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A remark on quadratic differentials vanishing at infinity

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For a Fuchsian group Γ acting on the upper half-plane H, let $B(\Gamma)$ denote the Banach space of all holomorphic functions on H satisfying $\gamma^*\varphi=\varphi$ for every $\gamma\in\Gamma$ and $\|\varphi\|<\infty$, where $(\gamma^*\varphi)(z):=\varphi(\gamma(z))\gamma'(z)^2$ and $\|\varphi\|:=\sup_{z\in H}4(\operatorname{Im}z)^2|\varphi(z)|$. An element in $B(\Gamma)$ is called a bounded holomorphic quadratic differential for Γ. Let $S(\Gamma)$ be a subset of $B(\Gamma)$ consisting of those $\varphi=S_f$, where S_f is the Schwarzian derivative for a Γ-compatible univalent function f on H. The Nehari theorem says that if $\varphi\in S(\Gamma)$ then $\|\varphi\|\leq 6$. Also $S(\Gamma)$ is closed in $B(\Gamma)$.

The boundary semi-norm for $\varphi\in B(\Gamma)$ is defined by

$$\|\varphi\|_0=\inf_V\|\varphi|_{H-\Gamma(V)}\|,$$

where the infimum is taken over all compact subsets $V\subset H$. It is said that $\varphi\in B(\Gamma)$ *vanishes at infinity* if $\|\varphi\|_0=0$. Let $B_0(\Gamma)$ be a Banach subspace of $B(\Gamma)$ consisting of all φ vanishing at infinity. An element $[\varphi]$ in the quotient Banach space $B(\Gamma)/B_0(\Gamma)$ is identified with the coset $\varphi+B_0(\Gamma)$ in $B(\Gamma)$. For each $\varphi\in B(\Gamma)$, we set

$$\|\varphi\|_{@(\Gamma)}=\inf\{\|\varphi+\psi\|\mid\psi\in B_0(\Gamma)\},$$

which induces the quotient norm for $[\varphi]$ in $B(\Gamma)/B_0(\Gamma)$.

The purpose of this note is to remark the following theorem. An idea of the proof is contained in [2]. This article as well as [1] studies the case where $\phi(z)=\frac{1}{2}z^{-2}$ and $\beta=2+\varepsilon$ in the statement below.

Theorem. *Let $\tilde{\phi}\in B(\Gamma)$ satisfy $\|\tilde{\phi}\|_{@(\Gamma)}<\beta$ for a positive constant $\beta>0$ and $\phi\in B(1)$ (for the trivial group 1) satisfy $r\phi\notin S(1)$ for all $r>1$. Assume that there exists a sequence $\{h_n\}$ of conformal automorphisms of H such that the orbit $\{h_n(z)\}$ eventually exits from $\Gamma(V)$ for any compact subset $V\subset H$ and such that $h_n^*\phi$ converge to ϕ locally uniformly. Then there exists $\varphi\in B(\Gamma)$ with $\|\varphi\|_{@(\Gamma)}<\beta$ satisfying*

$$(\varphi+B_0(\Gamma))\cap S(\Gamma)=\emptyset.$$

Proof. Suppose to the contrary that every $\varphi \in B(\Gamma)$ with $\|\varphi\|_{\mathfrak{B}(\Gamma)} < \beta$ satisfies $(\varphi + B_0(\Gamma)) \cap S(\Gamma) \neq \emptyset$. We take $(1+\delta)\tilde{\phi}$ as this φ, where $\delta > 0$ is chosen so that $(1+\delta)\|\tilde{\phi}\|_{\mathfrak{B}(\Gamma)} < \beta$. Then there exists some $\psi \in B_0(\Gamma)$ such that $(1+\delta)\tilde{\phi}+\psi \in S(\Gamma)$. Set $\tilde{\phi}_n = h_n^* \tilde{\phi}$ and $\psi_n = h_n^* \psi$. By assumption, $\tilde{\phi}_n$ converge to ϕ locally uniformly. Since ψ vanishes at infinity, ψ_n converge to 0 locally uniformly. Hence

$$(1+\delta)\tilde{\phi}_n + \psi_n \to (1+\delta)\phi.$$

On the other hand, since $(1+\delta)\tilde{\phi}_n + \psi_n$ belong to $S(1)$ for all n, there exist univalent functions f_n on Δ such that $(1+\delta)\tilde{\phi}_n + \psi_n = S_{f_n}$. We may give a certain normalization to f_n so that a subsequence converges to a univalent function f on Δ locally uniformly. Then $S_{f_n} \to S_f$ and hence $(1+\delta)\phi = S_f \in S(1)$. However, this contradicts the assumption that $r\phi \notin S(1)$ for all $r > 1$. □

Corollary. *Suppose that a Fuchsian group* Γ *is contained in another Fuchsian group* $\tilde{\Gamma}$ *as a normal subgroup of infinite index. Let* $\phi \in B_0(\Gamma)$ *satisfy* $r\phi \notin S(\Gamma)$ *for all* $r > 1$. *Then, for every* $\varepsilon > 0$, *there exists* $\varphi \in B(\Gamma)$ *with* $\|\varphi\|_{\mathfrak{B}(\Gamma)} < \|\phi\| + \varepsilon$ *satisfying*

$$(\varphi + B_0(\Gamma)) \cap S(\Gamma) = \emptyset.$$

Proof. Take a system of representatives $\{h_1, h_2, \dots\} \subset \tilde{\Gamma}$ for the coset decomposition of $\tilde{\Gamma}$ modulo Γ. Then the sequence $\{h_n\}$ of conformal automorphisms of H holds a property that the orbit $\{h_n(z)\}$ eventually exits from $\Gamma(V)$ for any compact subset $V \subset H$. Moreover, for a given $\varepsilon > 0$, we can choose a subsequence h_{n_k} so that

$$\tilde{\phi} = \sum_{k=1}^{\infty} (h_{n_k}^{-1})^* \phi \in B(\Gamma)$$

satisfies $(\|\tilde{\phi}\|_{\mathfrak{B}(\Gamma)} \leq)\ \|\tilde{\phi}\| < \|\phi\| + \varepsilon$ and so that $h_{n_k}^* \tilde{\phi}$ converge to ϕ locally uniformly. Then we can apply the above theorem. □

REFERENCES

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