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A remark on quadratic differentials vanishing at infinity

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For a Fuchsian group $\Gamma$ acting on the upper half-plane $H$, let $B(\Gamma)$ denote the Banach space of all holomorphic functions on $H$ satisfying $\gamma \varphi = \varphi$ for every $\gamma \in \Gamma$ and $\|\varphi\| < \infty$, where $(\gamma \varphi)(z) := \varphi(\gamma(z)) \gamma'(z)^2$ and $\|\varphi\| := \sup_{z \in H} 4(\text{Im } z)^2|\varphi(z)|$. An element in $B(\Gamma)$ is called a bounded holomorphic quadratic differential for $\Gamma$. Let $S(\Gamma)$ be a subset of $B(\Gamma)$ consisting of those $\varphi = S_f$. where $S_f$ is the Schwarzian derivative for a $\Gamma$-compatible univalent function $f$ on $H$. The Nehari theorem says that if $\varphi \in S(\Gamma)$ then $\|\varphi\| \leq 6$. Also $S(\Gamma)$ is closed in $B(\Gamma)$.

The boundary semi-norm for $\varphi \in B(\Gamma)$ is defined by

$$\|\varphi\|_0 = \inf_V \|\varphi|_{H-\Gamma(V)}\|,$$

where the infimum is taken over all compact subsets $V \subset H$. It is said that $\varphi \in B(\Gamma)$ vanishes at infinity if $\|\varphi\|_0 = 0$. Let $B_0(\Gamma)$ be a Banach subspace of $B(\Gamma)$ consisting of all $\varphi$ vanishing at infinity. An element $[\varphi]$ in the quotient Banach space $B(\Gamma)/B_0(\Gamma)$ is identified with the coset $\varphi + B_0(\Gamma)$ in $B(\Gamma)$. For each $\varphi \in B(\Gamma)$, we set

$$\|\varphi\|_{B(\Gamma)} = \inf \{\|\varphi + \psi\| \mid \psi \in B_0(\Gamma)\},$$

which induces the quotient norm for $[\varphi]$ in $B(\Gamma)/B_0(\Gamma)$.

The purpose of this note is to remark the following theorem. An idea of the proof is contained in [2]. This article as well as [1] studies the case where $\varphi(z) = \frac{1}{2} z^{-2}$ and $\beta = 2 + \varepsilon$ in the statement below.

**Theorem.** Let $\tilde{\varphi} \in B(\Gamma)$ satisfy $\|\tilde{\varphi}\|_{B(\Gamma)} < \beta$ for a positive constant $\beta > 0$ and $\varphi \in B(1)$ (for the trivial group 1) satisfy $r \varphi \not\in S(1)$ for all $r > 1$. Assume that there exists a sequence $\{h_n\}$ of conformal automorphisms of $H$ such that the orbit $\{h_n(z)\}$ eventually exits from $\Gamma(V)$ for any compact subset $V \subset H$ and such that $h_n^* \varphi$ converge to $\varphi$ locally uniformly. Then there exists $\varphi \in B(\Gamma)$ with $\|\varphi\|_{B(\Gamma)} < \beta$ satisfying

$$(\varphi + B_0(\Gamma)) \cap S(\Gamma) = \emptyset.$$
Proof. Suppose to the contrary that every $\varphi \in B(\Gamma)$ with $\|\varphi\|_{B(\Gamma)} < \beta$ satisfies $\varphi + B_0(\Gamma) \cap S(\Gamma) \neq \emptyset$. We take $(1 + \delta)\widetilde{\varphi}$ as this $\varphi$, where $\delta > 0$ is chosen so that $(1 + \delta)\|\varphi\|_{B(\Gamma)} < \beta$. Then there exists some $\psi \in B_0(\Gamma)$ such that $(1 + \delta)\widetilde{\varphi} + \psi \in S(\Gamma)$. Set $\phi_n = h_n^*\psi$ and $\psi_n = h_n^*\psi$. By assumption, $\phi_n$ converge to $\phi$ locally uniformly. Since $\psi$ vanishes at infinity, $\psi_n$ converge to 0 locally uniformly. Hence

$$(1 + \delta)\phi_n + \psi_n \to (1 + \delta)\phi.$$ 

On the other hand, since $(1 + \delta)\phi_n + \psi_n$ belong to $S(1)$ for all $n$, there exist univalent functions $f_n$ on $\Delta$ such that $(1 + \delta)\phi_n + \psi_n = S_{f_n}$. We may give a certain normalization to $f_n$ so that a subsequence converges to a univalent function $f$ on $\Delta$ locally uniformly. Then $S_{f_n} \to S_f$ and hence $(1 + \delta)\phi = S_f \in S(1)$. However, this contradicts the assumption that $r\phi \notin S(1)$ for all $r > 1$.

Corollary. Suppose that a Fuchsian group $\Gamma$ is contained in another Fuchsian group $\tilde{\Gamma}$ as a normal subgroup of infinite index. Let $\phi \in B_0(\Gamma)$ satisfy $r\phi \notin S(\Gamma)$ for all $r > 1$. Then, for every $\varepsilon > 0$, there exists $\varphi \in B(\Gamma)$ with $\|\varphi\|_{B(\Gamma)} < \|\phi\| + \varepsilon$ satisfying

$$(\varphi + B_0(\Gamma)) \cap S(\Gamma) = \emptyset.$$ 

Proof. Take a system of representatives $\{h_1, h_2, \ldots\} \subset \tilde{\Gamma}$ for the coset decomposition of $\tilde{\Gamma}$ modulo $\Gamma$. Then the sequence $\{h_n\}$ of conformal automorphisms of $H$ holds a property that the orbit $\{h_n(z)\}$ eventually exits from $\Gamma(V)$ for any compact subset $V \subset H$. Moreover, for a given $\varepsilon > 0$, we can choose a subsequence $h_{n_k}$ so that

$$\phi = \sum_{k=1}^{\infty}(h_{n_k}^{-1})^*\phi \in B(\Gamma)$$

satisfies $(\|\phi\|_{B(\Gamma)} \leq) \|\phi\| < \|\phi\| + \varepsilon$ and so that $h_{n_k}^*\phi$ converge to $\phi$ locally uniformly. Then we can apply the above theorem.

References


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