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SOME MEAN VALUE THEOREMS FOR THE SQUARE OF CLASS NUMBERS TIMES REGULATOR OF QUADRATIC EXTENSIONS

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ABSTRACT. In this article we give a survey of [T1] and [T2] and discuss related topics. Let k be a number field, and Δ_k, h_k and R_k the absolute discriminant, the class number and the regulator, respectively. In [T2] we found the asymptotic behavior of the mean values of $h_F^2R_F^2$ with respect to $|\Delta_F|$ for certain families of quadratic extensions F of a fixed number field k.

1. INTRODUCTION

This article is a survey of [T1] and [T2]. We start with our main result. We fix an algebraic number field k. Let $\mathfrak{M}$, $\mathfrak{M}_\infty$, $\mathfrak{M}_\mathrm{f}$, $\mathfrak{M}_\mathbb{R}$ and $\mathfrak{M}_\mathbb{C}$ denote respectively the set of all places of k, all infinite places, all finite places, all real places and all complex places. For $v \in \mathfrak{M}$ let k_v denotes the completion of k at v and if $v \in \mathfrak{M}_\mathrm{f}$ then let q_v denote the order of the residue field of k_v. We let r_1, r_2, and e_k be respectively the number of real places, the number of complex places, and the number of roots of unity contained in k. We denote by $\zeta_k(s)$ the Dedekind zeta function of k.

To state our result, we classify quadratic extensions of k via the splitting type at places of $\mathfrak{M}_\infty$. Note that if $[F:k]=2$, then $F\otimes k_v$ is isomorphic to either $\mathbb{R}\times\mathbb{R}$ or $\mathbb{C}$ for $v\in\mathfrak{M}_\mathbb{R}$ and is $\mathbb{C}\times\mathbb{C}$ for $v\in\mathfrak{M}_\mathbb{C}$. We fix an $\mathfrak{M}_\infty$-tuples $L_\infty=(L_v)_{v\in\mathfrak{M}_\infty}$ where $L_v\in\{\mathbb{R}\times\mathbb{R},\mathbb{C}\}$ for $v\in\mathfrak{M}_\mathbb{R}$ and $L_v=\mathbb{C}\times\mathbb{C}$ for $v\in\mathfrak{M}_\mathbb{C}$. We define

$$\mathfrak{Q}(L_\infty)=\{F \mid [F:k]=2, F\otimes k_v\cong L_v \text{ for all } v\in\mathfrak{M}_\infty\}.$$

Let $r_1(L_\infty)$ and $r_2(L_\infty)$ be the number of real places and complex places of $F\in\mathfrak{Q}(L_\infty)$, respectively. (This does not depend on the choice of F.) For $v\in\mathfrak{M}_\mathrm{f}$ we put

$$E_v=1-3q_v^{-3}+2q_v^{-4}+q_v^{-5}-q_v^{-6},\quad E_v'=2^{-1}(1-q_v^{-1})^3(1+2q_v^{-1}+4q_v^{-2}+2q_v^{-3}).$$

The following theorem is a special case of [T2, Theorem 10.12].

Theorem 1.1. *We fix an L_∞ and $v_1,\dots,v_n\in\mathfrak{M}_\mathrm{f}$ satisfying $r_2(L_\infty)-2r_2+n\geq 2$. Then the limit*

$$\lim_{X\to\infty}\frac{1}{X^2}\sum_{\substack{F\in\mathfrak{Q}(L_\infty)\\ F:\text{not split at } v_1,\dots,v_n\\ |\Delta_{F/k}|\leq X}} h_F^2R_F^2$$

exists, and the value is equal to

$$\frac{(\mathrm{Res}_{s=1}\zeta_k(s))^3\Delta_k^2e_k^2\zeta_k(2)^2}{2^{r_1+r_2+1}2^{2r_1(L_\infty)}(2\pi)^{2r_2(L_\infty)}}\cdot\prod_{1\leq i\leq n}E_{v_i}'\prod_{\substack{v\in\mathfrak{M}_\mathrm{f}\\ v\neq v_1,\dots,v_n}}E_v.$$

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Density theorems with local conditions at finite places are obtained simultaneously. For details, see [T2, Theorem 10.12].

Combined with the result of Kable-Yukie [KY2], we also obtain the limit of certain correlation coefficients. For simplicity we state in the case $k = \mathbb{Q}$. We state the full version of this theorem in Section 5.

Theorem 1.2. *We fix a prime l satisfying $l \equiv 1(4)$. For any quadratic field $F = \mathbb{Q}(\sqrt{m})$ other than $\mathbb{Q}(\sqrt{l})$, we put $F^* = \mathbb{Q}(\sqrt{ml})$. For a positive number X, we put*

$$\mathcal{A}_l(X) = \left\{ F \;\middle|\; \begin{array}{l} [F : \mathbb{Q}] = 2,\ -X < D_F < 0, \\ F \otimes \mathbb{Q}_l \text{ is the quadratic unramified extension of } \mathbb{Q}_l. \end{array} \right\}.$$

Then we have

$$\lim_{X\to\infty} \frac{\sum_{F\in\mathcal{A}_l(X)} h_F h_{F^*}}{\left(\sum_{F\in\mathcal{A}_l(X)} h_F^2\right)^{1/2} \left(\sum_{F\in\mathcal{A}_l(X)} h_{F^*}^2\right)^{1/2}} = \prod_{\left(\frac{p}{l}\right)=-1} \left(1 - \frac{2p^{-2}}{1+p^{-1}+p^{-2}-2p^{-3}+p^{-5}}\right),$$

where $\left(\frac{p}{l}\right)$ is the Legendre symbol and p runs through all the primes satisfying $\left(\frac{p}{l}\right) = -1$.

From this theorem, for example, we can observe that if we choose l so that $\left(\frac{p}{l}\right) = 1$ for all small primes p then h_F and h_{F^*} have strong relation, and if we choose l so that $\left(\frac{p}{l}\right) = -1$ for all small primes p then the relations between h_F and h_{F^*} become weak.

2. Tauberian theorem

Our approach to prove the theorems above are the use of global zeta functions of prehomogeneous vector spaces. Before giving a sketch of the proof, we briefly recall the Tauberian theorem to clarify the relation between Dirichlet series we consider and the asymptotic formulae in Theorems 1.1 and 1.2. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers. We put

$$a(s) = \sum_{n\geq 1} \frac{a_n}{n^s} \qquad (s \in \mathbb{C}),$$

$$A(X) = \sum_{n\leq X} a_n \qquad (X > 0).$$

Then, roughly speaking, the Tauberian theorem says that we can find some informations of asymptotic behavior of $A(X)$ as $X \to \infty$ from the analytic properties of $a(s)$. The following is a basic type of the Tauberian theorem.

Theorem 2.1. *Assume $a(s)$ is holomorphic for $\Re(s) > a$ except for a pole at $s = a$ of order b. Let $c/(s-a)^b$ be the leading coefficient of the Laurent expansion at $s = a$. Then*

$$\lim_{X\to\infty} \frac{A(X)}{X^a(\log X)^b} = \frac{c}{ab!}.$$

Hence to prove Theorem 1.1 it is enough to investigate the function

$$\sum_{\substack{F\in\mathfrak{Q}(L_\infty) \\ F:\text{not split at } v_1,\ldots,v_n}} \frac{h_F^2 R_F^2}{|\Delta_{F/k}|^s}.$$

On the other side this function is more or less the global zeta function of one specific prehomogeneous vector space, and we can study analytic properties of the zeta functions from the Fourier analysis.

3. Prehomogeneous vector spaces and global zeta functions

We briefly recall the definition of prehomogeneous vector spaces and their applications to number theory. For details, see [SS] or [Y1, Introduction]. For simplicity we here give a definition of a certain restricted class instead of the general. Let k be a field.

Definition 3.1. An irreducible representation of a connected reductive algebraic group (G, V) over k is called a *prehomogeneous vector space* if

(1) there exists a Zariski open G-orbit in V and
(2) there exists a non-constant polynomial $P \in k[V]$ and a rational character χ of G such that $P(gx) = \chi(g)P(x)$ for all $g \in G$ and $x \in V$.

The space of quadratic forms $(\mathrm{GL}(n), \mathrm{Sym}^2 k^n)$ is a classical example. Irreducible prehomogeneous vector spaces over an arbitrary characteristic 0 algebraically closed field were classified by Sato and Kimura in [SK]. Sato and Shintani [SS] defined global zeta functions for prehomogeneous vector spaces if (G, V) is defined over a number field. Let us recall the definition. Let k be a number field and $\mathbb{A}$ the ring of adeles. We denote by $|\cdot|_{\mathbb{A}}$ the idele norm. Let (G, V) be a prehomogeneous vector space defined over k. Let $P \in k[V]$ be of minimum degree satisfying (2) in Definition 3.1 (which is unique up to constant), and χ be the corresponding character. We put

$$T := \ker(G \to \mathrm{GL}(V)), \quad \tilde{G} := G/T, \quad V^{\mathrm{ss}} := \{x \in V \mid P(x) \neq 0\}.$$

Let $\mathscr{S}(V(\mathbb{A}))$ be the space of Schwartz-Bruhat functions on $V(\mathbb{A})$. We fix a Haar measure dg on $\tilde{G}(\mathbb{A})$.

Definition 3.2. For $\Phi \in \mathscr{S}(V(\mathbb{A}))$ and $s \in \mathbb{C}$ we define

$$Z(\Phi, s) := \int_{\tilde{G}(\mathbb{A})/\tilde{G}(k)} |\chi(g)|_{\mathbb{A}} \sum_{x \in V^{\mathrm{ss}}(k)} \Phi(gx)\, dg$$

and call it the *global zeta function*.

Remark 3.3. For $x \in V^{\mathrm{ss}}(k)$ let $\tilde{G}_x = \{g \in \tilde{G} \mid gx = x\}$ and $\tilde{G}_x^\circ$ its identity component. We denote by $\tau(\tilde{G}_x^\circ)$ the unnormalized Tamagawa number of $\tilde{G}_x^\circ$. Roughly speaking **the global zeta function is a counting function of rational orbits $\tilde{G}(k)\backslash V^{\mathrm{ss}}(k)$ with weight $\tau(\tilde{G}_x^\circ)$.** We do not give the details here, but mention that by a standard modification we have

$$Z(\Phi, s) = \sum_{x \in \tilde{G}(k)\backslash V^{\mathrm{ss}}(k)} \frac{\tau(\tilde{G}_x^\circ)}{[\tilde{G}_x(k) : \tilde{G}_x^\circ(k)]} \int_{\tilde{G}(\mathbb{A})/\tilde{G}_x^\circ(\mathbb{A})} |\chi(g_x)|_{\mathbb{A}}^s \Phi(g_x x) dg_x,$$

where dg_x be an appropriate left $\tilde{G}(\mathbb{A})$-invariant measure.

The interpretation of $\tilde{G}(k)\backslash V^{\mathrm{ss}}(k)$ and $\tilde{G}_x^\circ$ in terms of field extensions for prehomogeneous vector spaces is first established systematically in the celebrated work of Wright-Yukie [WY].

The remark above implies that if we know the information of pole structures of $Z(\Phi, s)$, one can obtain the density theorems of $\tilde{G}(k)\backslash V^{\mathrm{ss}}(k)$ with weight $\tau(\tilde{G}_x^\circ)$. On the other side it is in general a very difficult problem to describe the principal parts of $Z(\Phi, s)$ explicitly and is one of the central problem in the theory of prehomogeneous vector spaces.

4. The space of pairs of 2 × 2 matrices and its inner forms

Let $\mathcal{B}$ be a quaternion algebra over k. We denote by $\mathcal{B}^{\mathrm{op}}$ the opposite algebra of $\mathcal{B}$. Let us consider the representation $(G, V) = (G, \rho, V)$ where

$$G = \mathcal{B}^\times \times (\mathcal{B}^{\mathrm{op}})^\times \times \mathrm{GL}(2), \qquad V = \mathcal{B} \otimes k^2 = \mathcal{B} \oplus \mathcal{B}, \tag{4.1}$$

and

$$\rho(g)(a \otimes v) = (g_1 a g_2) \otimes (g_3 v) \quad \text{for} \quad g = (g_1, g_2, g_3) \in G, a \in \mathcal{B}, v \in k^2.$$

We regard this representation as a representation of the algebraic group G over k. If $\mathcal{B} = \mathrm{M}(2,2)$ then V is the space of pairs of 2×2 matrices, and in general (G, V) is an inner form of this split form. This is an example of prehomogeneous vector space, and there is an interesting interpretation of $\tilde{G}(k)\backslash V^{\mathrm{ss}}(k)$ and $\tilde{G}_x^\circ$.

Proposition 4.2. (1) *There exists a non-zero polynomial P of V and a rational character χ on G such that $P(gx) = \chi(g)P(x)$.*
(2) *There exists the canonical bijection between $\tilde{G}(k)\backslash V^{\mathrm{ss}}(k)$ and the set of isomorphism classes of separable quadratic algebras of k those are embeddable into $\mathcal{B}$. For $x \in V^{\mathrm{ss}}(k)$, we denote by $k(x)$ the corresponding algebra.*
(3) *For $x \in V^{\mathrm{ss}}(k)$, $\tilde{G}_x^\circ \cong (k(x)^\times/k^\times)^2$ as an algebraic group over k.*

By (3), $\tau(\tilde{G}_x^\circ)$ is equal to $h_{k(x)}^2 R_{k(x)}^2$ up to a constant multiple. This together with Remark 3.3 implies that the study of this zeta function yields Theorem 1.1.

On the principal parts formula, we proved the following in [T1].

Theorem 4.3. *Let $\mathcal{B}$ be a non-split quaternion algebra. Then*

$$\begin{aligned} Z(\Phi, s) &= Z_+(\Phi, s) + Z_+(\hat{\Phi}, 2-s) \\ &\quad + \tau(G/T)\left(\frac{\hat{\Phi}(0)}{s-2} - \frac{\Phi(0)}{s}\right) + \frac{Z_{\mathcal{B}}(R\hat{\Phi}, 1/2)}{s-3/2} - \frac{Z_{\mathcal{B}}(R\Phi, 1/2)}{s-1/2}. \end{aligned}$$

Here $Z_+(\Phi, s)$ is an integral entire as a function of s, $\tau(G/T)$ is the Tamagawa number of G/T, $R\Phi$ the suitable restriction of Φ to $\mathcal{B}(\mathbb{A})$, $\hat{\Phi}$ a suitable Fourier transform, and $Z_{\mathcal{B}}$ the zeta function of simple algebra associated to $\mathcal{B}$.

This is proved by using the Fourier analysis. The global theory includes rather case by case explicit computation. The k-rank of the group $\tilde{G}$ affects seriously on the complexity of this computation and if $\mathcal{B}$ is non-split then the computation becomes quite mild. This is the reason why we consider non-split forms. For details on this topic see Yukie's treatise [Y1]. We mention that H. Saito's method [Sa2] is an alternative strong tool to establish the global theory in some cases.

5. Correlation coefficients

Since the split form of (4.1)

$$G = \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2), \qquad V = k^2 \otimes k^2 \otimes k^2 \tag{5.1}$$

has a high symmetry, there are many k-forms of this representation. One interesting k-forms is studied by Kable-Yukie in a series of work [KY1, Y2, KY2, KY3, KY4]. We fix a separable quadratic algebra $\tilde{k}$. We denote by $H_2(\tilde{k})$ be the set of binary Hermitian forms over $\tilde{k}$. Let $\sigma : \tilde{k} \to \tilde{k}$ be the non-trivial k-automorphism. Then set theoretically,

$H_2(\tilde{k}) = \{x \in \mathrm{M}(2,2;\tilde{k}) \mid {}^t x^\sigma = x\}$. The group $\mathrm{GL}(2,\tilde{k})$ acts naturally on this space by $(g,x) \mapsto g x {}^t g^\sigma$. Now

$$G = \mathrm{GL}(2,\tilde{k}) \times \mathrm{GL}(2,k), \qquad V = H_2(\tilde{k}) \otimes k^2$$

is an outer form of (5.1). For this case the following proposition is proved in [KY1].

Proposition 5.2. (1) *There exists a non-zero polynomial P of V and a rational character χ on G such that $P(gx) = \chi(g)P(x)$.*

(2) *There exists the canonical bijection between $\tilde{G}(k)\backslash V^{\mathrm{ss}}(k)$ and separable quadratic algebras of k. For $x \in V^{\mathrm{ss}}(k)$ we denote by $k(x)$ the corresponding algebra.*

(3) *For $x \in V^{\mathrm{ss}}(k)$, $\tilde{G}_x^\circ \cong (\tilde{k} \otimes k(x))^\times / \tilde{k}^\times$ as an algebraic group over k.*

Let $\tilde{k}$ and $k(x)$ be different quadratic extensions of k. Then the biquadratic field $\tilde{k} \otimes k(x)$ contains another quadratic extension of k. We denote this field by $k(x)^*$. Then by (3) the Tamagawa number of $\tilde{G}_x^\circ$ is more or less the product $h_{k(x)}R_{k(x)} \times h_{k(x)^*}R_{k(x)^*}$.

We will state the full version of [T2, Section 11]. Let S denote a finite set of places of k containing $\mathfrak{M}_\infty$ and $L_S = (L_v)_{v\in S}$ an S-tuple of separable quadratic algebra L_v of k_v. Let F be a quadratic extension of k. We write $F \approx L_S$ to mean that $F \otimes k_v \cong L_v$ for all $v \in S$. Let X be a positive number. For convenience, we introduce the abbreviation

$$\mathfrak{Q}(L_S, X) = \left\{F \mid [F:k] = 2, F \approx L_S, \mathcal{N}(\Delta_{F/k}) \le X\right\}.$$

Here, $\mathcal{N}(\Delta_{F/k})$ is the ideal norm of the relative discriminant $\Delta_{F/k}$ of F/k.

Definition 5.3. We define

$$\mathrm{Cor}(L_S) = \lim_{X\to\infty} \frac{\sum_{F\in\mathfrak{Q}(L_S,X)} h_F R_F h_{F^*} R_{F^*}}{\left(\sum_{F\in\mathfrak{Q}(L_S,X)} h_F^2 R_F^2\right)^{1/2} \left(\sum_{F\in\mathfrak{Q}(L_S,X)} h_{F^*}^2 R_{F^*}^2\right)^{1/2}}$$

if the limit of the right hand side exists and call it the *correlation coefficient.*

The asymptotic behavior of the numerator in the right hand side as $X \to \infty$ was investigated in [KY2, KY3, KY4], while the denominator is considered in [T2]. Hence we could find the correlation coefficients for certain types of $\tilde{k}$ and L_S. Let $\mathfrak{M}_{\mathrm{rm}}$, $\mathfrak{M}_{\mathrm{in}}$ and $\mathfrak{M}_{\mathrm{sp}}$ be the sets of finite places of k which are respectively ramified, inert and split on extension to $\tilde{k}$. For $v \in \mathfrak{M}$ and a separable quadratic algebra L_v, we define the separable quadratic algebra L_v^* as follows. Let $\tilde{k}_v = \tilde{k} \otimes k_v$. If $\tilde{k}_v \cong k_v \times k_v$ then we define $L_v^* = L_v$ for any L_v. In the case $\tilde{k}_v$ is a field, if $L_v = k_v \times k_v$ then we let $L_v^* \cong \tilde{k}_v$, and if $L_v \cong \tilde{k}_v$ then we let $L_v^* = k_v \times k_v$. Finally in the case $\tilde{k}_v$ and L_v are distinct fields, we define L_v^* the same way as we defined F^* for number fields. Let $\mathfrak{M}_{\mathrm{dy}} = \{v \in \mathfrak{M}_{\mathrm{f}} \mid v \mid 2\}$. We proved the following in [T2, Section 11].

Theorem 5.4. *Assume $\mathfrak{M}_{\mathrm{rm}} \cap \mathfrak{M}_{\mathrm{dy}} = \emptyset$ and $S \supset \mathfrak{M}_{\mathrm{rm}}$. Let $L_S = (L_v)_{v\in S}$ is an S-tuple of separable quadratic algebras such that there are at least two places v with L_v are fields. Further assume that there are at least two places v with L_v^* are fields. Then we have*

$$\mathrm{Cor}(L_S) = \prod_{v\in\mathfrak{M}_{\mathrm{in}}\setminus S} \left(1 - \frac{2q_v^{-2}}{1 + q_v^{-1} + q_v^{-2} - 2q_v^{-3} + q_v^{-5}}\right).$$

6. Further problems

In the invaluable work [WY], Wright and Yukie found good interpretations of rational orbits for 8 cases including our case (5.1), and discussed the expected density theorems for those cases. On the other hand, in the process [T1] and [T2] to prove Theorem 1.1, the technical heart is to consider the inner form to handle the global theory. The k-forms of irreducible reduced regular prehomogeneous vector spaces over local and global fields are classified by H. Saito [Sa1], and we could see that some other cases treated in [WY] have inner forms. In this section, we will discuss the rational orbit decomposition for some inner form representations. The proof may be appear in a forthcoming paper. Let k be an arbitrary field. Let $\mathcal{E}_i$ be the set of isomorphism classes of separable algebras of k of degree i.

(I) The case $(\mathrm{GL}(3) \times \mathrm{GL}(3) \times \mathrm{GL}(2), k^3 \otimes k^3 \otimes k^2)$.
Let $\mathcal{D}$ be a simple algebra of degree 3 over k. Then

$$G = \mathcal{D}^\times \times (\mathcal{D}^{\mathrm{op}})^\times \times \mathrm{GL}(2), \qquad V = \mathcal{D} \otimes k^2$$

is an inner form. Let $\mathcal{E}_3(\mathcal{D})$ be the set of isomorphism classes of separable cubic algebras of k those are embeddable into $\mathcal{D}$. Then the following proposition holds.

Proposition 6.1. (1) *There exists a non-zero polynomial P of V and a rational character χ on G such that $P(gx) = \chi(g)P(x)$.*
(2) *Let $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$. Then there exists the canonical bijection between $G(k)\backslash V^{\mathrm{ss}}(k)$ and $\mathcal{E}_3(\mathcal{D})$. For $x \in V^{\mathrm{ss}}(k)$ we denote by $k(x) \in \mathcal{E}_3(\mathcal{D})$ be the corresponding algebra.*
(3) *For $x \in V^{\mathrm{ss}}(k)$, $G_x^\circ \cong k(x)^\times \times k^\times$ as an algebraic group over k.*

From this proposition, we may obtain the density of $h_F R_F$ of cubic extensions F of k. In the case $\mathcal{D}$ is not split, the principal parts of the global zeta function were described in [T1]. It has possible simple pole at $s = 0, 1/6, 4/3, 3/2$ and holomorphic elsewhere. The local theory and the filtering process necessary to obtain the density theorem are in progress.

(II) The case $(\mathrm{GL}(4) \times \mathrm{GL}(2), \wedge^2 k^4 \otimes k^2)$.
Let $\mathcal{B}$ be a quaternion algebra of k. We denote by $H_2(\mathcal{B})$ be the set of binary Hermitian forms over $\mathcal{B}$. Then

$$G = \mathrm{GL}(2, \mathcal{B}) \times \mathrm{GL}(2), \qquad V = H_2(\mathcal{B}) \otimes k^2$$

is an inner form. For this case the following proposition holds.

Proposition 6.2. (1) *There exists a non-zero polynomial P of V and a rational character χ on G such that $P(gx) = \chi(g)P(x)$.*
(2) *Let $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$. Then there exists the canonical bijection between $G(k)\backslash V^{\mathrm{ss}}(k)$ and $\mathcal{E}_2$. For $x \in V^{\mathrm{ss}}(k)$ we denote by $k(x) \in \mathcal{E}_2$ be the corresponding algebra.*
(3) *For $x \in V^{\mathrm{ss}}(k)$, $G_x^\circ \cong (\mathcal{B} \otimes k(x))^\times$ as an algebraic group over k.*

(III) The case $(\mathrm{GL}(6) \times \mathrm{GL}(2), \wedge^2 k^6 \otimes k^2)$.
Let $H_3(\mathcal{B})$ be the set of ternary Hermitian forms over $\mathcal{B}$. Then just the same as the above case,

$$G = \mathrm{GL}(3, \mathcal{B}) \times \mathrm{GL}(2), \qquad V = H_3(\mathcal{B}) \otimes k^2$$

is an inner form. For this case the following proposition holds.

Proposition 6.3. (1) *There exists a non-zero polynomial P of V and a rational character χ on G such that $P(gx) = \chi(g)P(x)$.*

(2) *Let $V^{ss} = \{x \in V \mid P(x) \neq 0\}$. Then there exists the canonical bijection between $G(k)\backslash V^{ss}(k)$ and $\mathcal{E}_3$. For $x \in V^{ss}(k)$ we denote by $k(x) \in \mathcal{E}_3$ be the corresponding algebra.*

(3) *For $x \in V^{ss}(k)$, $G_x^\circ \cong \{g \in (\mathcal{B} \otimes k(x))^\times \mid \mathcal{N}(g) \in k^\times\}$ as an algebraic group over k.*

The principal parts of the global zeta function for (II) and (III) are not known.

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