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Author(s)
MATSUZAKI, KATSUHIKO

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A remark on quadratic differentials vanishing at infinity

KATSUHIKO MATSUZAKI
松崎克彦
Okayama University
岡山大学自然科学研究科

For a Fuchsian group $\Gamma$ acting on the upper half-plane $H$, let $B(\Gamma)$ denote the Banach space of all holomorphic functions on $H$ satisfying $\gamma^*\varphi = \varphi$ for every $\gamma \in \Gamma$ and $\|\varphi\| < \infty$, where $(\gamma^*\varphi)(z) := \varphi(\gamma(z))\gamma'(z)^2$ and $\|\varphi\| := \sup_{z \in H} 4(\text{Im} z)^2|\varphi(z)|$. An element in $B(\Gamma)$ is called a bounded holomorphic quadratic differential for $\Gamma$. Let $S(\Gamma)$ be a subset of $B(\Gamma)$ consisting of those $\varphi = S_f$, where $S_f$ is the Schwarzian derivative for a $\Gamma$-compatible univalent function $f$ on $H$. The Nehari theorem says that if $\varphi \in S(\Gamma)$ then $\|\varphi\| \leq 6$. Also $S(\Gamma)$ is closed in $B(\Gamma)$.

The boundary semi-norm for $\varphi \in B(\Gamma)$ is defined by

$$\|\varphi\|_0 = \inf_V \|\varphi|_{H-\Gamma(V)}\|,$$

where the infimum is taken over all compact subsets $V \subset H$. It is said that $\varphi \in B(\Gamma)$ vanishes at infinity if $\|\varphi\|_0 = 0$. Let $B_0(\Gamma)$ be a Banach subspace of $B(\Gamma)$ consisting of all $\varphi$ vanishing at infinity. An element $[\varphi]$ in the quotient Banach space $B(\Gamma)/B_0(\Gamma)$ is identified with the coset $\varphi + B_0(\Gamma)$ in $B(\Gamma)$. For each $\varphi \in B(\Gamma)$, we set

$$\|\varphi\|_{B(\Gamma)} = \inf \{\|\varphi + \psi\| : \psi \in B_0(\Gamma)\},$$

which induces the quotient norm for $[\varphi]$ in $B(\Gamma)/B_0(\Gamma)$.

The purpose of this note is to remark the following theorem. An idea of the proof is contained in [2]. This article as well as [1] studies the case where $\varphi(z) = \frac{1}{2}z^{-2}$ and $\beta = 2 + \varepsilon$ in the statement below.

**Theorem.** Let $\tilde{\varphi} \in B(\Gamma)$ satisfy $\|\tilde{\varphi}\|_{B(\Gamma)} < \beta$ for a positive constant $\beta > 0$ and $\varphi \in B(1)$ (for the trivial group 1) satisfy $r\varphi \notin S(1)$ for all $r > 1$. Assume that there exists a sequence $\{h_n\}$ of conformal automorphisms of $H$ such that the orbit $\{h_n(z)\}$ eventually exits from $\Gamma(V)$ for any compact subset $V \subset H$ and such that $h_n\tilde{\varphi}$ converge to $\varphi$ locally uniformly. Then there exists $\varphi \in B(\Gamma)$ with $\|\varphi\|_{B(\Gamma)} < \beta$ satisfying

$$(\varphi + B_0(\Gamma)) \cap S(\Gamma) = \emptyset.$$
Proof. Suppose to the contrary that every \( \varphi \in B(\Gamma) \) with \( \|\varphi\|_{\Omega(\Gamma)} < \beta \) satisfies \( (\varphi + B_0(\Gamma)) \cap S(\Gamma) \neq \emptyset \). We take \((1 + \delta)\hat{\varphi}\) as this \( \varphi \), where \( \delta > 0 \) is chosen so that
\( (1 + \delta)\|\hat{\varphi}\|_{\Omega(\Gamma)} < \beta \). Then there exists some \( \psi \in B_0(\Gamma) \) such that \((1 + \delta)\hat{\varphi} + \psi \in S(\Gamma)\). Set \( \hat{\varphi}_n = h_n^*\hat{\varphi} \) and \( \psi_n = h_n^*\psi \). By assumption, \( \hat{\varphi}_n \) converge to \( \hat{\varphi} \) locally uniformly. Since \( \psi \) vanishes at infinity, \( \psi_n \) converge to 0 locally uniformly. Hence
\[
(1 + \delta)\hat{\varphi}_n + \psi_n \rightarrow (1 + \delta)\hat{\varphi}.
\]
On the other hand, since \((1 + \delta)\hat{\varphi}_n + \psi_n \) belong to \( S(1) \) for all \( n \), there exist univalent functions \( f_n \) on \( \Delta \) such that \((1 + \delta)\hat{\varphi}_n + \psi_n = S_{f_n} \). We may give a certain normalization to \( f_n \) so that a subsequence converges to a univalent function \( f \) on \( \Delta \) locally uniformly. Then \( S_{f_n} \rightarrow S_f \) and hence \((1 + \delta)\hat{\varphi} = S_f \in S(1) \). However, this contradicts the assumption that \( r\varphi \notin S(1) \) for all \( r > 1 \). \( \square \)

Corollary. Suppose that a Fuchsian group \( \Gamma \) is contained in another Fuchsian group \( \tilde{\Gamma} \) as a normal subgroup of infinite index. Let \( \varphi \in B_0(\Gamma) \) satisfy \( r\varphi \notin S(\Gamma) \) for all \( r > 1 \). Then, for every \( \varepsilon > 0 \), there exists \( \varphi \in B(\Gamma) \) with \( \|\varphi\|_{\Omega(\Gamma)} < \|\varphi\| + \varepsilon \) satisfying
\[
(\varphi + B_0(\Gamma)) \cap S(\Gamma) = \emptyset.
\]

Proof. Take a system of representatives \( \{h_1, h_2, \ldots\} \subset \tilde{\Gamma} \) for the coset decomposition of \( \tilde{\Gamma} \) modulo \( \Gamma \). Then the sequence \( \{h_n\} \) of conformal automorphisms of \( H \) holds a property that the orbit \( \{h_n(z)\} \) eventually exits from \( \Gamma(V) \) for any compact subset \( V \subset H \). Moreover, for a given \( \varepsilon > 0 \), we can choose a subsequence \( h_{n_k} \) so that
\[
\hat{\varphi} = \sum_{k=1}^{\infty} (h_{n_k}^{-1})^*\varphi \in B(\Gamma)
\]
satisfies \( \|\hat{\varphi}\|_{\Omega(\Gamma)} \leq \|\varphi\| < \|\varphi\| + \varepsilon \) and so that \( h_{n_k}^*\hat{\varphi} \) converge to \( \hat{\varphi} \) locally uniformly. Then we can apply the above theorem. \( \square \)

References

Department of Mathematics, Okayama University, Tsushima-naka 3-1-1, Okayama 700-8530, Japan
E-mail address: matsuzak@math.okayama-u.ac.jp