<table>
<thead>
<tr>
<th>Title</th>
<th>Decomposition of Link Complements (Methods of Transformation Group Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yamasaki, Masayuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1517: 20-24</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58720">http://hdl.handle.net/2433/58720</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Decomposition of Link Complements

岡山理科大学・理学部　山崎　正之 (Masayuki Yamasaki)
Faculty of Science, Okayama University of Science

1. Introduction

Suppose $K$ is a knot in $S^3$, and $E(K)$ denotes the exterior of $K$. Define a 4-manifold $M(K)$ to be $\partial(E(K) \times D^2)$. This 4-manifold has the same fundamental group as $E(K)$, but it is not aspherical. In a talk at the RIMS Conference "Methods of Transformation Group Theory", May 2006, I announced that the TOP surgery obstruction theory works for normal maps to $M(K)$. Later I extended the result to the cases of non-split links and non-split subcomplexes of a triangulation. Actually if $X$ is a connected compact orientable 3-manifold with nonempty boundary such that the assembly map $A : H_4(X; \mathbb{L}) \to L_4(\pi_1(X))$ is injective, then we have the same conclusion for $M = \partial(X \times D^2)$.

Then I learned from Jim Davis that, if the 3-manifold $X$ is aspherical, the following theorem of Qayum Khan [3] can be applied to these examples to show that the surgery obstruction theory works even in the $PL = DIFF$ category for normal maps to $M$:

**Theorem.** (Khan) Suppose $M$ is a closed connected orientable PL 4-manifold with fundamental group $\pi$ such that the assembly map

$$A : H_4(\pi; \mathbb{L}) \to L_4(\pi)$$

is injective, or more generally, the 2-dimensional component of its prime 2 localization

$$\kappa_2 : H_2(\pi; \mathbb{Z}_2) \to L_4(\pi)$$

is injective. Then any degree 1 normal map $(f, b) : N \to M$ with vanishing surgery obstruction in $L_4(\pi)$ is normally bordant to a homotopy equivalence $M \to M$.

So I decided to change the statement. Let $X$ be as above. $X$ has a handle decomposition, and a handle decomposition produces a CW-spine $B$ of $X$: $X$ is a mapping cylinder of some map $\partial X \to B$. The mapping cylinder structure induces a strong deformation retraction $q : X \to B$. Compose this with the projection $X \times D^2 \to X$ and restrict it to the boundary to get a map
$p : M = \partial(X \times D^2) \to B$. It turns out that, for any choice of the spine $B$, this map $p : M \to B$ is $UV^1$ (see [4] for the definition of $UV^1$-maps). So the following observation of Hegenbarth and Repovš [2] based on [5] can be applied to $p : M \to B$, if the assembly map is injective.

**Theorem.** (Hegenbarth-Repovš) Let $M$ be a closed oriented $TOP$ $4$-manifold and $p : M \to B$ be an $UV^1$-map to a finite CW-complex such that the assembly map

$$A : H_4(B; \mathbb{L}) \to L_4(\pi_1(B))$$

is injective. Then the following holds: if $(f, b) : N \to M$ is a degree $1$ $TOP$ normal map with trivial surgery obstruction in $L_4(\pi_1(M))$, then $(f, b)$ is $TOP$ normally bordant to a $p^{-1}(\epsilon)$-homotopy equivalence $f' : N' \to M$ for any $\epsilon > 0$. In particular $(f, b)$ is $TOP$ normally bordant to a homotopy equivalence.

For example, we have

**Theorem.** If $X$ is a compact connected orientable Haken $3$-manifold with boundary, and $B$ is any CW-spine of $X$, then there is a $UV^1$-map $p : M(X) \to B$, and the assembly map $A : H_4(B; \mathbb{L}) \to L_4(\pi_1(B))$ is an isomorphism. Therefore, if $(f, b) : N \to M$ is a degree $1$ $TOP$ normal map with trivial surgery obstruction in $L_4(\pi_1(M))$, then $(f, b)$ is $TOP$ normally bordant to a $p^{-1}(\epsilon)$-homotopy equivalence $f' : N' \to M$ for any $\epsilon > 0$.

See [8] for details.

In the talk at RIMS, I used an ideal cell decomposition of link complements to construct a spine for $X = E(K)$. This is now obsolete. But it may be of some interest, so I will discuss the construction in this note.

2. Ideal Cell Decomposition of Link Complements

Let $K$ be a knot in $S^3$. We show that $S^3 - K$ decomposes into ideal $3$-cells ($= 3$-cells whose vertices are removed). The following construction works equally well when $K$ is a link.

Identify $S^3$ with $S^2 \times (-\infty, \infty) \cup \{\pm \infty\}$, and consider a knot projection to $S^2 \times 0$, with $n$ crossings. We assume that $n \geq 1$ and that $K$ stays in $S^2 \times 0$ except at the overcrossings as in the next picture:
Consider the dual graph of the knot diagram:

The dual graph and the knot diagram together decompose $S^2 \times 0$ into $4n$-many quadrangles $R_i$. One such quadrangle is indicated in the picture above. Roughly speaking, $R_i \times (-\infty, \infty) - K$ are the desired ideal 3-cells:

Unfortunately their union is not $S^3 - K$, but $S^3 - \{\pm\infty\} - K$. So pick an intersection point of $K$ and the dual graph, and dig tunnels from that point to $\pm\infty$ along the edges. This affects four of the 3-cells as in the picture below and gives a decomposition of $S^3 - K$ into ideal cells:
Remark. A knot/link complement has a decomposition into ideal tetrahedra. Discussions on this topic can be found in [1][6][7][9], but these are all quite technical.

The dual spine of the ideal cell decomposition can be defined in the following way: Take one point from each 1-cell; the union of these points is the dual spine of the 1-skeleton and there is a collapsing map from the 1-skeleton to the spine. Next, take one point from the interior of each 2-cell, and take the topological join of the point and the spine of the boundary. The union of these joins is the spine of the 2-skeleton. The collapsing map of the 1-skeleton extends to the collapsing map of the 2-skeleton to the spine. Finally, take one point from the interior of each 3-cell, take the join of the point and the spine of the boundary. The union of these joins is the desired spine $B$, and the collapsing map of the 2-skeleton extends to a collapsing map $q : S^3 - K \to B$.

References


