数理解析研究所講究録
1517巻 2006年 60-68



# THE ABELIANIZATION OF THE CONGRUENCE IA-AUTOMORPHISM GROUP OF A FREE GROUP

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*Dedicated to Professor Kojun Abe on the occation of his 60th birthday*

ABSTRACT. In this paper we consider the abelianizations of some normal subgroups of the automorphism group of a finitely generated free group. Let $F_n$ be a free group of rank $n$. For $d \geq 2$, we consider a group consisting the automorphisms of $F_n$ which act trivially on the first homology group of $F_n$ with $\mathbf{Z}/d\mathbf{Z}$-coefficients. We call it the congruence IA-automorphism group of level $d$ and denote it by $IA_{n,d}$. Let $IO_{n,d}$ be the qutient group of the congruence IA-automorphism group of level $d$ by the inner automorphism group of a free group. In this paper we determine the abelianization of $IA_{n,d}$ and $IO_{n,d}$ for $n \geq 2$ and $d \geq 2$. Furthermore, for $n = 2$ and odd prime $p$, we compute the integral homology groups of $IA_{2,p}$ for any dimension.

## 1. INTRODUCTION

Let $F_n$ be a free group of rank $n$, and $\mathrm{Aut}\, F_n$ the automorphism group of the free group $F_n$. We denote the abelianization of $F_n$ by $H$. The abelianization homomorphism $F_n \to H$ induces a surjective homomorphism $\rho : \mathrm{Aut}\, F_n \to GL(n, \mathbf{Z})$. In this paper we consider the abelianization of the preimages of the congruence subgroups of $GL(n, \mathbf{Z})$ by $\rho$. For $n \geq 2$ and $d \geq 2$, let $GL(n, d)$ be the general linear group over $\mathbf{Z}/d\mathbf{Z}$, and $\pi_d : GL(n, \mathbf{Z}) \to GL(n, d)$ the natural homomorphism induced by the mod reduction $d$. We call the kernel $\Gamma(n, d)$ of $\pi_d$ the congruence subgroup of $GL(n, d)$ of level $d$. Classically, the congruence subgroups $\Gamma(n, d)$ have been studied by many authors, and there is a broad range remarkable results of them. In particular, it is well known that for $n \geq 3$ and odd prime integer $p$, Lee and Szczarba [10, Theorem 1.1] determined the structure of the abelianization of the congruence subgroup $\Gamma(n, p)$. More precisely, they showed that it is isomorphic to the Lie algebra $\mathfrak{sl}_n(\mathbf{F}_p)$ of trace-zero matrices over $\mathbf{F}_p$ as an $SL(n, \mathbf{F}_p)$-module where $\mathbf{F}_p$ is the finite field of order $p$.

2000 *Mathematics Subject Classification.* 20F28, 20F12, 20F14, 20F40, 16W25(Primary), 20F38, 57M05(Secondly).
*Key words and phrases.* the automorphism group of a free group, the Johnson homomorphism, Morita's trace.

In this paper we are also interested in the kernel $IA_n$ of the natural map $\rho$. We call it the IA-automorphism group of $F_n$. Then we have an exact sequence

$$1 \to IA_n \to \mathrm{Aut}\, F_n \xrightarrow{\rho} GL(n, \mathbf{Z}) \to 1.$$

This exact sequence plays important roles in the study of $\mathrm{Aut}\, F_n$. Although Magnus [11] obtained a finitely many generating set of $IA_n$ for $n \geq 3$, (See Subsection 2.1.) it is not known whether $IA_n$ is finitely presented or not for $n \geq 4$. We remark that Krstić and McCool [9] showed that $IA_3$ is not finitely presentable.

For a group $G$, we denote the abelianization of $G$ by $G^{\mathrm{ab}}$. Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [8] independently showed that the abelianization $IA_n^{\mathrm{ab}}$ of $IA_n$ is a free abelian group of rank $n^2(n-1)/2$. More precisely, it is isomorphic to $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ as a $GL(n, \mathbf{Z})$-module where $H^*$ is the dual group $\mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ of $H$. We remark that the $GL(n, \mathbf{Z})$-module structure of $IA_n^{\mathrm{ab}}$ is determined by using the first Johnson homomorphism of $\mathrm{Aut}\, F_n$.

Here we consider subgroups of $\mathrm{Aut}\, F_n$ which corresponds to the congruence subgroup of $GL(n, \mathbf{Z})$. Let $IA_{n,d}$ be the kernel of $\pi_d \circ \rho : \mathrm{Aut}\, F_n \to GL(n, d)$, and call it the congruence IA-automorphism group of a free group $F_n$ of level $d$. The first aim of this paper is to determine the structure of the abelianization of $IA_{n,d}$. Then our first result is

**Theorem 1.1.** *For $n \geq 2$ and $d \geq 2$,*

$$IA_{n,d}^{\mathrm{ab}} \simeq (IA_n^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) \bigoplus \Gamma(n, d)^{\mathrm{ab}}.$$

In Section 3, we prove this theorem using the "extended" Johnson homomorphism, introduced by Kawazumi [8], constructed from the $\mathbf{Z}/d\mathbf{Z}$-valued Magnus expansion of $\mathrm{Aut}\, F_n$. Considering the result of Lee and Szczarba [10] stated above, we see that for any odd prime $p$, the abelianization of $IA_n$ is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n-1)(n^2+2n+2)}$ as an abelian group.

Next we consider the outer automorphism group of a free group and the images of $IA_n$ and $IA_{n,d}$ by a natural projection. An automorphism $\iota$ of $F_n$ is called an inner automorphism of $F_n$ if there exists some element $y \in F_n$ such that $x^\iota = y^{-1}xy$ for any $x \in F_n$. Then the group $\mathrm{Inn}\, F_n$ of inner automorphisms of $F_n$ is a normal subgroup of $\mathrm{Aut}\, F_n$. Let $\mathrm{Out}\, F_n$ be the quotient group $\mathrm{Aut}\, F_n/\mathrm{Inn}\, F_n$. The groups $\mathrm{Inn}\, F_n$ and $\mathrm{Out}\, F_n$ are called the inner automorphism group and the outer automorphism group of the free group respectively. We define a group $IO_n$ to be the quotient group of $IA_n$ by $\mathrm{Inn}\, F_n$. The group $IO_n$ is the kernel of the natural map $\bar{\rho} : \mathrm{Out}\, F_n \to GL(n, \mathbf{Z})$ induced by $\rho$. It is also known that the abelianization $IO_n^{\mathrm{ab}}$ of $IO_n$ is given by $IO_n^{\mathrm{ab}} \simeq (H^* \otimes_{\mathbf{Z}} \Lambda^2 H)/H$. (See [8, Theorem 6.2].) For $n \geq 2$ and $d \geq 2$, we define $IO_{n,d}$ to be the quotient group of $IA_{n,d}$ by $\mathrm{Inn}\, F_n$. The group $IO_{n,d}$ is the kernel of $\pi_d \circ \bar{\rho}$. The second aim of this paper is to determine the structure of the abelianization $IO_{n,d}^{\mathrm{ab}}$. The result is

**Theorem 1.2.** *For $n \geq 2$ and $d \geq 2$,*

$$IO_{n,d}^{\mathrm{ab}} \simeq (IO_n^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) \bigoplus \Gamma(n,d)^{\mathrm{ab}}.$$

Finally, in Section 5, we compute the integral homology groups of $IA_{2,p}$ for an odd prime $p$. In general, to compute the integral homology groups of $IA_{n,d}$ is quite difficult as well as that of $IA_n$. In the case where $n = 2$ and $d = p$, we can compute those as follows:

**Theorem 1.3.** *For any prime $p$,*

$$H_q(IA_{2,p}, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } q = 0, \\ \mathbf{Z}^{\oplus \alpha(p)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2} & \text{if } q = 1, \\ \mathbf{Z}^{\oplus(2\alpha(p)-2)} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3 \end{cases}$$

*where $\alpha(p) = 1 + \frac{(p-1)p(p+1)}{12}$ is the rank of $\Gamma(2,p)$ as a free group.*

We remark that $IO_{2,p}$ is isomrphic to the congruence subgroup $\Gamma(2,p)$ since $IA_2 = \mathrm{Inn}\, F_2$ due to Nielsen [12], and hence, it is a free group of rank $\alpha(p)$.

## 2. Preliminaries

In this section we review the IA-automorphism group of a free group and the first Johnson homomorphism of the automorphism group of a free group. Throughout this paper we use the following notation and conventions.

- The group $\mathrm{Aut}\, F_n$ acts on $F_n$ from the right.
- For any $\sigma \in \mathrm{Aut}\, F_n$ and $x \in F_n$, the action of $\sigma$ on $x$ is denoted by $x^\sigma$.
- For elements $x$ and $y$ of a group, the commutator bracket $[x, y]$ of $x$ and $y$ is defined to be $[x, y] := xyx^{-1}y^{-1}$.

### 2.1. The IA-automorphism group.

In this subsection, we prepare generators of $IA_n$, and some basic exact sequences which is required to prove our main theorems.

Let $F_n$ be a free group on $\{x_1, \dots, x_n\}$. Magnus [11] showed that $IA_n$ is finitely generated by automorphisms

$$K_{ij} : \begin{cases} x_i & \mapsto x_j{}^{-1} x_i x_j, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct $i, j \in \{1, 2, \dots, n\}$ and

$$K_{klm} : \begin{cases} x_k & \mapsto x_k x_l x_m x_l{}^{-1} x_m{}^{-1}, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq k) \end{cases}$$

for distinct $k, l, m \in \{1, 2, \dots, n\}$ such that $l < m$. Since $IO_n$ is the qutient group $IA_n / \mathrm{Inn}\, F_n$, $IO_n$ is also generated by (the coset classes of) automorphisms $K_{ij}$ and $K_{ijk}$.

Next we give some basic exact sequences. Since the natural maps $\rho$ and $\bar{\rho}$ are surjective, for any $n \geq 2$ and $d \geq 2$, we have exact sequences

(1) $$1 \to IA_n \to IA_{n,d} \xrightarrow{\rho} \Gamma(n, d) \to 1$$

and

(2) $$1 \to IO_n \to IO_{n,d} \to \Gamma(n, d) \to 1$$

respectively. Furthermore, by definition, we have

(3) $$1 \to \mathrm{Inn}\, F_n \to IA_{n,d} \to IO_{n,d} \to 1.$$

These exact sequences are used in later sections.

### 2.2. The first Johnson homomorphism.

In this subsection, we review the first Johnson homomorphism of the automorphism group of a free group. For each $k \geq 1$, let $\Gamma_n(k)$ be the $k$-th subgroup of the lower central series of $F_n$ defined by

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 1.$$

We denote the graded quotients $\Gamma_n(k)/\Gamma_n(k+1)$ by $\mathcal{L}_n(k)$. Set $\mathcal{L}_n = \bigoplus_{k \geq 1} \mathcal{L}_n(k)$. Then it is well known that $\mathcal{L}_n$ naturally has a structure of a graded Lie algebra over $\mathbf{Z}$ induced from the commtator bracket on $F_n$ and it is naturally isomorphic to a graded free Lie algebra over $H$. In particular, we have $\mathcal{L}_n(1) = H$ and $\mathcal{L}_n(2) = \Lambda^2 H$. (For details, see [14].)

In this paper, for any $x \in \Gamma_n(k)$, we also denote by $x$ the coset class of $x$ in $\mathcal{L}_n(k)$.

For each $k \geq 1$, let

$$\tau' : IA_n \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(2)) = H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

be the homomorphism defined by

$$\sigma \mapsto (x \mapsto x^{-1} x^{\sigma}).$$

for $\sigma \in IA_n$ and $x \in H$. The map $\tau'$ naturally induces a homomorphism

$$\tau : IA_n^{\mathrm{ab}} \to H^* \otimes_{\mathbf{Z}} \Lambda^2 H.$$

These homomorphisms $\tau'$ and $\tau$ are called the first Johnson homomorphisms of $\mathrm{Aut}\, F_n$. The map $\tau$ is a $GL(n, \mathbf{Z})$-equivariant isomorphism. In particular, $IA_n^{\mathrm{ab}}$ is a free abelian group of rank $\frac{1}{2}n^2(n-1)$. (See [8, Theorem 6.1].)

Next, we consider $IO_n^{\mathrm{ab}}$. For any $y \in F_n$, we denote by $\iota_y$ the inner automorphism of $F_n$ such that $x^{\iota_y} = y^{-1} x y$ for any $x \in F_n$. Considering a natural isomorphim $\mathrm{Inn}\, F_n \to F_n$; $\iota \mapsto y$, we often identify $\mathrm{Inn}\, F_n$ with $F_n$. Then we have $(\mathrm{Inn}\, F_n)^{\mathrm{ab}} = H$. It is also known that the induced homomorphism $(\mathrm{Inn}\, F_n)^{\mathrm{ab}} = H \to H^* \otimes_{\mathbf{Z}} \Lambda^2 H = IA_n^{\mathrm{ab}}$ from the inclusion map $\mathrm{Inn}\, F_n \hookrightarrow IA_n$ is injective, and whose image, which we identify with $H$ by this map, is a direct summand of $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ as a $\mathbf{Z}$-module. Hence we see $IO_n^{\mathrm{ab}}$ is isomorphic to a free abelian group $(H^* \otimes_{\mathbf{Z}} \Lambda^2 H)/H$ of rank $\frac{1}{2}n(n+1)(n-2)$. For details, see [8, Theorem 6.2].

Now, the first Johnson homomorphism $\tau'$ induces a homomorphism

$$\tau'_d : IA_n \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

Finally, we recall that $\tau'_d$ is extended to a homomorphism from $IA_{n,d}$ to $(H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$. For any $\mathbf{Z}/d\mathbf{Z}$-valued Magnus expansion $\theta$, Kawazumi [8] constructed a crossed homomorphism

$$\tau^{\theta} : \mathrm{Aut}\, F_n \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$$

and showed that if we restrict it to $IA_{n,d}$, then the map $\tau^{\theta}$ is a homomorpism. Furthermore he also show that $\tau^{\theta} \equiv \tau'_d$ on $IA_n$. Especially the restriction $\tau^{\theta}|_{IA_n}$ is independent of the choice of the Magnus expansion $\theta$. For details, see [8, Theorem 3.1].

## 3. The abelianization of $IA_{n,d}$.

In this section we give a proof of Theorem 1.1. First, we see that since the first Johnson homomorphism $\tau$ is a $GL(n,\mathbf{Z})$-equvariant isomorphism, $\tau$ induces a surjective homomorphism

$$\tilde{\tau} : H_0(\Gamma(n,d), IA_n^{\mathrm{ab}}) \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

To show $\tilde{\tau}$ is an isomorphism, we use

**Lemma 3.1.** *For $n \geq 2$ and $d \geq 2$, we have*

$$d\,[K_{ij}] = 0 \quad \text{and} \quad d\,[K_{klm}] = 0$$

*in $H_0(\Gamma(n,d), IA_n^{\mathrm{ab}})$.*

Then considering the homological five term exact sequence

$$\begin{aligned} H_2(IA_n, \mathbf{Z}) \to H_2(IA_{n,d}, \mathbf{Z}) \to H_0(\Gamma(n,d), IA_n^{\mathrm{ab}}) \\ \xrightarrow{\eta} IA_{n,d}^{\mathrm{ab}} \to \Gamma(n,d)^{\mathrm{ab}} \to 0. \end{aligned}$$

of (1), and a homomorphism

$$\tau^{\theta} : IA_{n,d} \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$$

induced from a Magnus expansion $\theta$, we see $\eta$ is injective. Hence we have a split exact sequence

$$0 \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \xrightarrow{\eta} IA_n^{\mathrm{ab}} \to \Gamma(n,d)^{\mathrm{ab}} \to 0.$$

This completes the proof of Theorem 1.1.

Here we consider the case where $d$ equals to an odd prime integer $p$. For $n \geq 3$, Lee and Szczarba [10] showed that the abelianization $\Gamma(n,p)^{\mathrm{ab}}$ of the congruence subgroup $\Gamma(n,p)$ is a $\mathbf{Z}/p\mathbf{Z}$-vector space of dimension $n^2-1$. For $n=2$, Frasch [5] showed that the congruence subgroup $\Gamma(2,p)$ is a free group of rank $\alpha(p) := 1 + \frac{(p-1)p(p+1)}{12}$. Furthermore Nielsen [12] showed that $IA_2 = \mathrm{Inn}\, F_2$. Hence we have

**Corollary 3.1.**

$$IA_{n,p}^{\mathrm{ab}} = \begin{cases} \mathbf{Z}^{\oplus \alpha(p)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2} & \text{if} \quad n = 2, \\ (\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n-1)(n^2+2n+2)} & \text{if} \quad n \geq 3. \end{cases}$$

## 4. THE ABELIANIZATION OF $IO_{n,d}$.

In this section we give a proof of Theorem 1.2. Considering the homological five term exact sequence of (3), we have

$$H_2(IA_{n,d}, \mathbf{Z}) \to H_2(IO_{n,d}, \mathbf{Z}) \to H_0(IO_{n,d}, (\operatorname{Inn} F_n)^{\mathrm{ab}}) \xrightarrow{\delta} IA^{\mathrm{ab}}_{n,d} \to IO^{\mathrm{ab}}_{n,d} \to 0.$$

Since $H$ and $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ is free abelian groups, the injective homomorphism

$$H = (\operatorname{Inn} F_n)^{\mathrm{ab}} \hookrightarrow IA^{\mathrm{ab}}_n = H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

induces an injective homomorphism

$$\psi_d : H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \hookrightarrow (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

For each $i$, $1 \leq i \leq n$, set $\iota_i := \iota_{x_i}$. Then, $\operatorname{Inn} F_n$ is a free group on $\{\iota_1, \dots, \iota_n\}$.

**Lemma 4.1.** *For $n \geq 2$ and $d \geq 2$,*

$$d\,[\iota_i] = 0, \quad 1 \leq i \leq n$$

*in $H_0(IO_{n,d}, (\operatorname{Inn} F_n)^{\mathrm{ab}})$.*

Hence there exists an isomorphism

$$\bar{\xi} : H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \to H_0(IO_{n,d}, (\operatorname{Inn} F_n)^{\mathrm{ab}})$$

such that $\psi_d = \delta \circ \bar{\xi}$, and we have a short exact sequence

$$0 \to H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \xrightarrow{\delta} IA^{\mathrm{ab}}_{n,d} \to IO^{\mathrm{ab}}_{n,d} \to 0,$$

and hence

$$\begin{aligned} ((H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) \,/\, (H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) & \\ \simeq ((H^* \otimes_{\mathbf{Z}} \Lambda^2 H)/H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}, & \\ \simeq IO^{\mathrm{ab}}_n \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}. & \end{aligned}$$

This completes the proof of Theorem 1.2.

For $n \geq 2$ and an odd prime $p$, by an argument similar to that in Corollary 3.1, we obtain

**Corollary 4.1.**

$$IO^{\mathrm{ab}}_{n,p} = \begin{cases} \mathbf{Z}^{\oplus \alpha(p)} & \text{if} \quad n = 2, \\ (\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n+1)(n^2-2)} & \text{if} \quad n \geq 3. \end{cases}$$

## 5. The integral homology groups of $IA_{2,p}$

In this section, we compute the integral homology groups of $IA_{2,p}$ for any odd prime $p$. Since the groups $IA_2$ and $\Gamma(2,p)$ are free groups stated above, considering the homological Lyndon-Hochscild-Serre spectral sequence of (1) for $n = 2$ and $d = p$, we see the homological dimension of $IA_{2,p}$ is 2. On the other hand, since the first homology group $H_1(IA_{2,p}, \mathbf{Z})$ is obtained in Section 3, it suffices to compute the second homology group $H_2(IA_{2,p}, \mathbf{Z})$. Our result is

**Theorem 5.1.** *For any odd prime p,*

$$H_2(IA_{2,p}, \mathbf{Z}) = \mathbf{Z}^{\oplus(2\alpha(p)-2)}$$

*where* $\alpha(p) = 1 + \frac{(p-1)p(p+1)}{12}$.

To prove this theorem, first, we directly compute the second cohomology groups of $IA_{2,p}$. Then, using the universal coefficients theorem, we obtain the second homology group of $IA_{2,p}$.

**Proposition 5.1.** *For any odd prime integer p, we have*

$$H^2(IA_{2,p}, \mathbf{Z}) = \mathbf{Z}^{\oplus(2\alpha(p)-2)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2}.$$

Similarly, we obtain

**Proposition 5.2.** *For any odd prime integer p, we have*

$$H^2(IA_{2,p}, \mathbf{Z}/q\mathbf{Z}) \simeq \begin{cases} (\mathbf{Z}/q\mathbf{Z})^{\oplus(2\alpha(p)-2)} & \text{if } (q,p) = 1, \\ (\mathbf{Z}/q\mathbf{Z})^{\oplus(2\alpha(p)-2)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2} & \text{if } q = p^e. \end{cases}$$

Using Propositions 5.1 and 5.2, we obtain the second homology group $H_2(IA_{2,p}, \mathbf{Z})$ by the universal coefficients theorem.

## 6. Acknowledgments

The author would like to express his sincere gratitude to Professor Nariya Kawazumi for his valuable advice and especially access to his unpublished work. He is also grateful to Professor Fred Cohen and Professor Benson Farb for explaining their unpublished works for the abelianizaton of the IA-automorphism group of a free group. Finally he would like to express his thanks to the referee for helpful comments and suggestions. The author is supported by JSPS Research Fellowships for Young Scientists.

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