x^r(z)))}. \quad (9)$$

Let

$$f_{m,l(m,n+1)}(x) = \frac{A_{m,l(m,n+1)}(x)}{B_{m,l(m,n+1)}(x)} \Phi_{l(m,n+1)}(x)^{m} - 1,$$

where we may assume $A_{m,l(m,n+1)}(0) = B_{m,l(m,n+1)}(0) = 1$. Since $\Delta_{m,l(m,n+1)}(x) \neq 0$, we have $D_{m,l(m,n+1)}(x) \neq 0$ by Lemma 6. Therefore by Lemma 5

$$\text{ord} f_{m,l(m,n+1)}(x) \leq \text{ord} (A_{m,l(m,n+1)}(x)\Phi_{l(m,n+1)}(x)^{m} - B_{m,l(m,n+1)}(x))$$

$$\leq r(2mL + 1).$$

Applying Lemma 7, we see that there exists a positive number $\eta_m < 1$ such that

$$0 < |f_{m,l(m,n+1)}(\alpha^r(z)))| < \eta_m^{r/l(m,n+1)} \quad \text{for every large } n.$$  \quad (10)
we get by (9)

$$\begin{align*}
    f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}}) &= B_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}}) \prod_{k=0}^{l(m,n)} \left( \frac{E_{k}(\alpha^{k})}{F_{k}(\alpha^{k})} \right)^{m} \\
    &= \Phi_{0}(\alpha)^{m} - \prod_{k=0}^{l(m,n)} \left( \frac{E_{k}(\alpha^{k})}{F_{k}(\alpha^{k})} \right)^{m} \frac{B_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}})}{A_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}})}.
\end{align*}$$

If $\Phi_{0}(\alpha)^{m}$ is algebraic, then $f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})$ is also algebraic and so there exists a constant $C_{m} > 1$ such that

$$||f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})|| \leq C_{m}^{r^{l(m,n)}}.$$

These inequalities (10) and (11) contradict the fundamental inequality if $n$ is large. Hence $\Phi_{0}(\alpha)$ is transcendental, also in this case. The converse is trivial.

The next lemma together with Lemma 8 implies Theorem 1.

**Lemma 9.** $\Phi_{0}(x)$ is a rational function with coefficients in $K$ if and only if $\Phi_{0}(x)^{m}$ is so for some positive integer $m$.

**Proof.** Suppose that $\Phi_{0}(x)^{m} \in K(x)$ for some integer $m \geq 1$, then $\Phi_{0}(\alpha)$ is algebraic. By Lemma 8 there exists a positive integer $m'$ such that $\Delta_{m',n}(x) = 0$ for every large $n$, that is,

$$\left( \frac{E_{n}(x)}{F_{n}(x)} \right)^{m'} = B_{m',n}(x)A_{m',n+1}(x^{r}) / A_{m,n}(x)B_{m',n+1}(x), \quad n \geq N.$$

Hence we have

$$\Phi_{0}(x)^{mm'} = \left( \prod_{k=0}^{n-1} \frac{E_{k}(x^{r})}{F_{k}(x^{r})} \right)^{mm'} \left( \frac{B_{m',n}(x^{r})}{A_{m',n}(x^{r})} \right)^{m} = \left( \frac{P(x)}{Q(x)} \right)^{m'}, \quad n \geq N$$

for some $P(x), Q(x) \in K[x]$. We can put

$$\frac{B_{m',n}(x)}{A_{m',n}(x)} = C_{n}(x)p_{n}(x)^{m'}, \quad \frac{P(x)}{Q(x)} = R(x)q_{n}(x)^{m},$$

where $p_{n}(x), q_{n}(x) \in K(x)^{\times}$, $p_{n}(0) = 1$ and $C_{n}(x), R(x) \in K[x]$ with orders less than $m'$ and $m$ at each zero, respectively. Since $B_{m',n}(x)/A_{m',n}(x) = 1 + O(x)$, we may assume $C_{n}(0) = 1$. If $\deg C_{n}(x) \geq 1$, there exists an $\alpha \neq 0$ with $C_{n}(\alpha) = 0$. Since $C_{n}(x^{n})^{m} \in R(x)^{m'}(K(x)^{\times})^{mm'}$ and the order of $C_{n}(x^{n})$ at $\alpha^{\frac{1}{m'}}$ is less than $m'$, we see that $\alpha^{\frac{1}{m'}}$ is a root of $R(x)$. This implies $m'n \leq m'\deg R(x)$. Hence $C_{n}(x) = 1$ for every large $n$. Therefore we obtain

$$\frac{B_{m',n}(x)}{A_{m',n}(x)} = \left( \frac{B_{n}(x)}{A_{n}(x)} \right)^{m'}, \quad n \geq M.
for some $A_n(x), B_n(x) \in K[x]$ satisfying $A_n(0) = B_n(0) = 1$, $(A_n(x), B_n(x)) = 1$, and $\deg A_n(x), \deg B_n(x) \leq L$. Then we have

\[
\frac{E_n(x)}{F_n(x)} = \frac{B_n(x)A_{n+1}(x^r)}{A_n(x)B_{n+1}(x^r)}, \quad n \geq M,
\]

(12)

that is, $\Phi_0(x)$ is a rational function with coefficients in $K$. The converse is trivial. Hence the proof is completed.

References


