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Kyoto University
HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGY COBORDISMS OF A SURFACE

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1. Introduction

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 0$ with one boundary component. A homology cylinder (over $\Sigma_{g,1}$) consists of a homology cobordism from $\Sigma_{g,1}$ to itself with markings of its boundary. We denote by $C_{g,1}$ the set of all diffeomorphism classes of homology cylinders. Stacking two homology cylinders gives a new one, and by this, we can endow $C_{g,1}$ with a monoid structure. A systematic study of $C_{g,1}$ was initiated by Habiro in [4], where $C_{g,1}$ appeared as a nice collection of 3-manifolds to which his clasper surgery theory is applied. Later Garoufalidis-Levine [3] and Levine [9] introduced a group $H_{g,1}$ by taking a quotient of $C_{g,1}$ with respect to homology cobordant of homology cylinders. A feature of the monoid $C_{g,1}$ and the group $H_{g,1}$ is that they contain the mapping class group $\mathcal{M}_{g,1}$, which is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,1}$. Moreover some tools for studying $\mathcal{M}_{g,1}$ can be also used for $C_{g,1}$ and $H_{g,1}$ after appropriate generalizations. From these facts, we can consider $C_{g,1}$ and $H_{g,1}$ to be enlargements of $\mathcal{M}_{g,1}$.

Now we consider an application of higher-order Alexander invariants, which are numerical invariants of finitely presentable groups, to homology cylinders. Higher-order Alexander invariants were first defined by Cochran in [1] for knot groups, and then generalized for arbitrary finitely presentable groups by Harvey in [5, 6]. They are interpreted as degrees of "non-commutative Alexander polynomials", which have some unclear ambiguity except their degrees in difficulties of non-commutative rings. Using them, Harvey obtained various sharper results than those given by the ordinary Alexander invariants — lower bounds on the Thurston norm, necessary conditions for realizing a given group as the fundamental group of some compact oriented 3-manifold, and so on.

In the process of applying higher-order Alexander invariants to homology cylinders, we can see that the Magnus representation for homology cylinders [15] plays an important role. This representation generalizes not only the Magnus representation for $\mathcal{M}_{g,1}$ defined by Morita [11], but the Gassner representation for string links given by Le Dimet [8] and Kirk-Livingston-Wang [7]. In this paper, we begin by reviewing the definition and fundamental properties of the Magnus representation, and then study some relationships to higher-order Alexander invariants. Note that the paper [16] treats the same topics and complements the contents of this paper.
2. Homology cobordisms of surfaces

We proceed all our discussion in PL or smooth category.

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 0$ with one boundary component. We take a base point $p$ on the boundary of $\Sigma_{g,1}$, and take $2g$ loops $\gamma_1, \ldots, \gamma_{2g}$ of $\Sigma_{g,1}$ as shown in Figure 1. We consider them to be an embedded bouquet $R_{2g}$ of $2g$-circles tied at the base point $p \in \partial \Sigma_{g,1}$. Then $R_{2g}$ and the boundary loop $\zeta$ of $\Sigma_{g,1}$ together with one 2-cell make up a standard CW-decomposition of $\Sigma_{g,1}$. It is well-known that the fundamental group $\pi_1 \Sigma_{g,1}$ of $\Sigma_{g,1}$ is isomorphic to the free group $F_{2g}$ of rank $2g$ generated by $\gamma_1, \ldots, \gamma_{2g}$, in which

\[ \zeta = \prod_{i=1}^{g} [\gamma_i, \gamma_{g+i}] \]

Figure 1

A homology cylinder $(M, i_+, i_-)$ (over $\Sigma_{g,1}$), which has its origin in Habiro [4], Garoufalidis-Levine [3] and Levine [9], consists of a compact oriented 3-manifold $M$ and two embeddings $i_+, i_- : \Sigma_{g,1} \rightarrow \partial M$ satisfying that

1. $i_+$ is orientation-preserving and $i_-$ is orientation-reversing,
2. $\partial M = i_+ (\Sigma_{g,1}) \cup i_- (\Sigma_{g,1})$ and $i_+ (\Sigma_{g,1}) \cap i_- (\Sigma_{g,1}) = \partial (\Sigma_{g,1}) = i_- (\partial \Sigma_{g,1})$,
3. $i_+ |_{\Sigma_{g,1}} = i_- |_{\Sigma_{g,1}}$,
4. $i_+, i_- : H_i(\Sigma_{g,1}) \rightarrow H_i(M)$ are isomorphisms.

We denote $i_+(p) = i_-(p)$ by $p \in \partial M$ again and consider it to be the base point of $M$. We write a homology cylinder by $(M, i_+, i_-)$ or simply by $M$.

Two homology cylinders are said to be isomorphic if there exists an orientation-preserving diffeomorphism between the underlying 3-manifolds which is compatible with the markings. We denote the set of isomorphism classes of homology cylinders by $C_{g,1}$. Given two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$, we can define a new homology cylinder $M \cdot N$ by

\[ M \cdot N = (M \cup_{l \in \Sigma_{g,1}} N, i_+, j_-). \]

Then $C_{g,1}$ becomes a monoid with the identity element $1_{C_{g,1}} := (\Sigma_{g,1} \times I, \text{id} \times 1, \text{id} \times 0)$.

From the monoid $C_{g,1}$, we can construct the homology cobordism group $\mathcal{H}_{g,1}$ of homology cylinders as in the following way. Two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ are homology cobordant if there exists a compact oriented 4-manifold $W$ such that

1. $\partial W = M \cup (-N)/(i_+(x) = j_+(x), i_-(x) = j_-(x)) \ x \in \Sigma_{g,1}$,
2. the inclusions $M \hookrightarrow W, N \hookrightarrow W$ induce isomorphisms on the homology.
where \(-N\) is \(N\) with opposite orientation. We denote by \(\mathcal{H}_{g,1}\) the quotient set of \(C_{g,1}\) with respect to the equivalence relation of homology cobordism. The monoid structure of \(C_{g,1}\) induces a group structure of \(\mathcal{H}_{g,1}\). In the group \(\mathcal{H}_{g,1}\), the inverse of \((M, i_{+}, i_{-})\) is given by \((-M, i_{-}, i_{+})\).

**Example 2.1.** For each element \(\varphi\) of the mapping class group \(M_{g,1}\) of \(\Sigma_{g,1}\), we can construct a homology cylinder \(M_{\varphi} \in C_{g,1}\) defined by

\[M_{\varphi} := (\Sigma_{g,1} \times I, \text{id} \times 1, \varphi \times 0),\]

where collars of \(i_{+}(\Sigma_{g,1})\) and \(i_{-}(\Sigma_{g,1})\) are stretched half-way along \(\partial \Sigma_{g,1} \times I\). This gives injective homomorphisms \(M_{g,1} \hookrightarrow C_{g,1}\) and \(M_{g,1} \hookrightarrow \mathcal{H}_{g,1}\).

Let \(N_{k}(G) := G/\Gamma^{k}G\) be the \(k\)-th nilpotent quotient of a group \(G\), where we define \(\Gamma^{0}G = G\) and \(\Gamma^{k+1}G = [\Gamma^{k}G, G]\) for \(i \geq 1\). For simplicity, we write \(N_{k}(X)\) for \(N_{k}(\pi_{1}X)\) where \(X\) is a CW-complex, and write \(N_{k}\) for \(N_{k}(F_{2g}) = N_{k}(\Sigma_{g,1})\). It is known that \(N_{k}\) is a torsion-free nilpotent group for each \(k \geq 2\).

Let \((M, i_{+}, i_{-})\) be a homology cylinder. By definition, \(i_{+}, i_{-} : \pi_{1}\Sigma_{g,1} \to \pi_{1}M\) are both 2-connected, namely they induce isomorphisms on \(H_{1}\) and epimorphisms on \(H_{2}\). Then, by Stallings' theorem \([17]\), \(i_{+}, i_{-} : N_{k} \to N_{k}(M)\) are isomorphisms for each \(k \geq 2\). Using them, we obtain a monoid homomorphism

\[\sigma_{k} : C_{g,1} \to \text{Aut}N_{k} \quad ((M, i_{+}, i_{-}) \mapsto (i_{+})^{-1} \circ i_{-}).\]

It can be easily checked that \(\sigma_{k}\) induces a group homomorphism \(\sigma_{k} : \mathcal{H}_{g,1} \to \text{Aut}N_{k}\). We define filtrations of \(C_{g,1}\) and \(\mathcal{H}_{g,1}\) by

\[C_{g,1} := C_{g,1}, \quad C_{g,1}[k] := \text{Ker}\left(C_{g,1} \to \text{Aut}N_{k}\right) \text{ for } k \geq 2,\]

\[\mathcal{H}_{g,1} := \mathcal{H}_{g,1}, \quad \mathcal{H}_{g,1}[k] := \text{Ker}\left(\mathcal{H}_{g,1} \to \text{Aut}N_{k}\right) \text{ for } k \geq 2.\]

3. Magnus Representations for Homology Cylinders

We first summarize our notation. For a matrix \(A\) with entries in a ring \(R\), and a homomorphism \(\varphi : R \to R'\), we denote by \(^{\varphi}A\) the matrix obtained from \(A\) by applying \(\varphi\) to each entry. \(A^{T}\) denotes the transpose of \(A\). When \(R = \mathbb{Z}G\) for a group \(G\) or its right field of fractions (if exists), we denote by \(A^{T}\) the matrix obtained from \(A\) by applying the involution induced from \((x \mapsto x^{-1}, x \in G)\) to each entry. For a module \(M\), we write \(M^{n}\) (resp. \(M_{n}\)) for the module of column (resp. row) vectors with \(n\) entries.

For a finite CW-complex \(X\) and its regular covering \(X_{\Gamma}\) with respect to a homomorphism \(\pi_{1}X \to \Gamma, \Gamma\) acts on \(X_{\Gamma}\) from the right through its deck transformation group. Therefore we regard the \(\mathbb{Z}G\)-cellular chain complex \(C_{\cdot}(X_{\Gamma})\) of \(X_{\Gamma}\) as a collection of free right \(\mathbb{Z}G\)-modules consisting of column vectors together with differentials given by left multiplications of matrices. For each \(\mathbb{Z}G\)-bimodule \(A\), the twisted chain complex \(C_{\cdot}(X, A)\) is given by the tensor product of the right \(\mathbb{Z}G\)-module \(C_{\cdot}(X_{\Gamma})\) and the left \(\mathbb{Z}G\)-module \(A\), so that \(C_{\cdot}(X, A)\) and \(H_{\cdot}(X, A)\) are right \(\mathbb{Z}G\)-modules.

Now we define and study the Magnus representation for homology cylinders. The following construction is based on Kirk-Livingston-Wang's paper \([7]\).
Let $(M, i_+, i_-) \in C_{g,1}$ be a homology cylinder. By Stallings’ theorem, $N_k$ and $N_k(M)$ are isomorphic. Since $N_k$ is a finitely generated torsion-free nilpotent group for each $k \geq 2$, we can embed $\mathbb{Z}N_k$ into the right field of fractions $\mathcal{K}_{N_k} := \mathbb{Z}N_k(\mathbb{Z}N_k - \{0\})^{-1}$. (See Section 5.) Similarly, we obtain $\mathbb{Z}N_k(M) \leftrightarrow \mathcal{K}_{N_k(M)} := \mathbb{Z}N_k(M)(\mathbb{Z}N_k(M) - \{0\})^{-1}$. We consider $\mathcal{K}_{N_k}$ (resp. $\mathcal{K}_{N_k(M)}$) to be a local coefficient system on $\Sigma_{g,1}$ (resp. $M$).

By a standard argument using covering spaces, we have the following.

**Lemma 3.1.** $\kappa : H_*(\Sigma_{g,1}, p; \mathcal{K}_{N_k(M)}) \to H_*(M, p; \mathcal{K}_{N_k(M)})$ are isomorphisms as right $\mathcal{K}_{N_k(M)}$-vector spaces.

Since $R_{2g} \subset \Sigma_{g,1}$ is a deformation retract, we have

$$H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k(M)}) \cong H_1(R_{2g}, p; \mathcal{K}_{N_k(M)}) = C_1(\overline{R_{2g}}) \otimes F_{2g} \mathcal{K}_{N_k(M)} \cong \mathcal{K}_{N_k(M)}^{2g}$$

with a basis

$$\{[\gamma_1 \otimes 1, \ldots, \gamma_{2g} \otimes 1] \in C_1(\overline{R_{2g}}) \otimes F_{2g} \mathcal{K}_{N_k(M)}$$

as a right $\mathcal{K}_{N_k(M)}$-vector space, where $[\gamma]$ is a lift of $\gamma$ on the universal covering $\overline{R_{2g}}$.

**Definition 3.2.** (1) For each $M = (M, i_+, i_-) \in C_{g,1}$, we denote by $r_\kappa(M) \in GL(2g, \mathcal{K}_{N_k(M)})$ the representation matrix of the right $\mathcal{K}_{N_k(M)}$-isomorphism

$$\mathcal{K}_{N_k(M)}^{2g} \cong H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k(M)}) \overset{r_\kappa}{\longrightarrow} H_1(M, p; \mathcal{K}_{N_k(M)}) \cong \mathcal{K}_{N_k(M)}^{2g}$$

(2) The Magnus representation for $C_{g,1}$ is the map $r_\kappa : C_{g,1} \to GL(2g, \mathcal{K}_{N_k})$ which assigns to $M = (M, i_+, i_-) \in C_{g,1}$ the matrix $r_\kappa(M)$.

While we call $r_\kappa(M)$ the Magnus “representation”, it is actually a crossed homomorphism.

**Theorem 3.3** ([14, Theorem 7.12]). For $M_1 = (M_1, i_+, i_-), M_2 = (M_2, j_+, j_-) \in C_{g,1}$, we have

$$r_\kappa(M_1 \cdot M_2) = r_\kappa(M_1) \cdot \sigma(M_1) r_\kappa(M_2).$$

Moreover, we can show the following.

**Theorem 3.4** ([14, Theorem 7.13]). $r_\kappa : C_{g,1} \to GL(2g, \mathcal{K}_{N_k})$ factors through $H_{g,1}$.

Consequently, we obtain the Magnus representation $r_\kappa : H_{g,1} \to GL(2g, \mathcal{K}_{N_k})$, which is a crossed homomorphism. Note that if we restrict $r_\kappa$ to $C_{g,1}[k]$ (and $H_{g,1}[k]$), it becomes a homomorphism.

**Example 3.5.** For $\varphi \in M_{g,1} \leftrightarrow \text{Aut}F_{2g}$, we can obtain

$$r_\kappa(M_\varphi) = \frac{\partial \varphi(\gamma_i)}{\partial \gamma_j}$$

where $\rho_\kappa : \mathbb{Z}F_{2g} \to \mathbb{Z}N_k \subset \mathcal{K}_{N_k}$ is the natural map and $\partial/\partial \gamma_i$ are free differentials. From this, we see that $r_\kappa$ generalizes the original Magnus representation for $M_{g,1}$ in [11].

In general, the Magnus matrix $r_\kappa(M)$ of a homology cylinder $M$ can be obtained from a finite presentation of the form

$$\pi_1 M \cong \left\{ \begin{array}{c|c} i_-(\gamma_1), \ldots, i_-(\gamma_{2g}) & i_-(\gamma_1)s_1, \ldots, i_-(\gamma_{2g})s_{2g} \\ z_{1}, \ldots, z_{2g+1} & r_1, \ldots, r_l \\ i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) & i_+(\gamma_1)u_1, \ldots, i_+(\gamma_{2g})u_{2g} \end{array} \right\},$$
where $s_i, r_i$ and $u_i$ are words in $z_1, \ldots, z_{2g+1}$, by a purely algebraic calculation. Note that such a presentation does exist for each homology cylinder.

As in the case of $\mathcal{M}_{g,1}$ (see [11] and [18]), the Magnus representation for $\mathcal{H}_{g,1}$ satisfies the following “symplectic” property.

**Theorem 3.6.** For any homology cylinder $M$, we have the equality

$$r_k(M)^T \overline{J} r_k(M) = \sigma(M) \overline{J},$$

where $\overline{J} = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} \in GL(2g, \mathbb{Z}N_k)$ is defined by

$$J_1 = \begin{pmatrix}
1 - \gamma_1 \\
(1 - \gamma_2)(1 - \gamma_1^{-1}) & 1 - \gamma_2 \\
(1 - \gamma_3)(1 - \gamma_1^{-1})(1 - \gamma_3)(1 - \gamma_2^{-1}) & 1 - \gamma_3 \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \gamma_g)(1 - \gamma_1^{-1})(1 - \gamma_g)(1 - \gamma_2^{-1}) & \cdots & 1 - \gamma_g
\end{pmatrix},$$

$$J_2 = \begin{pmatrix}
\gamma_1 \gamma_{g+1}^{-1} \\
(1 - \gamma_2)(1 - \gamma_{g+1}^{-1}) & \gamma_2 \gamma_{g+1}^{-1} \\
(1 - \gamma_3)(1 - \gamma_{g+1}^{-1})(1 - \gamma_3)(1 - \gamma_{g+2}^{-1}) & \gamma_3 \gamma_{g+2}^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \gamma_g)(1 - \gamma_{g+1}^{-1})(1 - \gamma_g)(1 - \gamma_{g+2}^{-1}) & \cdots & \gamma_g \gamma_{2g}^{-1}
\end{pmatrix},$$

$$J_3 = \begin{pmatrix}
1 - \gamma_1^{-1} - \gamma_g \\
(1 - \gamma_g^{-1})(1 - \gamma_1^{-1}) & 1 - \gamma_1^{-2} - \gamma_g^{-2} \\
(1 - \gamma_g^{-1})(1 - \gamma_3^{-1})(1 - \gamma_g^{-1})(1 - \gamma_3^{-1}) & 1 - \gamma_3^{-1} - \gamma_g^{-3} \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \gamma_{2g}^{-1})(1 - \gamma_1^{-1})(1 - \gamma_{2g}^{-1})(1 - \gamma_2^{-1}) & \cdots & 1 - \gamma_{2g}^{-1} - \gamma_2^{-1}
\end{pmatrix},$$

$$J_4 = \begin{pmatrix}
1 - \gamma_{g+1}^{-1} \\
(1 - \gamma_g^{-1})(1 - \gamma_{g+1}^{-1}) & 1 - \gamma_g^{-2} \\
(1 - \gamma_g^{-1})(1 - \gamma_{g+2}^{-1})(1 - \gamma_g^{-1})(1 - \gamma_{g+2}^{-1}) & 1 - \gamma_{g+2}^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \gamma_{2g}^{-1})(1 - \gamma_{g+1}^{-1})(1 - \gamma_{2g}^{-1})(1 - \gamma_{2g}^{-1}) & \cdots & 1 - \gamma_{2g}^{-2}
\end{pmatrix}.$$

Note that the matrix $\overline{J}$ appeared in Papakyriakopoulos' paper [12], and that it is mapped to the ordinary symplectic matrix by the augmentation map $\mathbb{Z}N_k \to \mathbb{Z}$.

**Sketch of Proof.** First we define a natural pairing

$$\langle \cdot, \cdot \rangle : H_1(\Sigma_{g,1}, p; \mathcal{K}_N) \times H_1(\Sigma_{g,1}, p; \mathcal{K}_N) \to \mathcal{K}_N$$

satisfying

$$\langle af, b \rangle = \overline{f}(a, b), \quad \langle a, bf \rangle = \langle a, b \rangle f$$

for all $f \in \mathcal{K}_N$. This generalizes Suzuki's higher intersection form in [18]. To construct it, we use the following type of the Poincaré-Lefschetz duality: Let $X$ be a compact oriented $n$-manifold whose boundary $\partial M$ is decomposed as the union of two compact manifolds $A$ and $B$.
with $\partial A = \partial B = A \cap B$, and let $M$ be a local coefficient system on $X$. Then the cap product with a fundamental class gives isomorphisms $H^k(X; A; M) \cong H_{n-k}(X, B; M)$ for all $k$.

The naturality of the Poincaré-Lefschetz duality shows the equality

$$\langle r_k(M)a, r_k(M)b \rangle = \sigma_k(M)\langle a, b \rangle$$

for each homology cylinder $M$. By writing down this equality with respect to the basis $(\gamma_1 \otimes 1, \ldots, \gamma_{2g} \otimes 1)$ of $H_1(\Sigma_{g,1}, p; \mathcal{K}_n)$, where we use Papakyriakopoulos' argument in [12], we obtain the desired equality.

4. Example: Relationship to the Gassner representation for string links

In [9], Levine gave a method for constructing homology cylinders from pure string links. By this, we can obtain many homology cylinders not belonging to the subgroup $M_{g,1}$. Also, we can see a relationship between the Gassner representation for string links and our representation.

For a $g$-component pure string link $L \subset D^2 \times I$, we now construct a homology cylinder $M_L \in C_{g,1}$ as follows. Consider a closed tubular neighborhood of the loops $\gamma_{g+1}, \gamma_{g+2}, \ldots, \gamma_{2g}$ in Figure 1 to be the image of an embedding $\iota : D_g \hookrightarrow \Sigma_{g,1}$ of a $g$-holed disk $D_g$ as in Figure 2.

![Figure 2](image)

Let $C$ be the complement of an open tubular neighborhood of $L$ in $D^2 \times I$. For each choice a framing of $L$, a homeomorphism $h : \partial C \cong \partial(\iota(D_g) \times I)$ is fixed. Then the manifold $M_L$ given from $\Sigma_{g,1} \times I$ by removing $\iota(D_g) \times I$ and regluing $C$ by $h$ becomes a homology cylinder. This construction gives an injective monoid homomorphism $L_g \to C_{g,1}$ from the monoid $L_g$ of (framed) pure string links to $C_{g,1}$. Moreover it also induces an injective homomorphism $S_g \to \mathcal{H}_{g,1}$ from the concordance group of (framed) pure string links to $\mathcal{H}_{g,1}$. In particular, the (smooth) knot concordance group, which coincides with $S_1$, is embedded in $\mathcal{H}_{g,1}$. If we restrict these embeddings to the pure braid group, which is a subgroup of $L_g$ and $S_g$, their images are contained in $M_{g,1}$.

We fix an integer $k \geq 2$. By the Gassner representation, we mean the crossed homomorphism $r_{G,k} : L_g \to GL(g, \mathcal{K}_{N_k}(D_g))$ or $r_{G,k} : S_g \to GL(g, \mathcal{K}_{N_k}(D_g))$ given by a construction similar to that in the previous section. (In [8] and [7], only $r_{G,2}$ is treated.) Comparing methods for calculating the Gassner and the Magnus representations, we obtain the following.

**Theorem 4.1** ([14, Theorem 7.18]). For any pure string link $L \in L_g$, $r_k(M_L) = \begin{pmatrix} * & 0_g \\ * & r_{G,k}(L) \end{pmatrix}$. 
We mention two remarks about this theorem. First we identify $F_g = \pi_1 D_g$ with the subgroup of $F_{2g} = \pi_1 \Sigma_{g,1}$ generated by $\gamma_{g+1}, \ldots, \gamma_{2g}$. Then the maps $F_g = \langle \gamma_{g+1}, \ldots, \gamma_{2g} \rangle \hookrightarrow F_{2g} \twoheadrightarrow F_g$, where the second map sends $\gamma_1, \ldots, \gamma_g$ to 1, show that $N_k(F_g) \subset N_k$ and $\mathcal{K}_n(F_g) \subset \mathcal{K}_n$. Second, the embeddings $L_g \hookrightarrow C_{g,1}$ and $S_g \hookrightarrow \mathcal{H}_{g,1}$ have ambiguity with respect to Ramings. However we can check that the lower right part of $r_k(M_L)$ does not depend on the choice of framings.

**Corollary 4.2.** $M_{g,1}$ is not a normal subgroup of $\mathcal{H}_{g,1}$ for $g \geq 3$.

**Proof.** In [7], they gave 3-component pure string links denoted by $L_5$ and $L_6$ having the condition that $L_5$ is a pure braid, while the conjugate $L_6 L_5 L_5^{-1}$ is not. To show that $L_6 L_5 L_5^{-1}$ is not a pure braid, they use the fact that $r_{G,2}(L_6 L_5 L_5^{-1})$ has an entry not belonging to $\mathbb{Z} N_2(D_3)$. Then our claim follows from Theorem 4.1 with respect to this example. \[\square\]

**Example 4.3.** Let $L$ be a 2-component pure string link as depicted in Figure 3.

\[\text{Figure 3}\]

Then the presentation of $\pi_1 M_L$ is given by

\[
\pi_1 M_L \cong \left\langle i_-(\gamma_1), \ldots, i_-(\gamma_4), i_+(\gamma_1), \ldots, i_+(\gamma_4) \mid i_-(\gamma_1) i_-(\gamma_3) i_-(\gamma_4) i_-(\gamma_1)^{-1} i_+(\gamma_1) i_+(\gamma_3) i_+(\gamma_4) i_+(\gamma_1)^{-1}, i_-(\gamma_1) i_+(\gamma_3) i_+(\gamma_4) i_+(\gamma_1)^{-1} i_-(\gamma_3) i_-(\gamma_4) i_-(\gamma_1)^{-1}, z \right\rangle,
\]

where we use the blackboard framing. We identify $N_2$ and $N_2(M_L)$ by using $i_+$. Using the presentation, we have $z = i_-(\gamma_3) = \gamma_3$, $i_-(\gamma_4) = \gamma_4$, $i_-(\gamma_2) = \gamma_2 \gamma_3$ and $i_-(\gamma_1) = \gamma_1 \gamma_3^{-1} \gamma_4$ in $N_2$. Then we obtain

\[
r_2(M_L) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\gamma_1^{-1} & 1 & \gamma_3^{-1} & \gamma_4^{-1} \\
\gamma_2^{-1} & \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} \\
\gamma_2^{-1} \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} & \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} \\
\gamma_2^{-1} \gamma_3^{-1} & \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} \\
\gamma_2^{-1} \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} \\
\gamma_2^{-1} \gamma_3^{-1} & \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1} & \gamma_3^{-1} \gamma_4^{-1}
\end{pmatrix}
\]

Note that $\det r_2(M_L) = \frac{\gamma_3 + \gamma_4 - 1}{\gamma_3 \gamma_4 (\gamma_3^{-1} + \gamma_4^{-1} - 1)}$. 

5. **Higher-order Alexander invariants and torsion-degree functions**

Here we summarize the theory of higher-order Alexander invariants along the lines of Harvey’s papers [5, 6]. For our use, we generalize them to functions of matrices called *torsion-degree functions*.

A group $\Gamma$ is *poly-torsion-free-abelian* (PTFA, for short) if $\Gamma$ has a normal series of finite length whose successive quotients are all torsion-free abelian. In particular, free nilpotent quotients $N_k$ are PTFA for all $k \geq 2$. Note that any subgroup of a PTFA group is also PTFA.

For each PTFA group $\Gamma$, the group ring $\mathbb{Z}\Gamma$ is known to be an Ore domain, so that it can be embedded in the *right field of fractions* $\mathcal{K}_\Gamma := \mathbb{Z}(\mathbb{Z}\Gamma - \{0\})^{-1}$, which is a skew field. We refer to [2, 13] for localizations of non-commutative rings.

We will also use the following localizations of $\mathbb{Z}\Gamma$ placed between $\mathbb{Z}\Gamma$ and $\mathcal{K}_\Gamma$. Let $\psi \in H^1(\Gamma)$ be a primitive element. This means the corresponding homomorphism, which is denoted by $\psi$ again, under $H^1(\Gamma) \cong \text{Hom}(\Gamma, \mathbb{Z})$ is onto. Then we have an exact sequence

$$1 \longrightarrow (\mathcal{I}^\psi := \text{Ker} \psi) \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 1.$$ 

We take a splitting $\xi : \mathbb{Z} \rightarrow \Gamma$ of this sequence and put $t := \xi(1) \in \Gamma$. Since $\mathcal{I}^\psi$ is again a PTFA group, $\mathbb{Z}\mathcal{I}^\psi$ can be embedded in its right field of fractions $\mathcal{K}_{\mathcal{I}^\psi} := \mathbb{Z}\mathcal{I}^\psi(\mathbb{Z}\mathcal{I}^\psi - \{0\})^{-1}$. Moreover, we can construct a right quotient ring $\mathbb{Z}\mathcal{I}^\psi(\mathbb{Z}\mathcal{I}^\psi - \{0\})^{-1}$. Then the splitting $\xi$ gives an isomorphism between $\mathbb{Z}\mathcal{I}^\psi(\mathbb{Z}\mathcal{I}^\psi - \{0\})^{-1}$ and the skew Laurent polynomial ring $\mathcal{K}_{\mathcal{I}^\psi}[t^\pm]$, in which $at = t(r^{-1}at)$ holds for each $a \in \Gamma$. $\mathcal{K}_{\mathcal{I}^\psi}[t^\pm]$ is known to be a non-commutative right and left principal ideal domain. By definition, we have inclusions

$$\mathbb{Z}\Gamma \hookrightarrow \mathcal{K}_{\mathcal{I}^\psi}[t^\pm] \hookrightarrow \mathcal{K}_\Gamma.$$ 

$\mathcal{K}_{\mathcal{I}^\psi}[t^\pm]$ and $\mathcal{K}_\Gamma$ are known to be flat $\mathbb{Z}\Gamma$-modules. On $\mathcal{K}_{\mathcal{I}^\psi}[t^\pm]$, we have a map $\text{deg}^\psi : \mathcal{K}_{\mathcal{I}^\psi}[t^\pm] \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ assigning to each polynomial its degree. We put $\text{deg}^\psi(0) := \infty$. Note that the composite $\mathbb{Z}\Gamma(\mathbb{Z}\mathcal{I}^\psi - \{0\})^{-1} \rightarrow \mathcal{K}_{\mathcal{I}^\psi}[t^\pm] \xrightarrow{\text{deg}^\psi} \mathbb{Z}_{\geq 0} \cup \{\infty\}$ does not depend on the choice of the splitting $\xi$.

Harvey’s higher-order Alexander invariants [6] are defined as follows. Let $G$ be a finitely presentable group, and let $\varphi : G \rightarrow \mathbb{Z}$ be an epimorphism. For a PTFA group $\Gamma$ and an epimorphism $\varphi : G \rightarrow \Gamma$, $\langle \varphi, \varphi \rangle$ is called an *admissible pair* for $G$ if there exists an epimorphism $\psi : \Gamma \twoheadrightarrow \mathbb{Z}$ satisfying $\varphi = \psi \circ \varphi$. For each admissible pair $\langle \varphi, \varphi \rangle$ for $G$, we regard $\mathcal{K}_{\mathcal{I}^\psi}[t^\pm] = \mathbb{Z}\mathcal{I}^\psi(\mathbb{Z}\mathcal{I}^\psi - \{0\})^{-1}$ as a $\mathbb{Z}G$-module, and we define the higher-order Alexander invariant for $\langle \varphi, \varphi \rangle$ by

$$\delta^\psi_{\mathcal{I}^\psi}(G) = \dim \mathcal{K}_{\mathcal{I}^\psi}(H_1(G; \mathcal{K}_{\mathcal{I}^\psi}[t^\pm])) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$ 

$\delta^\psi_{\mathcal{I}^\psi}(G)$ is also called the $\Gamma$-*degree*. Note that the right $\mathcal{K}_{\mathcal{I}^\psi}[t^\pm]$-module $H_1(G; \mathcal{K}_{\mathcal{I}^\psi}[t^\pm])$ are decomposed into

$$H_1(G; \mathcal{K}_{\mathcal{I}^\psi}[t^\pm]) = (\mathcal{K}_{\mathcal{I}^\psi}[t^\pm])^0 \bigoplus_{\nu \neq 0} \frac{\mathcal{K}_{\mathcal{I}^\psi}[t^\pm]}{p(\nu)\mathcal{K}_{\mathcal{I}^\psi}[t^\pm]}.$$ 

---

1Our definition is slightly different from that in [6].
for some \( r \in \mathbb{Z}_{\geq 0} \) and \( p_i(t) \in \mathcal{K}_{r^k}[t^*] \), and then

\[
\overline{\sigma}^R_r(G) = \begin{cases} 
\sum_{i=1}^r \deg^R(p_i(t)) & (r = 0), \\
\infty & (r > 0)
\end{cases}
\]

For a space \( X \) and an admissible pair \( \pi_1 X \), we define \( \overline{\sigma}^R_r(X) := \overline{\sigma}^R_r(\pi_1 X) \).

For a finitely presentable group \( G \) and an admissible pair \( (\varphi, \varphi) \) for \( G \). The \( \Gamma \)-degree can be computed from any presentation matrix of the right \( \mathcal{K}_{r^k}[t^*] \)-module \( H_1(G; \mathcal{K}_{r^k}[t^*]) \). Therefore we can consider it to be a \( \mathbb{Z}_{\geq 0} \)-valued function on the set \( M(\mathcal{K}_{r^k}[t^*]) \) of all matrices with entries in \( \mathcal{K}_{r^k}[t^*] \). In [14] (see also [16]), we extended this function to

\[
\overline{\sigma}^R_r : M(\mathcal{K}_{r^k}) \to \mathbb{Z} \cup \{\infty\}
\]
called the (truncated) torsion-degree function by using Reidemeister torsions and the Dieudonné determinant \( \det : GL(\mathcal{K}_{r^k}) \to (\mathcal{K}_{r^k})_{ab} \), where \( (\mathcal{K}_{r^k})_{ab} \) is the abelianization of the multiplicative group \( \mathcal{K}_{r^k} = \mathcal{K}_{r^k} - \{0\} \). The torsion-degree function is defined for each pair of a PTFA group \( \Gamma \) and an epimorphism \( \psi : \Gamma \to \mathbb{Z} \). It can be regarded as a generalization of the extension of \( \deg^R : \mathcal{K}_{r^k}[t^*] \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \) to \( \deg^R : \mathcal{K}_{r^k} \to \mathbb{Z} \cup \{\infty\} \) by setting \( \deg^R(fg^{-1}) = \deg^R(f) - \deg^R(g) \) for \( f \in \mathcal{K}_{r^k}, g \in \mathcal{K}_{r^k} - \{0\} \) (see Proposition 9.1.1 in [2], for example). It induces a group homomorphism \( \deg^R : (\mathcal{K}_{r^k})_{ab} \to \mathbb{Z} \).

Torsion-degree functions have the following properties.

**Proposition 5.1.** (1) For \( A \in GL(\mathcal{K}_{r^k}) \), we have \( \overline{\sigma}^R_r(A) = \deg^R(\det A) \). In particular, \( \overline{\sigma}^R_r(A) = 0 \) for any \( A \in GL(\mathcal{K}_{r^k}[t^*]) \).

(2) Let \( M \) be a finitely generated right \( \mathcal{K}_{r^k}[t^*] \)-module presented by a matrix \( A \in M(\mathcal{K}_{r^k}[t^*]) \). Then

\[
\overline{\sigma}^R_r(A) = \begin{cases} 
\dim \mathcal{K}_{r^k}(T_{\mathcal{K}_{r^k}[t^*]} M) & (\text{rank} \mathcal{K}_{r^k}[t^*](F_{\mathcal{K}_{r^k}[t^*]} M) \leq 1), \\
\infty & \text{(otherwise)}
\end{cases}
\]

where \( T_{\mathcal{K}_{r^k}[t^*]} M \) (resp. \( F_{\mathcal{K}_{r^k}[t^*]} M \)) is the \( \mathcal{K}_{r^k}[t^*] \)-torsion (resp. \( \mathcal{K}_{r^k}[t^*] \)-free) part of \( M \).

Let \( G \) be a finitely presentable group and we take a presentation \( \langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle \) of \( G \). For each admissible pair \( (\varphi, \varphi) \) for \( G \), the Jacobi matrix \( A := \left( \frac{\partial r_j}{\partial x_i} \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \) at \( \mathcal{K}_{r^k}[t^*] \) gives a presentation matrix of \( H_1(G, \{1\}; \mathcal{K}_{r^k}[t^*]) \). Then the \( \Gamma \)-degree is given by

\[
\overline{\sigma}^R_r(G) = \dim \mathcal{K}_{r^k}(H_1(G; \mathcal{K}_{r^k}[t^*])) = \overline{\sigma}^R_r(A),
\]

where the second equality follows from the direct sum decomposition

\[
H_1(G, \{1\}; \mathcal{K}_{r^k}[t^*]) \cong H_1(G; \mathcal{K}_{r^k}[t^*]) \oplus \mathcal{K}_{r^k}[t^*]
\]
given by Harvey in [5].

6. **Applications of Torsion-Degree Functions to Homology Cylinders**

In this section, we study some invariants of homology cylinders arising from the Magnus representation, twisted homology groups of related manifolds and torsion-degree functions. In [14], we can see other applications.
6.1. Torsion-degrees of Magnus matrices. First, we consider torsion-degree functions associated to nilpotent quotients $N_k$ of $\pi_1\Sigma_{g,1}$, and apply them to Magnus matrices. Since $H_1(N_k) = H_1(\Sigma_{g,1})$ and $H^1(N_k) = H^1(\Sigma_{g,1})$, taking an epimorphism $N_k \to \mathbb{Z}$, which is needed in the definition of a torsion-degree function, is done by choosing a primitive element of $H^1(\Sigma_{g,1})$.

**Theorem 6.1.** Let $M$ be a homology cylinder. For any $k \geq 2$ and any primitive element $\psi \in H^1(\Sigma_{g,1})$, the torsion-degree $\overline{d}^\psi_{N_k}(r_k(M))$ is always zero.

**Proof.** Proposition 5.1 (1) shows that torsion-degrees are additive for products of invertible matrices and vanish for those in $GL(\mathbb{Z}N_k)$. It can be also checked that they are invariant under taking the transpose and operating the involution. Hence, by applying the torsion-degree function to the equality $\overline{r}_k(M)^\top \overline{J} \overline{r}_k(M) = \sigma^{(M)} \overline{J}$ in Theorem 3.6, we obtain $2 \overline{d}^\psi_{N_k}(r_k(M)) = 0$. This completes the proof. \[\square\]

**Example 6.2.** Consider the homology cylinder $M_L$ in Example 4.3. $\overline{d}^\psi_{N_k}(r_2(M_L))$ is given by the degree of $\det r_2(M_L) = \frac{2^{g-1}g^{-1}}{2^{g}-1}$ with respect to $\psi$. It can be easily checked that it is zero.

**Remark 6.3.** In [14], we defined the Magnus representation $r_k : \text{Aut}^{\text{acy}}_n \to GL(n, \mathcal{K}_{N_k}(\alpha))$ for $\text{Aut}^{\text{acy}}_n$, where $\mathcal{K}^{\text{acy}}_n$ is a completion of $F_n$ in a certain sense and is called the acyclic closure of $F_n$. The natural map $F_n \to \mathcal{K}^{\text{acy}}_n$ is known to be injective and 2-connected. In particular, $N_k(F_n) = N_k(F_n^{\text{acy}})$. $\text{Aut}^{\text{acy}}_n$ can be regarded as an enlargement of $\text{Aut}^{\text{acy}}_n$, and we have the enlarged Dehn-Nielsen homomorphism $\sigma^{\text{acy}} : \mathcal{H}_{g,1} \to \text{Aut}^{\text{acy}}_{2g}$ extending the classical one $\sigma : \mathcal{M}_{g,1} \to \text{Aut}^{\text{acy}}_{2g}$. (Note that $\sigma^{\text{acy}}$ is not injective.) That is, we have the following commutative diagram.

$$\begin{array}{ccc}
\text{Aut}^{\text{acy}}_{2g} & \xrightarrow{\uparrow \sigma} & \text{Aut}^{\text{acy}}_{2g} \\
\uparrow \sigma & & \uparrow \sigma^{\text{acy}} \\
\mathcal{M}_{g,1} & \xrightarrow{\sigma} & \mathcal{H}_{g,1}
\end{array}$$

The Magnus representation for homology cylinders is nothing other than the composite $\mathcal{H}_{g,1}^{\text{acy}} \xrightarrow{\sigma^{\text{acy}}} \text{Aut}^{\text{acy}}_{2g} \xrightarrow{\uparrow r_k} GL(2g, \mathcal{K}_{N_k})$. We can easily check that $\overline{d}^\psi_{N_k} \circ r_k : \text{Aut}^{\text{acy}}_{2g} \to GL(2g, \mathcal{K}_{N_k})$ is non-trivial. Therefore $\overline{d}^\psi_{N_k} \circ r_k$ gives an invariant of $\text{Aut}^{\text{acy}}_n$ which vanishes on $\mathcal{M}_{g,1}$, $\text{Aut}^{\text{acy}}_n$ and $\mathcal{H}_{g,1}$ for each $k \geq 2$ and each primitive element $\psi \in H^1(N_k)$.

6.2. Factorization formula of $N_k,T$-degree for the mapping torus of a homology cylinder. For each homology cylinder $M = (M, i_+, i_-)$, we can construct a closed 3-manifold $T_M$ as follows. First we attach a 2-handle $I \times D^2$ along $I \times i_+(\partial \Sigma_{g,1})$, so that we obtain a homology cylinder $(M', i_+', i_-')$ over a closed surface $\Sigma_g$, which corresponds to the embedding $\Sigma_{g,1} \hookrightarrow \Sigma_g$. Then we put $T_M := M'/(i_+(x) = i_-'(x)), \quad x \in \Sigma_g$.

We call $T_M$ the mapping torus of $M$. Indeed, for $M_{g,1} \in \mathcal{C}_{g,1}$, the resulting manifold $T_M$ is nothing other than the usual mapping torus of $\varphi$ extended naturally to the mapping class of $\Sigma_g$. If $M \in \mathcal{C}_{g,1}[[k]]$, we have natural isomorphisms $N_k(\Sigma_g) \cong N_k(M')$ and $N_k(T_M) \cong N_k(\Sigma_g) \times \langle \lambda \rangle$.
Note that these groups are torsion-free nilpotent (hence PTFA). We consider $N_k(Σ_g)$ to be a subgroup of $N_k(T_M)$. For simplicity, we denote $N_k(T_M)$ by $N_{k,T}$. By an argument similar to that in Lemma 3.1, we can show that $H_*(M,i_*(Σ_g,1);K_{N_{k,T}}) = 0$. Hence we can define the Reidemeister torsion

$$\tau_{N_{k,T}}(M) := \tau(C_*(M,i_*(Σ_g,1);K_{N_{k,T}})) \in K_1(K_{N_{k,T}})/(±N_{k,T})$$

(See [10], [19] for generalities of Reidemeister torsions) Then we obtain the following factorization formula of $N_{k,T}$-degree for the mapping torus of a homology cylinder.

**Theorem 6.4** ([14, Theorem 11.6]). Let $M$ be a homology cylinder belonging to $C_{g,1}[k]$. (1) For each primitive element $ψ \in H^1(N_{k,T}) = H^1(T_M)$, the $N_{k,T}$-degree $δ_{N_{k,T}}(M)$ is finite.

(2) We have the equality

$$δ_{N_{k,T}}(M) = δ_{N_{k,T}}(ψ) + δ_{N_{k,T}}(λI_g - r_{k,T}(M)\overline{r_{k,T}(M)}) - 2|ψ(λ)|,$$

where $r_{k,T} : H_{g,1} → GL(2g,K_{N_{k,T}})$ is defined similarly to the Magnus representation $r_k$. 

**Remark 6.5** (The case of $k = 2$). Since $ZN_{2,T} = ZN_2(T_M)$ and $K_{N_{2,T}} = K_{N_2(T_M)}$ are commutative, we can use the ordinary determinant to calculate the invariants seen above. For $M \in C_{g,1}[2]$, we write $Δ_{T_M}$ for the Alexander polynomial of $T_M$. By a straightforward computation, we have

$$Δ_{T_M} = \tau_{N_{2,T}}(M) \cdot \det (λI_g - r_{2,T}(M)\overline{r_{2,T}(M)}) \cdot (1 - λ)^{-2},$$

where $\approx$ means that these equalities hold in $K_{N_2(T_M)}$ up to $±N_2(T_M)$. 

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### References


[14] T. Sakasai, Mapping class groups, groups of homology cobordisms of surfaces and invariants of 3-manifolds. Part II: Groups of homology cobordisms of a surface, Doctoral dissertation, the University of Tokyo (2006)

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