On Lie algebras of vector fields of manifolds with singularities

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§1. Introduction

In this talk we shall consider Pursell-Shanks type theorem for some manifolds with singularities.

Let $\mathcal{X}(M)$ be the Lie algebra of smooth vector fields on a connected smooth manifold $M$ with compact support. Then Pursell and Shanks proved the following.

**Theorem 1.1** (Pursell-Shanks [PS])

Let $M$ and $N$ be connected smooth manifolds. If $\mathcal{X}(M)$ and $\mathcal{X}(N)$ are isomorphic as a Lie algebra, then $M$ and $N$ are diffeomorphic.

There are many analogous results on the Lie algebra of smooth vector fields which preserve a geometric structures (c.f. [AM], [BA], [FU], [GP], [GR], [OM], [KO]). We extended Theorem 1.1 to the case of smooth orbifold.

**Theorem 1.2** (K. Abe [AB2])

Let $M$ and $N$ be connected smooth orbifold. If $\mathcal{X}(M)$ and $\mathcal{X}(N)$ are isomorphic as a Lie algebra, then $M$ and $N$ are diffeomorphic.

Note that a smooth orbifold is locally diffeomorphism to the orbit space $V/\Gamma$ of a representation space $V$ of a finite group $\Gamma$. In this paper we consider when $\Gamma$ is a discrete subgroup of $SL(2, \mathbb{Z})$.

§2. Statement of the result

Let $\mathcal{H}$ denote the upper half complex plane. Let $SL(2, \mathbb{R})$ be the group of real matrix with determinant 1. Then $SL(2, \mathbb{R})$ acts on $\mathcal{H}$ by the Möbius as the following.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $z \in \mathcal{H}$,

$$g \cdot z = \frac{az + b}{cz + d}.$$
Then $SL(2, \mathbb{R})$ acts transitively on $\mathcal{H}$ and the isotropy subgroup at $i = \sqrt{-1}$ is

$$SL(2, \mathbb{R}), i = SO(2).$$

The kernel of the action is $\mathbb{Z}_2 = \{\pm 1\}$ and $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$ acts effectively on $\mathcal{H}$ and

$$\mathcal{H} \cong SL(2, \mathbb{R})/SO(2).$$

The action can be extended to the Riemannian sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in \mathcal{H},$

$$g \cdot z = \begin{cases} \frac{az+b}{cz+d} & (z \neq -\frac{d}{c}, \infty) \\ \infty & (z = -\frac{d}{c}, z = d = 0) \\ \frac{a}{c} & (z = \infty) \end{cases}.$$

Set

$$R_1 = \left\{ \pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\}$$

and

$$R_2 = \left\{ \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then each $g \in SL(2, \mathbb{R})$ is conjugate to one of the elements of $SO(2) \cup R_1 \cup R_2$, and $g \neq \pm 1$ is called elliptic, hyperbolic and parabolic if $g$ is conjugate to an element in $SO(2)$, $R_1$ and $R_2$, respectively.

Let $\Gamma$ denote a discrete subgroup of $SL(2, \mathbb{R})$. $z \in \mathcal{H}$ is called elliptic point if there exits an elliptic element $g \in \Gamma$ such that $g \cdot z = z$. $x \in \mathbb{R} \cup \{\infty\}$ is called cusp point if there exists a parabolic element $g \in \Gamma$ such that $g \cdot z = z$.

**Proposition 2.1**

1. If $z$ is a elliptic point, then $\Gamma_z$ is a cyclic group which is conjugate to a cyclic subgroup of $SO(2)$.

2. If $x$ is a cusp point, then $\Gamma_x$ is isomorphic to $\mathbb{Z}$ which is conjugate to a subgroup of the group

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \quad (\exists k \in \mathbb{Z}).$$

Let $E_\Gamma$ denote the set of all elliptic points in $\mathcal{H}$ and $C_\Gamma$ be the set of cusp points of $\Gamma$. Set $\mathcal{H}^* = \mathcal{H} \cup C_\Gamma,$
We shall give the following topology on $\mathcal{H}^*$. 

(1) We give the canonical topology on $\mathcal{H}$.
(2) Let $x \in C_{\Gamma}$.

(2.1) If $x \neq \infty$, then we take all the family of the form
\[
\{x\} \cup \{\text{the interior of a circle in } \mathcal{H} \text{ tangent to the real axis at } x\}
\]
as a fundamental system of open neighborhoods of $x$.

(2.2) If $x = \infty$, then
\[
\{\infty\} \cup \cup_{c>0}\{z \in \mathcal{H} | \Im z > c\}
\]
as a fundamental system of open neighborhood of the point $\infty$. Then $\Gamma$ acts on $\mathcal{H}^*$ as a topological transformation group. Set
\[
\mathcal{R}_{\Gamma} = \mathcal{H}^*/\Gamma = \mathcal{H}/\Gamma \cup C_{\Gamma}/\Gamma
\]
Then $\mathcal{R}_\Gamma$ is a Hausdorff space.

Lemma 2.2 For each $x \in C_{\Gamma}$, there exists an open neighborhood $\tilde{U}_x$ of $x$ in $\mathcal{H}^*$ such that
\[
\Gamma_x = \{\gamma \in \Gamma | \gamma \cdot \tilde{U}_x \cap \tilde{U}_x \neq \emptyset\}.
\]
Take $x \in C_{\Gamma}$. Let $\iota_x : \tilde{U}_x/\Gamma_x \rightarrow \mathcal{R}_\Gamma$ be a map defined by $\iota_x(\Gamma_x \cdot z) = \Gamma \cdot z$ for $z \in \tilde{U}_x$. Put $p = \Gamma \cdot x$. Then $U_p = \iota_x(\tilde{U}_x/\Gamma_x)$ is an open neighborhood of $p$ in $\mathcal{R}_\Gamma$. For $x \in C_{\Gamma}$, there exist $g \in SL(2, \mathbb{R})$ and integer $k$ such that $g \cdot x = \infty$ and
\[
g\Gamma_x g^{-1} = \left\{ \pm \begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.
\]

Proposition 2.3 Let $\varphi_p : \tilde{U}_x/\Gamma_x \rightarrow \mathbb{C}$ be a map given by
\[
\varphi_p(\Gamma_x z) = \begin{cases} \exp\left(\frac{2\pi\sqrt{-1}}{k} (g \cdot z)\right) & (z \in \tilde{U}_x \setminus \{x\}), \\ 0 & (z = x). \end{cases}
\]
Then $\varphi_p$ is homeomorphic to an open subset $W_p$ of $\mathbb{C}$.

By Proposition 2.3, the map $\psi_p = \varphi_p \circ \iota_x^{-1} : U_p \rightarrow \tilde{U}_x/\Gamma_x \rightarrow W_p$ is regarded as a local coordinate of $\mathcal{R}_\Gamma$ around $p$.

$\bar{C}_\Gamma = C_{\Gamma}/\Gamma$

Definition 2.4 $f : \mathcal{R}_\Gamma \rightarrow \mathbb{R}$ is defined to be smooth if
(1) $f \circ \pi_\Gamma$ is smooth, where $\pi_\Gamma : \mathcal{H} \rightarrow \mathcal{H}/\Gamma$ is the natural projection,
(2) for each $p \in \bar{C}_\Gamma$, $f \circ \psi_p^{-1}$ is smooth.
Definition 2.4 (2) does not depend on the choice of \( x \) with \( \Gamma \cdot x = p \).

Let \( C^\infty(\mathcal{R}_\Gamma) \) denote the set of all real valued smooth functions on \( \mathcal{R}_\Gamma \).

**Definition 2.5** For discrete subgroups \( \Gamma, \Gamma' \) of \( SL(2, \mathbb{R}) \), \( h : \mathcal{R}_\Gamma \to \mathcal{R}_{\Gamma'} \) is said smooth if for each real valued smooth function \( f : \mathcal{R}_{\Gamma'} \to \mathbb{R} \) \( f \circ h \) is smooth. \( h \) is said diffeomorphic if \( h \) and \( h^{-1} \) are smooth.

**Definition 2.6** A derivation \( \mathcal{X} \) of \( C^\infty(\mathcal{R}_\Gamma) \) is called a smooth vector field on \( \mathcal{R}_\Gamma \) if \( \mathcal{X} \) vanishes on \( C_\Gamma \). Let \( \mathcal{L}(\mathcal{R}_\Gamma) \) denote the set of all smooth vector field on \( \mathcal{R}_\Gamma \) and let \( \mathcal{X}(\mathcal{R}_\Gamma) \) be the subalgebra of \( \mathcal{L}(\mathcal{R}_\Gamma) \) which consists of vector fields with compact support.

Then we have the following.

**Theorem 2.7** Let \( \Gamma \) and \( \Gamma' \) be discrete subgroups of \( SL(2, \mathbb{R}) \). Then \( \mathcal{R}_\Gamma \) and \( \mathcal{R}_{\Gamma'} \) are diffeomorphic if and only if \( \mathcal{X}(\mathcal{R}_\Gamma) \) and \( \mathcal{X}(\mathcal{R}_{\Gamma'}) \) are isomorphic as a Lie algebra.

§3. Maximal ideals of \( \mathcal{X}(\mathcal{R}_\Gamma) \)

In order to prove Theorem 2.7 we investigate the maximal ideals of \( \mathcal{X}(\mathcal{R}_\Gamma) \). Let \( \Gamma \) be a discrete subgroup of \( SL(2, \mathbb{R}) \). Let \( \mathcal{E}_\Gamma = E_\Gamma/\Gamma \) and \( \mathcal{C}_\Gamma \) denote the set of elliptic singularities and cusp singularities in \( \mathcal{R}_\Gamma \), respectively. Set \( \mathcal{S}_\Gamma = \mathcal{E}_\Gamma \cup \mathcal{C}_\Gamma \) which is the set of singularities in \( \mathcal{R}_\Gamma \). We abbreviate \( \mathcal{R}_\Gamma, \mathcal{S}_\Gamma \) and \( \mathcal{E}_\Gamma \) to \( \mathcal{R}, \mathcal{S} \) and \( \mathcal{E} \), respectively. Let \( \mathcal{R}_1 = \mathcal{R} \setminus \mathcal{S} \) be the set of regular points of \( \mathcal{R} \). For each \( p \in \mathcal{R}_1 \), set

\[
\mathcal{X}_p(\mathcal{R}) = \{ X \in \mathcal{X}(\mathcal{R}) | X(p) = 0 \}.
\]

**Proposition 3.1** For each \( p \in \mathcal{R}_1 \), there exists a unique maximal ideal \( \mathcal{M}_p \) of \( \mathcal{X}(\mathcal{R}) \) which is contained in \( \mathcal{X}_p(\mathcal{R}) \). Moreover \( \mathcal{M}_p \) is an infinite codimensional subalgebra in \( \mathcal{X}(\mathcal{R}) \).

Next we shall find the maximal ideals of \( \mathcal{X}(\mathcal{R}) \) which correspond to the singularities in \( \mathcal{R} \). Here we recall the results by Bierstone and Schwarz. Let \( G \) be a finite group and \( V \) be a representation space of \( G \). Let \( \pi : V \to V/G \) be the natural projection. \( \mathcal{X}_G(V) \) denotes the Lie algebra of \( G \)-invariant smooth vector fields on \( V \) with compact support.

**Theorem 3.2** (Bierstone [BI] and Schwarz [SC])

The induced map \( \pi_* : \mathcal{X}_G(V) \to \mathcal{X}(V/G) \) is a Lie algebra isomorphism.
For each \( p \in \overline{E} \), take \( x_p \in E \) with \( \Gamma \cdot x_p = p \). Let \( V_{x_p} \) be the linear slice at \( x_p \). Then \( V_{x_p} \) is a \( \Gamma_{x_p} \)-module. Let

\[
(\pi_{x_p})_* : \mathcal{X}_{\Gamma_{x_p}}(V_{x_p}) \rightarrow \mathcal{X}(V_{x_p}/\Gamma_{x_p}) \hookrightarrow \mathcal{X}(\mathcal{R})
\]

be the natural Lie algebra homomorphism. By Theorem 3.2, for each \( X \in \mathcal{X}(\mathcal{R}) \) there exists \( Y_{x_p} \in \mathcal{X}_{\Gamma_{x_p}}(V_{x_p}) \) such that \( (\pi_{x_p})_*(Y_{x_p}) = X \) on a neighborhood of \( p \) in \( \mathcal{R} \). Let \( \mathfrak{gl}_{\Gamma_{x_p}}(V_{x_p}) \) be the set of \( \Gamma_{x_p} \)-invariant linear endmorphisms. Let

\[
J_p : \mathcal{X}(\mathcal{R}) \rightarrow \mathfrak{gl}(2, \mathbb{R})
\]

be the homomorphism defined by \( J_p(X) = j_{x_p}^1(X|_{U_p}) \), where \( j_{x_p}^1(X|_{U_p}) \) is the 1-jet of \( Y_{x_p} \) at \( x_p \).

(II) For \( p \in \overline{C} \) there is a chart \( \psi_p : U_p \rightarrow W_p \subset \mathbb{C} = \mathbb{R}^2 \) around the open neighborhood \( U_p \) of \( p \) in \( \mathcal{R} \). Let

\[
J_p : \mathcal{X}(\mathcal{R}) \rightarrow \mathfrak{gl}(2, \mathbb{R})
\]

be the Lie algebra homomorphism defined by \( J_p(X) = j_{x_p}^1(X|_{U_p}) \).

Combining (I) and (II) we set

\[
J(\mathcal{R}) = \bigoplus_{p \in \overline{E}} \mathfrak{gl}_{\Gamma_{x_p}}(V_{x_p}) \oplus \bigoplus_{p \in \overline{C}} \mathfrak{gl}(2, \mathbb{R}).
\]

Let \( J : \mathcal{X}(\mathcal{R}) \rightarrow J(\mathcal{R}) \) be a Lie algebra homomorphism defined by

\[
J(X) = \bigoplus_{p \in \overline{E}} J_p(X) \oplus \bigoplus_{p \in \overline{C}} J_p(X).
\]

**Lemma 3.3** \( J \) is an onto Lie algebra homomorphism.

**Proposition 3.4** If \( \mathfrak{M} \) is a maximal ideal of \( \mathcal{X}(\mathcal{R}) \), then we have the following.

1. If \( \mathfrak{M} \) is contained in \( \mathcal{X}_p(\mathcal{R}) \) for some \( p \in \mathcal{R}_1 \), then \( \mathfrak{M} = \mathfrak{M}_p \), and \( \mathfrak{M} \) is an infinite codimensional subalgebra of \( \mathcal{X}(\mathcal{R}) \).
2. If \( \mathfrak{M} \not\subset \mathcal{X}_p(\mathcal{R}) \) for any \( p \in \mathcal{R}_1 \), then there exists a maximal ideal \( \mathfrak{L} \) of \( J(\mathcal{R}) \) such that \( \mathfrak{M} = J^{-1}(\mathfrak{L}) \), and \( \mathfrak{M} \) is a finite codimensional subalgebra of \( \mathcal{X}(\mathcal{R}) \).
§4. Stone topology of the maximal ideals

Let $\mathcal{R}^*$ be the set of all maximal ideals of $\mathcal{X}(\mathcal{R})$.

**Definition 4.1** The Stone topology on $\mathcal{R}^*$ is defined by the closure operator $\text{Cl}$ as following.

1. $\text{Cl}(\emptyset) = \emptyset$,
2. For a subset $B$ of $\mathcal{R}^*$ with $B \neq \emptyset$,
   
   $$\text{Cl}(B) = \left\{ \mathfrak{M} \in \mathcal{R}^* \mid \mathfrak{M} \supset \bigcap_{\mathfrak{M}' \in B} \mathfrak{M}' \right\}.$$ 

Let $\mathcal{O}(S)$ denote the family of all subsets of $S$. We define a map

$$\tau_{\mathcal{R}} : \mathcal{R}^* \to \mathcal{R}_1 \cup \mathcal{O}(S)$$

by the following way.

1. For $p \in \mathcal{R}_1$, $\tau_{\mathcal{R}}(\mathcal{M}_p) = p$.
2. If $\mathfrak{M} \in \mathcal{R}^*$ such that $\mathfrak{M} \not\subset \mathcal{X}_p(\mathcal{R})$ for any $p \in \mathcal{R}_1$, then
   
   $$\tau_{\mathcal{R}}(\mathfrak{M}) = \{p \in S \mid J(\mathfrak{M}) \not\supset J_p(\mathcal{X}(\mathcal{R}))\}.$$ 

Set $\mathcal{R}_1^* = \{\mathcal{M}_p \in \mathcal{R}^* \mid p \in \mathcal{R}_1\}$.

**Proposition 4.2**

The map $\tau_{\mathcal{R}} : \mathcal{R}_1^* \to \mathcal{R}_1$ is homeomorphic.

**Definition 4.3** (End)

Let $\mathcal{A}(\mathcal{R}_1) = \{K_i \mid i \in I\}$ denote the family of compact subset in $\mathcal{R}_1$. For each $K \in \mathcal{A}(\mathcal{R}_1)$, let $C_K : \text{be the set of connected component of } \mathcal{R}_1 \setminus K$.

$$\prod_{K_i \in \mathcal{A}(\mathcal{R}_1)} C_{K_i} \in \prod_{K_i \in \mathcal{A}(\mathcal{R}_1)} \mathcal{E}_{K_i}$$

is said to be an end of $\mathcal{R}_1$ if $C_{K_i} \subset C_{K_j}$ for any pair $i, j \in I$ with $K_j \subset K_i$.

$\mathcal{E}(\mathcal{R}_1)$ : the set of all ends of $\mathcal{R}_1$

For each $p \in S$ there exists a unique end $\mathcal{E}_p = \prod_{K_i \in \mathcal{A}(\mathcal{R}_1)} C_{K_i}$ in $\mathcal{R}_1$ such that

$$\bigcap_{K_i \in \mathcal{A}(\mathcal{R}_1)} \text{cl}(C_{K_i}) = \{p\},$$

where $\text{cl}(C_{K_i})$ is the closure of $C_{K_i}$ in $\mathcal{R}$. Set

$$\mathcal{E}_0(\mathcal{R}_1) = \{\mathcal{E}_p \mid p \in S\}, \quad \bar{\mathcal{R}}_1 = \mathcal{R}_1 \cup \mathcal{E}(\mathcal{R}_1).$$
Then $\overline{\mathcal{R}}_1$ has the natural topology such that

$$\{C_{K_j} \cup \prod_{K_i \in \mathcal{R}(\mathcal{R}_1)} C_{K_i} | K_j \in \mathcal{R}(\mathcal{R}_1)\}$$

is the fundamental system of neighborhood of a point $\prod_{K_i \in \mathcal{R}(\mathcal{R}_1)} C_{K_i} \in \mathcal{E}(\mathcal{R}_1)$.

Put $\overline{\mathcal{R}}_0 = \mathcal{R}_1 \cup \mathcal{E}_0(\mathcal{R}_1)$. Let $\kappa_{\mathcal{R}} : \mathcal{R} \to \overline{\mathcal{R}}_0$ be the natural map defined by

$$\kappa_{\mathcal{R}}(p) = \begin{cases} p & \text{for } p \in \mathcal{R}_1 \\ \mathcal{E}_p & \text{for } p \in S \end{cases}$$

**Lemma 4.4** The map $\kappa_{\mathcal{R}} : \mathcal{R} \to \overline{\mathcal{R}}_0$ is a homeomorphism.

## §5. Outline of the proof of Theorem 2.7

Let $\Gamma, \Gamma'$ be discrete subgroups. Assume that there exists a Lie algebra isomorphism $\Phi : \mathcal{X}(\mathcal{R}_\Gamma) \to \mathcal{X}(\mathcal{R}_{\Gamma'})$. We abbreviate $\mathcal{R}_{\Gamma'}, S_{\Gamma'}, \overline{E}_{\Gamma'}, ...$ to $\mathcal{R}', S', \overline{E}', ...$, respectively. By Proposition 4.2 we have.

**Proposition 5.1**

1. $\Phi_* : \mathcal{R}^* \to \mathcal{R}'^*$ is homeomorphic.
2. The composition $\sigma_1 = \tau_{\mathcal{R}'} \circ \Phi_* \circ \tau_{\mathcal{R}}^{-1} : \mathcal{R}_1 \to \mathcal{R}'_1$ is homeomorphic.

By Proposition 5.1 we have.

**Corollary 5.2** There exists a homeomorphism $\overline{\sigma} : \overline{\mathcal{R}} \to \overline{\mathcal{R}}'$ which is an extension of $\sigma_1$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{R}^* & \overset{\Phi_*}{\longrightarrow} & \mathcal{R}'^* \\
\tau_{\mathcal{R}} \downarrow & & \tau_{\mathcal{R}'} \\
\overline{\mathcal{R}} & \overset{\overline{\sigma}}{\longrightarrow} & \overline{\mathcal{R}}'
\end{array}
\]

**Lemma 5.3** For $p \in S$ let $U$ be a neighborhood of $p$ in $\mathcal{R}$ such that $\text{cl}(U) \cap S = \{p\}$. Then we have

$$\text{cl}(\tau_{\mathcal{R}}^{-1}(U)) = \tau_{\mathcal{R}}^{-1}(\text{cl}(U))$$

From Corollary 5.2, Lemma 5.3 and Lemma 4.4, we have the following.
Proposition 5.4  We can extend the homeomorphism $\sigma_1 : \mathcal{R}_1 \rightarrow \mathcal{R}'_1$ to the homeomorphism $\sigma : \mathcal{R} \rightarrow \mathcal{R}'$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\sigma} & \mathcal{R}' \\
\kappa_{\mathcal{R}} \downarrow & & \kappa_{\mathcal{R}'} \\
\mathcal{R}_0 & \xrightarrow{\overline{\sigma}} & \mathcal{R}'_0
\end{array}
\]

Lemma 5.5  Let $p \in \mathcal{R}_1$ and $X \in \mathcal{X}(\mathcal{R})$. Then $X_p \neq 0$ if and only if $[X, \mathcal{X}(\mathcal{R})] + \mathcal{M}_p = \mathcal{X}(\mathcal{R})$.

Corollary 5.6  $\sigma_1 : \mathcal{R}_1 \rightarrow \mathcal{R}'_1$ is diffeomorphic.

By the method Koriyama [KO] and Abe [AB1] we can prove that $\sigma : \mathcal{R} \rightarrow \mathcal{R}'$ is diffeomorphic.

References


