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Kyoto University
ON THE JOHNSON HOMOMORPHISM OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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ABSTRACT. In this paper we construct new obstructions for the surjectivity of the Johnson homomorphism of the automorphism group of a free group. We also determine the structure of the cokernel of the Johnson homomorphism for degrees 2 and 3.

1. Introduction

Let $F_n$ be a free group of rank $n \geq 2$ and $F_n = \Gamma_n(1), \Gamma_n(2), \ldots$ its lower central series. We denote by $\text{Aut} F_n$ the group of automorphisms of $F_n$. For each $k \geq 0$, let $A_n(k)$ be the group of automorphisms of $F_n$ which induce the identity on the quotient group $F_n/\Gamma_n(k+1)$. Then we have a descending filtration

$$\text{Aut} F_n = A_n(0) \supset A_n(1) \supset A_n(2) \supset \cdots$$

of $\text{Aut} F_n$. This filtration was introduced in 1963 with a remarkable pioneer work by S. Andreadakis [1] who showed that $A_n(1)$, $A_n(2)$, \ldots is a descending central series of $A_n(1)$ and each graded quotient $\text{gr}^k(A_n) = A_n(k)/A_n(k+1)$ is a free abelian group of finite rank. He [1] also computed that $\text{rank}_Z \text{gr}^k(A_2)$ for all $k \geq 1$ and $\text{rank}_Z \text{gr}^2(A_3)$, and asserted $\text{rank}_Z \text{gr}^3(A_3) = 44$. In Section 5, however, we show that $\text{gr}^3(A_3) = 43$. Moreover, by a recent remarkable work by A. Pettet [15] we have $\text{rank}_Z \text{gr}^2(A_n) = \frac{1}{3}n^2(n^2-4) + \frac{1}{2}n(n-1)$ for all $n \geq 3$. However, it is difficult to compute the rank of $\text{gr}^k(A_n)$.

Let $H$ be the abelianization of $F_n$ and $H^* = \text{Hom}_Z(H, Z)$ the dual group of $H$. Let $\mathcal{L}_n = \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ be the free graded Lie algebra generated by $H$. Then for each $k \geq 1$, a $GL(n, Z)$-equivariant injective homomorphism

$$\tau_k : \text{gr}^k(A_n) \to H^* \otimes Z \mathcal{L}_n(k+1)$$

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is defined. (For definition, see Section 2.) This is called the $k$-th Johnson homomorphism of Aut $F_n$. The theory of the Johnson homomorphism of a mapping class group of a compact Riemann surface began in 1980 by D. Johnson [6] and has been developed by many authors. There is a broad range of remarkable results for the Johnson homomorphism of a mapping class group. (For example, see [5] and [13].) However, the properties of the Johnson homomorphism of Aut $F_n$ are far from being well understood.

The main interest of this paper is to determine the structure of the cokernel of the Johnson homomorphism $\tau_k$ as a $GL(n, \mathbb{Z})$-module. For $k = 1$, it is a well known fact that the first Johnson homomorphism $\tau_1$ is an isomorphism. (See [8].) For $k \geq 2$, the Johnson homomorphism $\tau_k$ is not surjective. In fact, a recent remarkable work by Shigeyuki Morita indicates that there is a symmetric product $S^kH_{\mathbb{Q}}$ of $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$ in the cokernel of $\tau_{k, \mathbb{Q}} = \tau_k \otimes id_{\mathbb{Q}}$ for each $k \geq 2$. To show this, he introduced a homomorphism

\[ \text{Tr}_k : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k + 1) \to S^kH, \]

called the trace map, and showed that $\text{Tr}_k$ vanishes on the image of $\tau_k$ and is surjective after tensoring with $\mathbb{Q}$ for all $k \geq 2$.

The trace maps were introduced in the 1993 by Morita [12] for a Johnson homomorphism of a mapping class group of a surface. He called these maps traces because they were defined using the trace of some matrix representation. Morita's traces are very important to study the Lie algebra structure of the target $H^* \otimes_{\mathbb{Z}} \mathcal{L}_n = \text{Der}(\mathcal{L}_n)$ of the Johnson homomorphisms. Here $\text{Der}(\mathcal{L}_n)$ denotes the graded Lie algebra of derivations of $\mathcal{L}_n$. Morita conjectured that for any $n \geq 3$, the abelianization of the Lie algebra $\text{Der}(\mathcal{L}_n)$ is given by

\[ H_1(\text{Der}(\mathcal{L}_n^\mathbb{Q})) \simeq (H^*_\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda^2 H_{\mathbb{Q}}) \oplus \bigoplus_{k \geq 2} S^kH_{\mathbb{Q}} \]

where $\mathcal{L}_n^\mathbb{Q} = \mathcal{L}_n \otimes_{\mathbb{Z}} \mathbb{Q}$ and the right hand side is understood to be an abelian Lie algebra. Recently, combining a work of Kassabov [7] with the concept of the traces, he [14] showed that the isomorphism above holds up to degree $n(n - 1)$.

The subgroup $\mathcal{A}_n(1)$ is called the IA-automorphism group of $F_n$ and denoted by $IA_n$. The group $IA_n$ is the kernel of the natural map $\text{Aut} F_n \to GL(n, \mathbb{Z})$ which is given by the action of $\text{Aut} F_n$ on $H$. The structures
of $IA_n$ plays an important role in the study $Aut F_n$. W. Magnus [10] showed that $IA_n$ is finitely generated for all $n \geq 3$. However, it is not known whether $IA_n$ is finitely presented or not for any $n \geq 4$. For $n = 3$, by a remarkable work by S. Krstić and J. McCool [9], it is known that $IA_3$ is not finitely presented. On the other hand, the abelianization of $IA_n$ is given by

$$IA_n^{ab} \simeq H^* \otimes_{\mathbb{Z}} \Lambda^2 H$$

as a $GL(n, \mathbb{Z})$-module. (See [8].

Now let $\mathcal{A}_n'(1)$, $\mathcal{A}_n'(2)$, ... be the lower central series of $IA_n = \mathcal{A}_n(1)$ and $\text{gr}^k(\mathcal{A}_n')$ its graded quotient of it for each $k \geq 1$. In Section 2, we define a $GL(n, \mathbb{Z})$-equivariant homomorphism

$$\tau^k_k : \text{gr}^k(\mathcal{A}_n') \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k + 1)$$

which is also called the $k$-th Johnson homomorphism of $Aut F_n$. In this paper, we construct new obstructions of the surjectivity of the Johnson homomorphism $\tau^k_k$. Let us denote the tensor products with $\mathbb{Q}$ of a $\mathbb{Z}$-module by attaching a subscript $\mathbb{Q}$ to the original one. For example, $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathcal{L}_{\mathbb{Q}}^n(k) := \mathcal{L}_n(k) \otimes_{\mathbb{Z}} \mathbb{Q}$. Similarly, for a $\mathbb{Z}$-linear map $f : A \to B$ we denote by $f_{\mathbb{Q}}$ the $\mathbb{Q}$-linear map $A_{\mathbb{Q}} \to B_{\mathbb{Q}}$ induced by $f$.

It is conjectured that $\text{Coker} \tau^k_{k, \mathbb{Q}} = \text{Coker} \tau^k_{k, \mathbb{Q}}$ for $k \geq 1$. It is true for $1 \leq k \leq 3$. In fact, $\mathcal{A}_n(1) = \mathcal{A}_n'(1)$ by definition. We have $\mathcal{A}_n(2) = \mathcal{A}_n'(2)$ from the result stated above. (See [8].) Moreover, Pettet [15] showed that $\mathcal{A}_n'(3)$ has a finite index in $\mathcal{A}_n(3)$. Hence, $\text{Coker} \tau^k_{k, \mathbb{Q}} = \text{Coker} \tau^k_{k, \mathbb{Q}}$ for $1 \leq k \leq 3$. Our main result is

**Theorem 1.**

1. $\Lambda^k H_{\mathbb{Q}} \subset \text{Coker} \tau^k_{k, \mathbb{Q}}$ for odd $k$ and $3 \leq k \leq n$.

2. $H_{\mathbb{Q}}^{[2, 1^{k-2}]} \subset \text{Coker} \tau^k_{k, \mathbb{Q}}$ for even $k$ and $4 \leq k \leq n - 1$.

Here $\Lambda^k H_{\mathbb{Q}}$ denotes the $k$-th exterior product of $H_{\mathbb{Q}}$, and $H_{\mathbb{Q}}^{[2, 1^{k-2}]}$ denotes the Schur-Weyl module of $H_{\mathbb{Q}}$ corresponding to the partition $[2, 1^{k-2}]$.

In order to prove this, in Section 3, we introduce homomorphisms defined by

$$\text{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k + 1) \to \Lambda^k H,$$

$$\text{Tr}_{[2, 1^{k-1}]} := (id_H \otimes f_{[1^{k-1}]}) \circ \Phi_2^k : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k + 1) \to H \otimes_{\mathbb{Z}} \Lambda^{k-1} H$$

and show that these maps vanish on the image of the Johnson homomorphism $\tau^k_k$. Since these maps are constructed in a way similar to that of Morita's trace $\text{Tr}_k$, we also call these maps traces.
In Section 5, we determine the $GL(n, \mathbb{Z})$-module structure of the cokernel of the Johnson homomorphism $\tau_k$ for 2 and 3. Our result is

**Theorem 2.** We have $GL(n, \mathbb{Z})$-equivariant exact sequences

\[ 0 \to \text{gr}^2(A_n) \xrightarrow{\tau_2} H^* \otimes \mathcal{L}_n(3) \to S^2H \to 0 \]

and

\[ 0 \to \text{gr}^3(A_n) \xrightarrow{\tau_3} H^*_Q \otimes \mathcal{L}_n^Q(4) \to S^3H_Q \oplus \Lambda^3H_Q \to 0 \]

for $n \geq 3$.

Thus we have

**Corollary 1.** For $n \geq 3$,

\[ \text{rank}_\mathbb{Z} \text{gr}^3(A_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8). \]

2. Preliminaries

In this section we review some basic facts. First, we note that the group $\text{Aut} F_n$ acts on $F_n$ on the right. For any $\sigma \in \text{Aut} F_n$ and $x \in F_n$, the action of $\sigma$ on $x$ is denoted by $x^\sigma$.

**2.1. Commutators of higher weight.**

In this paper, we often use basic facts of commutator calculus. The reader is referred to [11] and [16], for example. Let $G$ be a group. For any elements $x$ and $y$ of $G$, the element

\[ xyx^{-1}y^{-1} \]

is called the commutator of $x$ and $y$, and denoted by $[x, y]$. In general, a commutator of higher weight is recursively defined as follows. First, a commutator of weight 1 is an element of $G$. For $k > 1$, a commutator of weight $k$ is an element of the type $C = [C_1, C_2]$ where $C_j$ is a commutator of weight $a_j$ ($j = 1, 2$) such that $a_1 + a_2 = k$. The weight of the commutator $C$ is denoted by $\text{wt}(C) = k$. The commutator which has elements $g_1, \ldots, g_t \in G$ in the bracket components is called the commutator among the components $g_1, \ldots, g_t$. For elements $g_1, \ldots, g_t \in G$, a commutator of weight $k$ among the components $g_1, \ldots, g_t$ of the type

\[ [[\cdots [[g_{i_1}, g_{i_2}], g_{i_3}], \cdots], g_{i_k}], \quad i_j \in \{1, \ldots, t\} \]

with all of its brackets to the left of all the elements occurring is called a simple $k$-fold commutator and is denoted by

\[ [g_{i_1}, g_{i_2}, \cdots, g_{i_k}]. \]
For each $k \geq 1$, the subgroups $\Gamma_G(k)$ of the lower central series of $G$ are defined recursively by

$$\Gamma_G(1) = G, \quad \Gamma_G(k + 1) = [\Gamma_G(k), G].$$

We use the following basic lemma in later sections.

**Lemma 2.1.** If a group $G$ is generated by $g_1, \ldots, g_t$, then each of the graded quotients $\Gamma_G(k)/\Gamma_G(k + 1)$ for $k \geq 1$ is generated by the cosets of the simple $k$-fold commutators

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}], \quad i_j \in \{1, \ldots, t\}.$$

Now, for each $k \geq 1$, let $\Gamma_n(k)$ be the $k$-th subgroup $\Gamma_{F_n}(k)$ of the lower central series of a free group $F_n$ of rank $n$ and $\text{gr}^k(\Gamma_n)$ its graded quotient $\Gamma_n(k)/\Gamma_n(k + 1)$. We denote by $\text{gr}(\Gamma_n) = \bigoplus_{k \geq 1} \text{gr}^k(\Gamma_n)$ the associated graded sum. Then the set $\text{gr}(\Gamma_n)$ naturally has a structure of a graded Lie algebra over $\mathbb{Z}$ induced by the commutator bracket on $F_n$. Let $H$ be the abelianization of $F_n$ and $\mathcal{L}_n = \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ the free graded Lie algebra generated by $H$. It is well known that the Lie algebra $\text{gr}(\Gamma_n)$ is isomorphic to $\mathcal{L}_n$ as a graded Lie algebra over $\mathbb{Z}$. Thus, in this paper, we identify $\text{gr}(\Gamma_n)$ with $\mathcal{L}_n$. For any element $x \in \Gamma_n(k)$, we also denote by $x$ the coset class of $x$ in $\mathcal{L}_n(k) = \Gamma_n(k)/\Gamma_n(k + 1)$. Let $T(H)$ be the tensor algebra of $H$ over $\mathbb{Z}$. Then the algebra $T(H)$ is the universal envelopping algebra of the free Lie algebra $\mathcal{L}_n$ and the natural map $\mathcal{L}_n \rightarrow T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for $X, Y \in \mathcal{L}_n$ is an injective Lie algebra homomorphism. Hence we also regard $\mathcal{L}_n(k)$ as a submodule of $H^\otimes k$ for each $k \geq 1$.

### 2.2. IA-automorphism group.

The kernel of the natural map $\text{Aut} F_n \rightarrow GL(n, \mathbb{Z})$ which is given by the action of $\text{Aut} F_n$ on $H$ is called the IA-automorphism group of $F_n$ and denoted by $IA_n$. Let $\{x_1, \ldots, x_n\}$ be a basis of a free group $F_n$. Magnus [10] showed that $IA_n$ is finitely generated by automorphisms

$$K_{ab} : \left\{ \begin{array}{c}
 x_a \mapsto x_b^{-1}x_ax_b, \\
 x_t \mapsto x_t, \quad (t \neq a)
 \end{array} \right.$$
and

\[ K_{abc} : \begin{cases} 
  x_a & \mapsto x_a x_b x_c x_b^{-1} x_c^{-1}, \\
  x_t & \mapsto x_t, \quad (t \neq a) 
\end{cases} \]

for any distinct \( a, b \) and \( c \in \{1, 2, \ldots, n\} \). It is known that the abelianization \( IA_n^{ab} \) of the IA-automorphism group is free abelian group with generators \( K_{ab} \) for distinct \( a \) and \( b \), and \( K_{abc} \) for distinct \( a, b, c \) and \( b < c \). More precisely, if we denote by \( H^* = \text{Hom}_\mathbb{Z}(H, \mathbb{Z}) \) the dual group of \( H \), we have a \( GL(n, \mathbb{Z}) \)-module isomorphism \( IA_n^{ab} \simeq H^* \otimes \mathbb{Z} \Lambda^2 H \). (For details, see [8].)

2.3. The associated graded Lie algebra.

Here we consider two descending filtrations of \( IA_n \). The first one is \( \{A_n(k)\}_{k \geq 1} \) defined as above. Since the series \( A_n(1), A_n(2), \ldots \) is central, the associated graded sum \( \text{gr}(A_n) = \bigoplus_{k \geq 1} \text{gr}^k(A_n) \) naturally has a structure of a graded Lie algebra over \( \mathbb{Z} \) induced from the commutator bracket on \( A_n(1) \). For each \( k \geq 1 \), the group \( A_n(0) = \text{Aut} F_n \) naturally acts on \( A_n(k) \) by conjugation, hence on \( \text{gr}^k(A_n) \). Since the group \( A_n(1) = IA_n \) trivially acts on \( \text{gr}^k(A_n) \), we see that the group \( GL(n, \mathbb{Z}) \simeq A_n(0)/A_n(1) \) naturally acts on \( \text{gr}^k(A_n) \).

The other is the lower central series \( \mathcal{A}_n'(1), \mathcal{A}_n'(2), \ldots \) of \( A_n(1) \). Let \( \text{gr}^k(\mathcal{A}_n') = \mathcal{A}_n'(k)/\mathcal{A}_n'(k+1) \) be the graded quotient for each \( k \geq 1 \). Similarly the associated graded sum \( \text{gr}(\mathcal{A}_n') = \bigoplus_{k \geq 1} \text{gr}^k(\mathcal{A}_n') \) has a structure of a graded Lie algebra structure on \( \mathbb{Z} \). Moreover, each graded quotient \( \text{gr}^k(\mathcal{A}_n') \) is a \( GL(n, \mathbb{Z}) \)-module. It is clear that \( \mathcal{A}_n'(k) \subset A_n(k) \) for every \( k \geq 1 \). In particular, we have \( \mathcal{A}_n'(k) = \mathcal{A}_n(k) \) for \( 1 \leq k \leq 2 \) and Pettet [15] showed that \( \mathcal{A}_n'(3) \) has finite index in \( A_n(3) \) as mentioned in section 1. From Lemma 2.1, for each \( k \geq 1 \), the graded quotient \( \text{gr}^k(\mathcal{A}_n') \) is generated by (the cosets of) the simple \( k \)-fold commutators among the components \( K_{ab} \) and \( K_{abc} \).

2.4. Johnson homomorphism.

Here we define the Johnson homomorphisms of \( \text{Aut} F_n \). For each \( k \geq 1 \), let \( \tau_k : A_n(k) \to \text{Hom}_\mathbb{Z}(H, \mathcal{L}_n(k+1)) \) be the map defined by

\[ \sigma \mapsto (x \mapsto x^{-1} x^\sigma) \]

for \( \sigma \in A_n(k) \) and \( x \in H \). Then the map \( \tau_k \) is a homomorphism and the kernel of \( \tau_k \) is just \( A_n(k+1) \). Hence, identifying \( \text{Hom}_\mathbb{Z}(H, \mathcal{L}_n(k+1)) \) with \( H^* \otimes \mathbb{Z} \mathcal{L}_n(k+1) \), we obtain an injective \( GL(n, \mathbb{Z}) \)-equivariant
homomorphism, also denoted by $\tau_k$,

$$\tau_k : \text{gr}^k(A_n) \to H^* \otimes_{\mathbb{Z}} L_n(k+1).$$

This homomorphism is called the $k$-th Johnson homomorphism of $\text{Aut} F_n$. Similarly, for each $k \geq 1$, we can define a homomorphism $\tau'_k : A'_n(k) \to \text{Hom}_{\mathbb{Z}}(H, L_n(k+1))$ as (1). Since $A'_n(k+1)$ is contained in the kernel of $\tau'_k$, we obtain a $GL(n, \mathbb{Z})$-equivariant homomorphism, also denoted by $\tau'_k$,

$$\tau'_k : \text{gr}^k(A'_n) \to H^* \otimes_{\mathbb{Z}} L_n(k+1).$$

We also call the map $\tau'_k$ the Johnson homomorphism of $\text{Aut} F_n$.

Let $\{x_1, \ldots, x_n\}$ be a basis of $F_n$. It defines a basis of $H$ as a free abelian group, also denoted by $\{x_1, \ldots, x_n\}$. Let $\{x_1^*, \ldots, x_n^*\}$ be the dual basis of $H^*$. For any $\sigma \in A'_n(k)$, if we set $s_i(\sigma) := x_i^{-1}x_i^\sigma \in L_n(k+1)$ ($1 \leq i \leq n$) then we have

$$\tau_k(\sigma) = \tau'_k(\sigma) = \sum_{i=1}^{n} x_i^* \otimes s_i(\sigma) \in H^* \otimes_{\mathbb{Z}} L_n(k+1).$$

Let $\text{Der}(L_n)$ be the graded Lie algebra of derivations of $L_n$. The degree $k$ part of $\text{Der}(L_n)$ is expressed as $\text{Der}(L_n)(k) = H^* \otimes_{\mathbb{Z}} L_n(k)$. Thus we sometimes identify $\text{Der}(L_n)$ with $H^* \otimes_{\mathbb{Z}} L_n$. Then the Johnson homomorphism $\tau = \bigoplus_{k \geq 1} \tau_k$ is a graded Lie algebra homomorphism. In fact, if we denote by $\partial \sigma$ the element of $\text{Der}(L_n)$ corresponding to an element $\sigma \in H^* \otimes_{\mathbb{Z}} L_n$ and write the action of $\partial \sigma$ on $X \in L_n$ as $X^{\partial \sigma}$ then we have

(2) $$\tau'_{k+l}([\sigma, \tau]) = \sum_{i=1}^{n} x_i^* \otimes (s_i(\sigma)^{\partial \tau} - s_i(\tau)^{\partial \sigma}).$$

for any $\sigma \in A'_n(k)$ and $\tau \in A'_n(l)$.

In general, each $s_i(\sigma) \in L_n(k+1)$ cannot be uniquely written as a sum of commutators among the components $x_1, \ldots, x_n$. In this paper, each $s_i(\sigma)$ is recursively computed in the following way. First, for $\sigma = K_{abc}$, we can set

$$s_a(K_{abc}) = [x_b, x_c], \quad s_t(K_{abc}) = 0 \text{ if } t \neq a.$$  

For $\sigma = K_{ab}$, we see that

$$x_t^{-1}x_t^\sigma = \begin{cases} [x_a^{-1}, x_b^{-1}] & \text{if } t = a, \\ 1 & \text{if } t \neq a \end{cases}$$
in $F_n$. Since $[x_a^{-1}, x_b^{-1}] = [x_a, x_b]$ in $L_n(2)$, so we can set

$$s_a(K_{ab}) = [x_a, x_b], \ s_t(K_{ab}) = 0 \text{ if } t \neq a.$$  

Next, if $\sigma = [\tau, K_{ab}]$ for $k$-fold simple commutator $\tau$, following from (2), we can set

$$s_i(\sigma) = s_i(\tau)^{\partial K_{ab}} - s_i(K_{ab})^{\partial \tau}$$  

for each $i$. Furthermore, since a commutator bracket of weight $l$ is considered as a $l$-fold multilinear map from the cartesian product of $l$ copies of $L_n(1)$ to $L_n(l)$, we can also set

$$s_i(\sigma) = \sum_{p=1}^{a(i)} (-1)^{e_{i,p}} C_{i,p}$$  

where $e_{i,p} = 0$ or 1, and $C_{i,p}$ is a commutator of degree $k + 1$ among the components $x_1, \ldots, x_n$. We compute $s_i([\tau, K_{abc}])$ for $\sigma = [\tau, K_{abc}]$ similarly. These computations are perhaps easiest explained with examples, so we give two here. For distinct $a, b, c$ and $d$, we have

$$\tau'_2([K_{ab}, K_{bac}]) = x_a^* \otimes ([x_a, x_b])^{\partial K_{bac}} - x_b^* \otimes ([x_a, x_c])^{\partial K_{ab}},$$

$$= x_a^* \otimes [x_a, [x_a, x_c]] - x_b^* \otimes [[x_a, x_b], x_c]$$

and

$$\tau'_3([K_{ab}, K_{bac}, K_{ad}]) = x_a^* \otimes ([x_a, [x_a, x_c]])^{\partial K_{ad}} - x_b^* \otimes ([x_a, x_b], x_c)^{\partial K_{ad}}$$

$$- x_a^* \otimes ([x_a, x_d])^{\partial [K_{ab}, K_{bac}]},$$

$$= x_a^* \otimes [[x_a, x_d], [x_a, x_c]] + x_a^* \otimes [x_a, [[x_a, x_d], x_c]]$$

$$- x_b^* \otimes [[[x_a, x_d], x_b], x_c]$$

$$- x_a^* \otimes [[x_a, [x_a, x_c]], x_d].$$

### 3. The contractions

For $k \geq 1$ and $1 \leq l \leq k + 1$, let $\varphi^k : H^* \otimes_Z H^\otimes(k+1) \to H^\otimes k$ be the contraction map defined by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^* (x_{j_1}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_l} \otimes x_{j_{l+1}} \otimes \cdots \otimes x_{j_{k+1}}.$$  

For the natural embedding $\iota^{k+1}_n : L_n(k + 1) \to H^\otimes(k+1)$, we obtain a $GL(n, Z)$-equivariant homomorphism

$$\Phi^k_I = \varphi^k \circ (id_H \otimes \iota^{k+1}_n) : H^* \otimes_Z L_n(k + 1) \to H^\otimes k.$$  

We also call the map $\Phi^k_I$ contraction.
Here we introduce one of methods of the computation of $\Phi^k_l(x_i^* \otimes C)$ for a commutator $C \in \mathcal{L}_n(k+1)$ among the components $x_1, \ldots, x_n$. In this paper, whenever we compute $\Phi^k_l(x_i^* \otimes C)$, we use the following method. First, if $x_i$ does not appear among the components of $C$, then $\Phi^k_l(x_i^* \otimes C) = 0$. On the other hand, if $x_i$ appears among the components of $C$ $m$ times, then we distinguish them and write such $x_i$'s as $x_{i_1}, \ldots, x_{i_m}$ in $C$. Then $\Phi^k_l(x_i^* \otimes C)$ is given by rewriting $x_{i_1}, \ldots, x_{i_m}$ as $x_i$ in

$$\sum_{j=1}^{m} \Phi^k_l(x_{i_j}^* \otimes C).$$

Thus it suffices to compute $\Phi^k_l(x_i^* \otimes C)$ for a commutator $C$ which has only one $x_i$ in its components. Now, $C$ is written as $[X, Y]$ for some commutators $X$ and $Y$. Rewriting the commutator $C$ as $-[Y, X]$ if $x_i$ appears in $Y$, we may always consider $C = \pm[X, Y]$ such that $x_i$ appears among the components of $X$. By a recursive argument, we have $C = \pm[x_{i_1}, C_1, \ldots, C_t]$ where each $C_j (1 \leq j \leq t)$ is a commutator of weight $d_j$ and $d_1 + \cdots + d_t = k$.

**Lemma 3.1.** For a commutator $[x_i, C_1, \ldots, C_t] \in \mathcal{L}_n(k+1)$ as above,

$$\Phi^k_l(x_i^* \otimes [x_i, C_1, \ldots, C_t]) = C_1 \otimes \cdots \otimes C_t.$$

**Lemma 3.2.** For a commutator $[x_i, C_1, \ldots, C_t] \in \mathcal{L}_n(k+1)$ as above,

$$\Phi^k_l(x_i^* \otimes [x_i, C_1, \ldots, C_t]) = - \sum_{\text{wt}(C_j) = 1} C_j \otimes C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_t.$$

Let $T(H) = \bigoplus_{k \geq 1} H^{\otimes k}$ and $S(H) = \bigoplus_{k \geq 1} S^k H$ be the tensor algebra and the symmetric algebra on $H$ respectively. Then the kernel of a natural map $T(H) \to S(H)$ is a graded ideal of $T(H)$, and denoted by $I(H) = \bigoplus_{k \geq 1} I^k(H)$. For each $k \geq 2$, let $\mathcal{U}_n(k)$ be the $GL(n, \mathbb{Z})$-submodule of $H^{\otimes k}$ generated by elements type of

$$[A, B] := A \otimes B - B \otimes A$$

for $A \in H^{\otimes a}$, $B \in H^{\otimes b}$ and $a + b = k$. If we put $\mathcal{U}_n = \bigoplus_{k \geq 1} \mathcal{U}_n(k)$, then $\mathcal{U}_n$ is the kernel of the abelianization $T(H) \to T(H)^{ab}$ as a Lie algebra. We have

$$\mathcal{L}_n(k) \subset \mathcal{U}_n(k) \subset I^k(H) \subset H^{\otimes k}.$$
3.1. The image of $\Phi_{l}^{k} \circ \tau_{k}'$.

Considering the image of any simple $k$-fold commutator $\sigma$ among the components $K_{ab}$ and $K_{abc}$, we prove the following propositions.

Proposition 3.1. For $n \geq 3$ and $k \geq 2$, $\text{Im} (\Phi_{1}^{k} \circ \tau_{k}') \subset \mathcal{U}_{n}(k)$.

Proposition 3.2. For $n \geq 3$ and $k \geq 3$, $\text{Im} (\Phi_{2}^{k} \circ \tau_{k}') \subset H \otimes_{\mathbb{Z}} \mathcal{U}_{n}(k - 1)$.

4. The trace maps

In this section, using the contractions defined in Section 3, we define a homomorphisms called the trace map which vanishes on the image of the Johnson homomorphism. Here we use some basic facts of the representation theory of $GL(n, \mathbb{Z})$. The reader is referred to, for example, Fulton-Harris [4] and Fulton [3].

For any $k \geq 1$ and any partition $\lambda$ of $k$, we denote by $H^{\lambda}$ the Schur-Weyl module of $H$ corresponding to the partition $\lambda$ of $k$. Let $f_{\lambda} : H^{\otimes k} \to H^{\lambda}$ be a natural homomorphism. In this paper, we mainly consider the case for $\lambda = [k]$ or $[1^k]$. The modules $H^{[k]}$ and $H^{[1^k]}$ are the symmetric product $S^{k}H$ and the exterior product $\Lambda^{k}H$ respectively. Using the natural map $\iota_{n}^{k} : \mathcal{L}_{n}(k) \to H^{\otimes k}$, we denote $f_{[1^k]} \circ \iota_{n}^{k}(C)$ by $\hat{C}$ for any $C \in \mathcal{L}_{n}(k)$.

Lemma 4.1. For any commutator $C$ of weight $k \geq 3$, $\hat{C} = 0$ in $\Lambda^{k}H$.

Lemma 4.2. For $1 \leq k \leq n - 2$ and any commutator $C$ of weight $k + 1$ among the components $x_{1}, \ldots, x_{n}$ except for $x_{i}$, there exists an element $\sigma \in A_{n}'(k)$ such that

$$\tau_{k}'(\sigma) = x_{i}^{*} \otimes C \in H^{*} \otimes_{\mathbb{Z}} \mathcal{L}_{n}(k + 1).$$

4.1. Morita's trace (Trace map for $S^{k}H$).

Here we consider the map

$$\text{Tr}_{[k]} = f_{[k]} \circ \Phi_{1}^{k} : H^{*} \otimes_{\mathbb{Z}} \mathcal{L}_{n}(k + 1) \to S^{k}H.$$  

By definition, this map coincides with the Morita's trace $\text{Tr}_{k}$. For $n \geq 3$ and $k \geq 2$, Morita defined the trace map $\text{Tr}_{k}$ using the Magnus representation of Aut $F_{n}$ and showed that $\text{Tr}_{k}$ vanishes on the image of $\tau_{k}$. By a recent work, he showed that $\text{Tr}_{k}^{Q}$ is surjective. Hence we have

Theorem 4.1. (Morita) For $n \geq 3$ and $k \geq 2$,

$$S^{k}H_{Q} \subset \text{Coker} \tau_{k,Q}.$$
Corollary 4.1. For $n \geq 3$ and $k \geq 2$,
\[
\text{rank}_\mathbb{Z}(\text{Coker } (\tau_k)) \geq \binom{n+k-1}{k}.
\]

4.2. Trace map for $\Lambda^k H$.
Here we consider the map
\[ \text{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi^k_1 : H^* \otimes \mathcal{L}_n (k+1) \to \Lambda^k H. \]

**Theorem 4.2.**
(1) For $3 \leq k \leq n$, $\text{Tr}_{[1^k]}$ is surjective,
(2) $\text{Im } (\text{Tr}_{[1^k]} \circ \tau'_k) = 0$ if $k$ is odd and $3 \leq k \leq n$,
(3) $\text{Im } (\text{Tr}_{[1^k]} \circ \tau'_k) = 2(\Lambda^k H) \subset \Lambda^k H$ if $k$ is even and $4 \leq k \leq n-2$.

**Corollary 4.2.** For an odd $k$ and $3 \leq k \leq n$,
\[ \Lambda^k H_Q \subset \text{Coker } \tau'_{k,Q}. \]

**Corollary 4.3.** For an odd $k$ and $3 \leq k \leq n$,
\[ \text{rank}_\mathbb{Z}(\text{Coker } (\tau'_{k})) \geq \binom{n}{k}. \]

4.3. Trace map for $H^{[2,1^{k-2}]}$.
Here we consider the map
\[ \text{Tr}_{[2,1^{k-1}]} := (\text{id}_H \otimes f^{k-1}_{[1^{k-1}]} \circ \Phi^k_2 : H^* \otimes \mathcal{L}_n (k+1) \to H \otimes \Lambda^{k-1} H. \]

Let $I$ be the $GL(n, \mathbb{Z})$-submodule of $H \otimes \Lambda^{k-1} H$ defined by
\[ I = \langle x \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge y + y \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge x \mid x, y, z_t \in H \rangle. \]

**Theorem 4.3.** For an even $k$ and $4 \leq k \leq n-1$,
(1) $\text{Im } (\text{Tr}^Q_{[2,1^{k-1}]}) = I_Q$,
(2) $\text{Im } (\text{Tr}^Q_{[2,1^{k-1}]} \circ \tau'_k) = 0$.

Now we have $H_Q \otimes \Lambda^{k-1} H_Q \simeq H^{[2,1^{k-2}]}_Q \oplus \Lambda^k H_Q$ from the representation theory of $GL(n, \mathbb{Z})$. For even $k$, since $I_Q$ is contained in the kernel of a natural map $H_Q \otimes \Lambda^{k-1} H_Q \to \Lambda^k H_Q$ defined by $x \otimes y_1 \wedge \cdots \wedge y_{k-1} \mapsto x \wedge y_1 \wedge \cdots \wedge y_{k-1}$, we have $I_Q \simeq H^{[2,1^{k-2}]}_Q$.

**Corollary 4.4.** For an even $k$ and $4 \leq k \leq n-1$,
\[ H^{[2,1^{k-2}]}_Q \subset \text{Coker } \tau'_{k,Q}. \]
Corollary 4.5. For an even $k$ and $4 \leq k \leq n-1$,
\[
\text{rank}_Z(\text{Coker} (\tau'_k)) \geq (k-1)\binom{n+1}{k}.
\]

5. The cokernel of the Johnson homomorphism $\tau_k$ for $k = 2$ and 3

5.1. The case $k = 2$.
In this subsection we consider the case where $n \geq 3$. From Theorem 4.1 and $\text{rank}_Z(\text{Coker} (\tau_2)) = \binom{n+1}{2}$ by Pettet [15], we have a $GL(n, Z)$-equivariant exact sequence
\[
0 \to \text{gr}^2_Q(A_n) \xrightarrow{\tau_2} H^*_Q \otimes Z L^Q_n(3) \to S^2 H_Q \to 0.
\]
We show that the exact sequence above holds before tensoring with $Q$. Namely,

Theorem 5.1. For $n \geq 3$,
\[
0 \to \text{gr}^2(A_n) \xrightarrow{\tau_2} H^* \otimes Z L_n(3) \to S^2 H \to 0
\]
is a $GL(n, Z)$-equivariant exact sequence.

5.2. The case $k = 3$.
Next we compute the cokernel of the Johnson homomorphism $\tau_{3,Q}$ for $n \geq 3$ using the fact that $\text{Coker} \tau_{3,Q} = \text{Coker} \tau'_{3,Q}$. We have

Theorem 5.2. For $n \geq 3$,
\[
0 \to \text{gr}^3_Q(A_n) \xrightarrow{\tau_{3,Q}} H^*_Q \otimes Z L^Q_n(4) \to S^3 H_Q \oplus \Lambda^3 H_Q \to 0
\]
is a $GL(n, Z)$-equivariant exact sequence.

Corollary 5.1. For $n \geq 3$,
\[
(3) \quad \text{rank}_Z \text{gr}^3(A_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8).
\]
In particular, substituting $n = 3$ into (3), we have $\text{rank}_Z \text{gr}^3(A_3) = 43$.

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