数理解析研究所講究録
1512 巻 2006 年 118-123

Asymptotic expansions of the non-holomorphic Eisenstein series II

Takumi Noda　野田　工
College of Engineering Nihon University　日本大学　工学部

Abstract

This report is revised version of [8] (2002). We describe one asymptotic formula of the non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ on the critical line by using Airy functions. The t-aspect of the Eisenstein series is not simple contrary to the good behavior of its constant term. We employ uniform expansions of the Bessel function due to Olver derived from the theory of asymptotic solutions of differential equations. Our result is maybe regarded as Voronoi-Atkinson type formula.

1 Eisenstein series

Let $i=\sqrt{-1}$, $s=\sigma+it\in\mathbb{C}$ and H is the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ with weight 0 is defined by

$$E(z,s)=y^s\sum_{\{c,d\}}|cz+d|^{-2s}. \tag{1}$$

Here z is a point of H, s is a complex variable and the summation is taken over $\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$, a complete system of representation of $SL_2(\mathbb{Z})$ over the stabilizer of the point at infinity. The right-hand side of (1) converges absolutely and locally uniformly on $\{(z,s)\,|\,z\in H,\sigma>1\}$, and Eisenstein series has the Fourier expansion:

$$\begin{aligned}\zeta(2s)E(z,s)&=\zeta(2s)y^s+\sqrt{\pi}\zeta(2s-1)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s}\\&+\frac{4\pi^s}{\Gamma(s)}\sqrt{y}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\sigma_{1-2s}(n)K_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx),\end{aligned} \tag{2}$$

where $\zeta(s)$ is the Riemann zeta-function, $\Gamma(s)$ is the Gamma function, $K_\nu(\tau)$ is the modified Bessel function and $\sigma_s(n)$ is the sum of s-th powers of positive divisors of n.

It is well-known that The Fourier expansion (2) gives the holomorphic continuation of this function to the whole s-plane except for the simple pole at $s=1$, and gives the functional equation:

$$\pi^{-s}\Gamma(s)\zeta(2s)E(z,s)=\pi^{-1+s}\Gamma(1-s)\zeta(2-2s)E(z,1-s).$$

We call the first two terms of (2) are the constant term of $E(z,s)$.

Remark 1. It is well-known that the constant term represents the y-aspect of $E(z,s)$ when y tends to ∞. Non-constant terms decay rapidly, because the Bessel function in the Fourier expansion decaies exponentially. Therefore we have the following estimation except on the poles:

$$|E(z,s)| \leqq A_1 y^{\mathrm{Re}(s)} + A_2 y^{1-\mathrm{Re}(s)} \qquad (y \to \infty). \tag{3}$$

In addition, modularity of $E(z,s)$ gives the y-aspect when $y \to 0$. So we see for every positive y,

$$|E(z,s)| \leqq \begin{cases} A_1(y^{-\mathrm{Re}(s)} + y^{\mathrm{Re}(s)}) & (\mathrm{Re}(s) > \frac{1}{2}) \\ A_2(y^{-1+\mathrm{Re}(s)} + y^{1-\mathrm{Re}(s)}) & (\mathrm{Re}(s) \leqq \frac{1}{2}) \end{cases} \tag{4}$$

except on the poles.

The t-aspect of $E(z,s)$ is not simple. **The non-constant terms in (2) are not negligible when $t \to \infty$.** More precisely, the non-constant terms are not exponential decay when the (imaginary) order of the Bessel function is less than or nearly equal to its parameter $2\pi ny$.

Remark 2. Matsumoto [7] (2003) gave asymptotic expansions (respect to z) of the holomorphic Eisenstein series. Asymptotic expansions associated with Epstein zeta-functions and its weighted mean value (as $y \to \infty$) are investigated precisely by Katsurada [6] (2004), which give several new proofs on analytic properties of the Eisenstein series.

2 Airy Functioin

On the calculation of the Fourier coefficients of the automorphic forms, the integral following type plays a fundamental role:

$$a_m(y,s) = \int_{-\infty}^{\infty} \exp(-2\pi imu)(u+iy)^{-k}|u+iy|^{-2s}du.$$

It should be noted that the initial work on this integral is due to Hecke (1927). And the representation of this integral into special functions was originally investigated by Maass (1964) in terms of the Whittaker function. Our case is weight zero, as is well-known, this integral is written by using K-Bessel function.

There are several definitions of the modified Bessel function. One way is using the integral like this:

$$K_\nu(\tau) = \frac{1}{2}\int_0^{\infty} u^{\nu-1} \exp\left(-\frac{1}{2}\tau(u+\frac{1}{u})\right) du.$$

It is also known that the K-Bessel function $K_\nu(\tau)$ satisfies the differential equation called modified Bessel equation:

$$\frac{d^2w}{d\tau^2}+\frac{1}{\tau}\frac{dw}{d\tau}-\left(1+\frac{\nu^2}{\tau^2}\right)w=0.$$

In order to describe the asymptotic expansion of the K-Bessel function, we introduce the **Airy function.** For real variable, the Airy function $\mathrm{Ai}(\tau)$ is defined by the following integral, or by the K-Bessel function with real order one third:

$$\mathrm{Ai}(\tau)=\frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}u^3+\tau u\right)du=\frac{1}{\sqrt{3}\pi}\tau^{\frac{1}{2}}K_{\frac{1}{3}}\left(\frac{2}{3}\tau^{\frac{3}{2}}\right).$$

For $\tau\in\mathbb{C}$ except negative real number, $\mathrm{Ai}(\tau)$ is defined by

$$\mathrm{Ai}(\tau)=\frac{\exp\left(-\frac{2}{3}\tau^{\frac{3}{2}}\right)}{2\pi}\int_0^\infty \exp\left(-\tau^{\frac{1}{2}}u\right)\cos\left(\frac{1}{3}u^{\frac{3}{2}}\right)u^{-\frac{1}{2}}du.$$

In which fractional powers take their principal values.

The differential equation satisfied by $\mathrm{Ai}(\tau)$ is as follows:

$$\frac{d^2w}{d\tau^2}=\tau w.$$

We also introduce some fundamental properties of $\mathrm{Ai}(\tau)$. For positive real variable, $\mathrm{Ai}(\tau)>0$ and its derivative ($\mathrm{Ai}'(\tau)$) is negative, and

$$\mathrm{Ai}(0)=3^{-\frac{2}{3}}\Gamma(\tfrac{2}{3})^{-1}.$$

The Airy function satisfies the identity

$$\mathrm{Ai}(-\tau)=e^{\frac{\pi}{3}i}\mathrm{Ai}(\tau e^{\frac{\pi}{3}i})+e^{-\frac{\pi}{3}i}\mathrm{Ai}(\tau e^{-\frac{\pi}{3}i}),$$

which will be emploied for the negative real axis.

Remark. The Airy function was originally investigated and defined by Sir George Biddell Airy who was English astronomer and geophysicist. (He was the chief of Royal Greenwich Observatry.) Definition was given in 1838 and 1849. The notation $\mathrm{Ai}(\tau)$ are due to H. Jeffreys and J. C. P. Miller around 1940's. (On the Airy function, see [13], Chapters 2, 4 and 11.)

3 Olver's result

The asymptotic expansion of $K_{it}(\tau)$ for the cases $\tau/t\not\to 1$ or $\tau-t=o(\tau^{\frac{1}{3}})$ are obtainable by using saddle-point method. However, in the transitional regions, namely τ/t is nealy equal to 1 while $|\tau - t|$ is large, the investigation becomes much more involved. As an another approach, the theory of asymptotic solutions of differential equations are emploied (see [4], §7.4 and §7.13). Balogh [2], [3] gave one uniform asymptotic expansion of the modified Bessel function by using Airy functions. Balogh's result is based on Olver's works [9]-[12]. The following proposition (Olver [13], Chap.11, p. 425) is the uniform asymptotic expansion of the modified Bessel function of imaginary order, which is crucial in this report.

Proposition 1 *(Balogh [3](1967), Olver [13] p.425) For $t \in \mathbb{R}_{>0}$, $m \geqq 0$ and $u \in \mathbb{C}$ with $|\arg(u)| < \pi$,*

$$K_{it}(tu) = \frac{\pi}{t^{\frac{1}{3}}} \exp\left(-\frac{\pi}{2}t\right) \left(\frac{4\xi}{1-u^2}\right)^{\frac{1}{4}} \left\{ \mathrm{Ai}(-t^{\frac{2}{3}}\xi) \sum_{k=0}^{m} \frac{A_k(\xi)}{t^{2k}} \right. \\ \left. + t^{-\frac{4}{3}} \mathrm{Ai}'(-t^{\frac{2}{3}}\xi) \sum_{k=0}^{m-1} \frac{B_k(\xi)}{t^{2k}} + \varepsilon_{2m+1}(t,\xi) \right\}. \tag{5}$$

The error term is estimated as

$$|\varepsilon_{2m+1}(t,\xi)| \leqq 4v_2 \frac{M_j(-t^{\frac{2}{3}}\xi)}{E_0(-t^{\frac{2}{3}}\xi)} \exp\left\{ \frac{4v_1}{t} V_{-\infty,\xi}(\xi^{\frac{1}{2}}B_0) \right\} \frac{V_{-\infty,\xi}(\xi^{\frac{1}{2}}B_m)}{t^{2m+1}}.$$

Here the path of variation being chosen so that $\mathrm{Im}(\xi^{3/2})$ is monotomic, and the suffix j on M being -1 if $0 \leqq \arg(u) < \pi$ and 1 if $-\pi < \arg(u) \leqq 0$.

Remark. Notations in the above Proposition are defined in [13] (Chap. 11). The variation $V(\xi^{\frac{1}{2}}B_m)$ converges in sutable region, for example, $|\arg(-\xi)| \leqq \frac{2}{3}\pi - \delta$ with any $\delta > 0$ or $\xi \in \mathbb{R}$.

4 Statement of the results

Combining Proposition 1 and the estimation of the sum of the divisor function, we have the following main theorem:

Theorem 1 *Let $z = x + iy \in H$ and $t > 289$. Suppose $t - (4\log t)^{\frac{2}{3}} t^{\frac{1}{3}} \leqq 2\pi y N \leqq t$. Define*

$$\frac{2}{3}\tau_n = t \log \frac{t + \{t^2 - (2\pi y)^2\}^{\frac{1}{2}}}{2\pi y} - \{t^2 - (2\pi n y)^2\}^{\frac{1}{2}}.$$

Then for every $\varepsilon > 0$,

$$E(z, \tfrac{1}{2} + it) \\ = \frac{4\sqrt{2}\pi^{\frac{1}{2}+it} y^{\frac{1}{2}}}{\zeta(1+2it)} \sum_{n=1}^{N} n^{-it} \sigma_{2it}(n) \{t^2 - (2\pi n y)^2\}^{-\frac{1}{4}} \cos(2\pi n x) \tau_n^{\frac{1}{4}} \mathrm{Ai}(-\tau_n^{\frac{2}{3}}) \\ + y^{\frac{1}{2}+it} + y^{\frac{1}{2}-it} e^{i\theta} + O\left(y^{-\frac{3}{2}} t^{-\frac{1}{3}} (\log t)^{\frac{1}{2}+\varepsilon} + y^{-\frac{1}{2}} (\log t)^{\frac{4}{3}+\varepsilon} \log(t/y) \right).$$

Here $e^{i\theta} = \pi^{-2it} \zeta(2it)\Gamma(it) / \overline{\zeta(2it)\Gamma(it)}$.

Corollary 1 *Suppose* $t - t^{\frac{1}{3}+\frac{1}{4}} \leqq 2\pi y M \leqq t$. *For every* $\varepsilon > 0$,

$$\begin{aligned} &E(z, \tfrac{1}{2}+it) \\ &= \frac{4\sqrt{2}\pi^{it}y^{\frac{1}{2}}}{\zeta(1+2it)} \sum_{n=1}^{M} n^{-it}\sigma_{2it}(n)\left\{t^2-(2\pi n y)^2\right\}^{-\frac{1}{4}}\cos(2\pi n x)\cos(\tfrac{2}{3}\tau_n - \tfrac{\pi}{4}) \\ &+ y^{\frac{1}{2}+it} + y^{\frac{1}{2}-it}e^{i\theta} + O\left(y^{-\frac{3}{2}}t^{-\frac{1}{3}}(\log t)^{\frac{1}{2}+\varepsilon} + y^{-\frac{1}{2}}t^{\frac{1}{4}}(\log t)^{\frac{4}{3}+\varepsilon}\log(t/y)\right). \end{aligned}$$

5 Jutila's formula

Our asymptotic formula have some analogy with the Voronoi-Atkinson type formula by Jutila on the square of the Riemann zeta-function. Jutila's formula is expressed as follows:

Theorem (Jutila [5] (1984)) *Let* $t \geqq 12\pi$, δ *be a fixed positive number,* $t^{\delta} \leqq N \leqq t/12\pi$, *and*

$$N' = t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{\frac{1}{2}}.$$

Define

$$f(t,n) = 2t\operatorname{arcsinh}\sqrt{\pi n/2t} + (\pi^2 n^2 + 2\pi n t)^{\frac{1}{2}} + \pi/4.$$

Then,

$$\begin{aligned} &\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 \\ &= 2^{\frac{1}{2}} \sum_{n=1}^{N} (-1)^n d(n) n^{-\frac{1}{2}} \left(\tfrac{1}{4} + \tfrac{t}{2\pi n}\right)^{-\frac{1}{4}} \cos(f(t,n)) \\ &+ 2\sum_{n=1}^{N'} d(n) n^{-\frac{1}{2}} \cos\left(t\log(t/2\pi n) - t - \tfrac{\pi}{4}\right) + O\left(N^{\frac{1}{4}} t^{-\frac{1}{4}} (\log t)^2 + \log t\right). \end{aligned}$$

Remark 1. Jutila's formula is a differentiated version of Atkinson's formula. In the proofs of these formulas, Voronoi's summation formula and the saddle-point method are the main instruments.

Remark 2. We have to make mention of differences between Theorem 1 and Jutila's formula. In the case of square of the Riemann zeta-function, Atkinson type formulas usually have two summations, whereas my Theorem 1 and Corollary 1 consist of one summation $\sum_{1 \leqq n \leqq N}$. This difference is explained by each approximate functional equations;

$$\zeta^2(s) = \sum_{n=1}^{N} d(n) n^{-s} + \pi^{2s-1}\frac{\Gamma^2(\frac{1}{2}-\frac{s}{2})}{\Gamma^2(\frac{s}{2})}\sum_{n=1}^{N'} d(n) n^{s-1} + O(N^{\frac{1}{2}-\sigma}\log t),$$

where $0 \leqq \sigma \leqq 1, NN' = (t/2\pi)^2, N \geqq 1, N' \geqq 1$.

For the case of $E(z,s)$, the Fourier expansion (2) itself may be regarded as one self dual (approximate) functional equation.

References

[1] F. V. Atkinson, *The mean-value of the Riemann zeta-function*, Acta Math. 81, 1949, 353-376.

[2] C. B. Balogh, *Uniform asymptotic expansions of the modified Bessel function of the third kind of large imaginary order*, Bull.Amer. Math. Soc., 72, 1966, 40-43.

[3] C. B. Balogh, *Asymptotic expansions of the modified Bessel function of the third kind of imaginary order*, SIAM-J.-Appl.-Math. 15, 1967, 1315-1323.

[4] A. Erdélyi, et al. *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953.

[5] M. Jutila, *Transformation formulae for Dirichlet polynomials*, J. Number Theory, 18, 1984, 135-156.

[6] M. Katsurada, *Complete asymptotic expansions associated with Epstein zeta-functions*, to appear

[7] K. Matsumoto, *Asymptotic expansions of doble zeta-functions of Barnes, of Shintani, and Eisenstein series*, Nagoya Math. J. 172, 2003, 59-102

[8] T. Noda, *Asymptotic expansions of the non-holomorphic Eisenstein series*, R.I.M.S. kokyuuroku 1314, 2003, 29-32

[9] F. W. J. Olver, *The asymptotic solutions of linear differential equations of the second order for large values of a parameter*, Philos. Trans. Roy. Soc. London, Ser. A, 247, 1954, 307-327.

[10] F. W. J. Olver, *The asymptotic expansions of Bessel functions of large order*, Ibid., 247, 1954, 328-368.

[11] F. W. J. Olver, *Error bounds for first approximations in turning-point problems*, SIAM-J.-Appl.-Math. 11, 1963, 748-772.

[12] F. W. J. Olver, *Error bounds for asymptotic expansions in turning-point problems*, Ibid., 12, 1964, 200-214.

[13] F. W. J. Olver, *Asymptotics and special functions*, A K Peters, 1997. (Academic Press, 1974.)