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Kyoto University
ON THE JOHNSON HOMOMORPHISM OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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Abstract. In this paper we construct new obstructions for the surjectivity of the Johnson homomorphism of the automorphism group of a free group. We also determine the structure of the cokernel of the Johnson homomorphism for degrees 2 and 3.

1. Introduction

Let $F_n$ be a free group of rank $n \geq 2$ and $F_n = \Gamma_n(1), \Gamma_n(2), \ldots$ its lower central series. We denote by $\text{Aut} F_n$ the group of automorphisms of $F_n$. For each $k \geq 0$, let $A_n(k)$ be the group of automorphisms of $F_n$ which induce the identity on the quotient group $F_n/\Gamma_n(k+1)$. Then we have a descending filtration

$$\text{Aut} F_n = A_n(0) \supset A_n(1) \supset A_n(2) \supset \cdots$$

of $\text{Aut} F_n$. This filtration was introduced in 1963 with a remarkable pioneer work by S. Andreadakis [1] who showed that $A_n(1), A_n(2), \ldots$ is a descending central series of $A_n(1)$ and each graded quotient $\text{gr}^k(A_n) = A_n(k)/A_n(k+1)$ is a free abelian group of finite rank. He [1] also computed that $\text{rank}_\mathbb{Z} \text{gr}^k(A_2)$ for all $k \geq 1$ and $\text{rank}_\mathbb{Z} \text{gr}^2(A_3)$, and asserted $\text{rank}_\mathbb{Z} \text{gr}^3(A_3) = 44$. In Section 5, however, we show that $\text{gr}^3(A_3) = 43$. Moreover, by a recent remarkable work by A. Pettet [15] we have $\text{rank}_\mathbb{Z} \text{gr}^2(A_n) = \frac{1}{3} n^2(n^2 - 4) + \frac{1}{2} n(n - 1)$ for all $n \geq 3$. However, it is difficult to compute the rank of $\text{gr}^k(A_n)$.

Let $H$ be the abelianization of $F_n$ and $H^* = \text{Hom}_\mathbb{Z}(H, \mathbb{Z})$ the dual group of $H$. Let $L_n = \bigoplus_{k \geq 1} L_n(k)$ be the free graded Lie algebra generated by $H$. Then for each $k \geq 1$, a $GL(n, \mathbb{Z})$-equivariant injective homomorphism

$$\tau_k : \text{gr}^k(A_n) \to H^* \otimes \mathbb{Z} L_n(k + 1)$$

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is defined. (For definition, see Section 2.) This is called the \( k \)-th Johnson homomorphism of \( \text{Aut} F_n \). The theory of the Johnson homomorphism of a mapping class group of a compact Riemann surface began in 1980 by D. Johnson [6] and has been developed by many authors. There is a broad range of remarkable results for the Johnson homomorphism of a mapping class group. (For example, see [5] and [13].) However, the properties of the Johnson homomorphism of \( \text{Aut} F_n \) are far from being well understood.

The main interest of this paper is to determine the structure of the cokernel of the Johnson homomorphism \( \tau_k \) as a \( GL(n, \mathbb{Z}) \)-module. For \( k = 1 \), it is a well known fact that the first Johnson homomorphism \( \tau_1 \) is an isomorphism. (See [8].) For \( k \geq 2 \), the Johnson homomorphism \( \tau_k \) is not surjective. In fact, a recent remarkable work by Shigeyuki Morita indicates that there is a symmetric product \( S^k H \) of \( H = H \otimes \mathbb{Z} Q \) in the cokernel of \( \tau_k, Q = \tau_k \otimes \text{id}_Q \) for each \( k \geq 2 \). To show this, he introduced a homomorphism

\[
\text{Tr}_k : H^* \otimes \mathcal{L}_n (k + 1) \rightarrow S^k H,
\]
called the trace map, and showed that \( \text{Tr}_k \) vanishes on the image of \( \tau_k \) and is surjective after tensoring with \( \mathbb{Q} \) for all \( k \geq 2 \).

The trace maps were introduced in the 1993 by Morita [12] for a Johnson homomorphism of a mapping class group of a surface. He called these maps traces because they were defined using the trace of some matrix representation. Morita's traces are very important to study the Lie algebra structure of the target \( H^* \otimes \mathcal{L}_n = \text{Der}(\mathcal{L}_n) \) of the Johnson homomorphisms. Here \( \text{Der}(\mathcal{L}_n) \) denotes the graded Lie algebra of derivations of \( \mathcal{L}_n \). Morita conjectured that for any \( n \geq 3 \), the abelianization of the Lie algebra \( \text{Der}(\mathcal{L}_n) \) is given by

\[
H_1(\text{Der}(\mathcal{L}_n^Q)) \simeq (H^*_Q \otimes \mathbb{Z} \Lambda^2 H_Q) \oplus \bigoplus_{k \geq 2} S^k H_Q
\]

where \( \mathcal{L}_n^Q = \mathcal{L}_n \otimes \mathbb{Z} Q \) and the right hand side is understood to be an abelian Lie algebra. Recently, combining a work of Kassabov [7] with the concept of the traces, he [14] showed that the isomorphism above holds up to degree \( n(n - 1) \).

The subgroup \( \mathcal{A}_n(1) \) is called the \( \text{IA} \)-automorphism group of \( F_n \) and denoted by \( \text{IA}_n \). The group \( \text{IA}_n \) is the kernel of the natural map \( \text{Aut} F_n \rightarrow GL(n, \mathbb{Z}) \) which is given by the action of \( \text{Aut} F_n \) on \( H \). The structures
of $IA_n$ plays an important role in the study $\text{Aut} F_n$. W. Magnus [10] showed that $IA_n$ is finitely generated for all $n \geq 3$. However, it is not known whether $IA_n$ is finitely presented or not for any $n \geq 4$. For $n = 3$, by a remarkable work by S. Krstić and J. McCool [9], it is known that $IA_3$ is not finitely presented. On the other hand, the abelianization of $IA_n$ is given by

$$IA_n^{ab} \simeq H^* \otimes \mathbb{Z} \Lambda^2 H$$

as a $GL(n, \mathbb{Z})$-module. (See [8].)

Now let $\mathcal{A}_n'(1), \mathcal{A}_n'(2), \ldots$ be the lower central series of $IA_n = \mathcal{A}_n(1)$ and $\text{gr}^k (\mathcal{A}_n')$ its graded quotient of it for each $k \geq 1$. In Section 2, we define a $GL(n, \mathbb{Z})$-equivariant homomorphism

$$\tau_k' : \text{gr}^k (\mathcal{A}_n') \to H^* \otimes \mathbb{Z} \mathcal{L}_n(k + 1)$$

which is also called the $k$-th Johnson homomorphism of $\text{Aut} F_n$. In this paper, we construct new obstructions of the surjectivity of the Johnson homomorphism $\tau_k'$. Let us denote the tensor products with $\mathbb{Q}$ of a $\mathbb{Z}$-module by attaching a subscript $\mathbb{Q}$ to the original one. For example, $H_{\mathbb{Q}} := H \otimes \mathbb{Z} \mathbb{Q}$ and $\mathcal{L}_n(\mathbb{Q})(k) := \mathcal{L}_n(k) \otimes \mathbb{Z} \mathbb{Q}$. Similarly, for a $\mathbb{Z}$-linear map $f : A \to B$ we denote by $f_{\mathbb{Q}}$ the $\mathbb{Q}$-linear map $A_{\mathbb{Q}} \to B_{\mathbb{Q}}$ induced by $f$.

It is conjectured that $\text{Coker} \tau_{k, \mathbb{Q}}^k = \text{Coker} \tau_{k, \mathbb{Q}}^k$ for $k > 1$. It is true for $1 \leq k \leq 3$. In fact, $A_n(1) = \mathcal{A}_n'(1)$ by definition. We have $\mathcal{A}_n(2) = \mathcal{A}_n'(2)$ from the result stated above. (See [8].) Moreover, Pettet [15] showed that $\mathcal{A}_n'(3)$ has a finite index in $\mathcal{A}_n(3)$. Hence, $\text{Coker} \tau_{k, \mathbb{Q}}^k = \text{Coker} \tau_{k, \mathbb{Q}}^k$ for $1 \leq k \leq 3$. Our main result is

**Theorem 1.**

1. $\Lambda^k H_{\mathbb{Q}} \subset \text{Coker} \tau_{k, \mathbb{Q}}^k$ for odd $k$ and $3 \leq k \leq n$.
2. $H_{\mathbb{Q}}^{[2, 1^{k-2}]} \subset \text{Coker} \tau_{k, \mathbb{Q}}^k$ for even $k$ and $4 \leq k \leq n - 1$.

Here $\Lambda^k H_{\mathbb{Q}}$ denotes the $k$-th exterior product of $H_{\mathbb{Q}}$, and $H_{\mathbb{Q}}^{[2, 1^{k-2}]}$ denotes the Schur-Weyl module of $H_{\mathbb{Q}}$ corresponding to the partition $[2, 1^{k-2}]$.

In order to prove this, in Section 3, we introduce homomorphisms defined by

$$\text{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes \mathbb{Z} \mathcal{L}_n(k + 1) \to \Lambda^k H,$$

$$\text{Tr}_{[2, 1^{k-2}]} := (id_H \otimes f_{[1^{k-1}]}) \circ \Phi_2^k : H^* \otimes \mathbb{Z} \mathcal{L}_n(k + 1) \to H \otimes \mathbb{Z} \Lambda^{k-1} H$$

and show that these maps vanish on the image of the Johnson homomorphism $\tau_k'$. Since these maps are constructed in a way similar to that of Morita's trace $\text{Tr}_k$, we also call these maps traces.
In Section 5, we determine the $GL(n, \mathbb{Z})$-module structure of the cokernel of the Johnson homomorphism $\tau_k$ for 2 and 3. Our result is

**Theorem 2.** We have $GL(n, \mathbb{Z})$-equivariant exact sequences

$$0 \to \text{gr}^2(A_n) \xrightarrow{\tau_2} H^* \otimes \mathcal{L}_n(3) \to S^2 H \to 0$$

and

$$0 \to \text{gr}^3(A_n) \xrightarrow{\tau_3, \mathbb{Q}} H^*_Q \otimes \mathcal{L}_n^Q(4) \to S^3 H_Q \oplus \Lambda^3 H_Q \to 0$$

for $n \geq 3$.

Thus we have

**Corollary 1.** For $n \geq 3$,

$$\text{rank}_\mathbb{Z} \text{gr}^3(A_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8).$$

2. Preliminaries

In this section we review some basic facts. First, we note that the group $\text{Aut} \ F_n$ acts on $F_n$ on the right. For any $\sigma \in \text{Aut} \ F_n$ and $x \in F_n$, the action of $\sigma$ on $x$ is denoted by $x^\sigma$.

2.1. Commutators of higher weight.

In this paper, we often use basic facts of commutator calculus. The reader is referred to [11] and [16], for example. Let $G$ be a group. For any elements $x$ and $y$ of $G$, the element

$$xyx^{-1}y^{-1}$$

is called the commutator of $x$ and $y$, and denoted by $[x, y]$. In general, a commutator of higher weight is recursively defined as follows. First, a commutator of weight 1 is an element of $G$. For $k > 1$, a commutator of weight $k$ is an element of the type $C = [C_1, C_2]$ where $C_j$ is a commutator of weight $a_j$ ($j = 1, 2$) such that $a_1 + a_2 = k$. The weight of the commutator $C$ is denoted by $\text{wt}(C) = k$. The commutator which has elements $g_1, \ldots, g_t \in G$ in the bracket components is called the commutator among the components $g_1, \ldots, g_t$. For elements $g_1, \ldots, g_t \in G$, a commutator of weight $k$ among the components $g_1, \ldots, g_t$ of the type

$$[[\cdots [[g_{i_1}, g_{i_2}], g_{i_3}], \cdots], g_{i_k}], \quad i_j \in \{1, \ldots, t\}$$

with all of its brackets to the left of all the elements occurring is called a simple $k$-fold commutator and is denoted by

$$[g_{i_1}, g_{i_2}, \cdots, g_{i_k}].$$
For each $k \geq 1$, the subgroups $\Gamma_{G}(k)$ of the lower central series of $G$ are defined recursively by

$$\Gamma_{G}(1) = G, \quad \Gamma_{G}(k + 1) = [\Gamma_{G}(k), G].$$

We use the following basic lemma in later sections.

**Lemma 2.1.** If a group $G$ is generated by $g_1, \ldots, g_t$, then each of the graded quotients $\Gamma_{G}(k)/\Gamma_{G}(k + 1)$ for $k \geq 1$ is generated by the cosets of the simple $k$-fold commutators

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}], \quad i_j \in \{1, \ldots, t\}.$$

Now, for each $k \geq 1$, let $\Gamma_n(k)$ be the $k$-th subgroup $\Gamma_{\mathcal{F}_{n}}(k)$ of the lower central series of a free group $\mathcal{F}_{n}$ of rank $n$ and $\text{gr}^k(\Gamma_n)$ its graded quotient $\Gamma_n(k)/\Gamma_n(k + 1)$. We denote by $\text{gr}(\Gamma_n) = \bigoplus_{k \geq 1} \text{gr}^k(\Gamma_n)$ the associated graded sum. Then the set $\text{gr}(\Gamma_n)$ naturally has a structure of a graded Lie algebra over $\mathbb{Z}$ induced on $\mathcal{F}_{n}$.

Let $H$ be the abelianization of $\mathcal{F}_{n}$ and $\mathcal{L}_{n} = \bigoplus_{k>1} \mathcal{L}_{n}(k)$ the free graded Lie algebra generated by $H$. It is well known that the Lie algebra $\text{gr}(\Gamma_n)$ is isomorphic to $\mathcal{L}_{n}$ as a graded Lie algebra over $\mathbb{Z}$. Thus, in this paper, we identify $\text{gr}(\Gamma_n)$ with $\mathcal{L}_{n}$. For any element $x \in \Gamma_n(k)$, we also denote by $x$ the coset class of $x$ in $\mathcal{L}_{n}(k) = \Gamma_n(k)/\Gamma_n(k + 1)$. Let $T(H)$ be the tensor algebra of $H$ over $\mathbb{Z}$. Then the algebra $T(H)$ is the universal enveloping algebra of the free Lie algebra $\mathcal{L}_{n}$ and the natural map $\mathcal{L}_{n} \rightarrow T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for $X, Y \in \mathcal{L}_{n}$ is an injective Lie algebra homomorphism. Hence we also regard $\mathcal{L}_{n}(k)$ as a submodule of $H^\otimes k$ for each $k \geq 1$.

### 2.2. IA-automorphism group.

The kernel of the natural map $\text{Aut} \mathcal{F}_{n} \rightarrow GL(n, \mathbb{Z})$ which is given by the action of $\text{Aut} \mathcal{F}_{n}$ on $H$ is called the IA-automorphism group of $\mathcal{F}_{n}$ and denoted by $IA_{n}$. Let $\{x_1, \ldots, x_n\}$ be a basis of a free group $\mathcal{F}_{n}$. Magnus [10] showed that $IA_{n}$ is finitely generated by automorphisms

$$K_{ab} : \begin{cases} 
   x_a &\mapsto x_b^{-1}x_a x_b, \\
   x_t &\mapsto x_t, \quad (t \neq a)
\end{cases}$$

for $a, b \in \{1, \ldots, n\}$.
and

\[ K_{abc} : \begin{cases} 
  x_a & \mapsto x_a x_b x_c x_b^{-1} x_c^{-1}, \\
  x_t & \mapsto x_t, \quad (t \neq a)
\end{cases} \]

for any distinct \( a, b \) and \( c \in \{1, 2, \ldots, n\} \). It is known that the abelianization \( IA_n^{ab} \) of the IA-automorphism group is free abelian group with generators \( K_{ab} \) for distinct \( a \) and \( b \), and \( K_{abc} \) for distinct \( a, b, c \) and \( b < c \). More precisely, if we denote by \( H^* = \text{Hom}_\mathbb{Z}(H, \mathbb{Z}) \) the dual group of \( H \), we have a \( GL(n, \mathbb{Z}) \)-module isomorphism \( IA_n^{ab} \simeq H^* \otimes_\mathbb{Z} \Lambda^2 H \). (For details, see [8].)

### 2.3. The associated graded Lie algebra.

Here we consider two descending filtrations of \( IA_n \). The first one is \( \{A_n(k)\}_{k \geq 1} \) defined as above. Since the series \( A_n(1), A_n(2), \ldots \) is central, the associated graded sum \( \text{gr}(A_n) = \bigoplus_{k \geq 1} \text{gr}^k(A_n) \) naturally has a structure of a graded Lie algebra over \( \mathbb{Z} \) induced from the commutator bracket on \( A_n(1) \). For each \( k \geq 1 \), the group \( A_n(0) = \text{Aut} F_n \) naturally acts on \( A_n(k) \) by conjugation, hence on \( \text{gr}^k(A_n) \). Since the group \( A_n(1) = IA_n \) trivially acts on \( \text{gr}^k(A_n) \), we see that the group \( GL(n, \mathbb{Z}) \simeq A_n(0)/A_n(1) \) naturally acts on \( \text{gr}^k(A_n) \).

The other is the lower central series \( A_n'(1), A_n'(2), \ldots \) of \( A_n(1) \). Let \( \text{gr}^k(A_n') = A_n'(k)/A_n'(k+1) \) be the graded quotient for each \( k \geq 1 \). Similarly the associated graded sum \( \text{gr}(A_n') = \bigoplus_{k \geq 1} \text{gr}^k(A_n') \) has a structure of a graded Lie algebra structure on \( \mathbb{Z} \). Moreover, each graded quotient \( \text{gr}^k(A_n') \) is a \( GL(n, \mathbb{Z}) \)-module. It is clear that \( A_n'(k) \subset A_n(k) \) for every \( k \geq 1 \). In particular, we have \( A_n'(k) = A_n(k) \) for \( 1 \leq k \leq 2 \) and Pettet [15] showed that \( A_n'(3) \) has finite index in \( A_n(3) \) as mentioned in section 1. From Lemma 2.1, for each \( k \geq 1 \), the graded quotient \( \text{gr}^k(A_n') \) is generated by (the cosets of) the simple \( k \)-fold commutators among the components \( K_{ab} \) and \( K_{abc} \).

### 2.4. Johnson homomorphism.

Here we define the Johnson homomorphisms of \( \text{Aut} F_n \). For each \( k \geq 1 \), let \( \tau_k : A_n(k) \to \text{Hom}_\mathbb{Z}(H, \mathcal{L}_n(k+1)) \) be the map defined by

\[ \sigma \mapsto (x \mapsto x^{-1} x^\sigma) \]

for \( \sigma \in A_n(k) \) and \( x \in H \). Then the map \( \tau_k \) is a homomorphism and the kernel of \( \tau_k \) is just \( A_n(k+1) \). Hence, identifying \( \text{Hom}_\mathbb{Z}(H, \mathcal{L}_n(k+1)) \) with \( H^* \otimes_\mathbb{Z} \mathcal{L}_n(k+1) \), we obtain an injective \( GL(n, \mathbb{Z}) \)-equivariant
homomorphism, also denoted by $\tau_k$,  

$$\tau_k : \text{gr}^k(A_n) \to H^* \otimes \mathcal{L}_n(k + 1).$$

This homomorphism is called the $k$-th Johnson homomorphism of $\text{Aut} F_n$. Similarly, for each $k \geq 1$, we can define a homomorphism $\tau'_k : A'_n(k) \to \text{Hom}_\mathbb{Z}(H, \mathcal{L}_n(k + 1))$ as (1). Since $A'_n(k + 1)$ is contained in the kernel of $\tau'_k$, we obtain a $GL(n, \mathbb{Z})$-equivariant homomorphism, also denoted by $\tau'_k$,  

$$\tau'_k : \text{gr}^k(A'_n) \to H^* \otimes \mathcal{L}_n(k + 1).$$

We also call the map $\tau'_k$ the Johnson homomorphism of $\text{Aut} F_n$.

Let $\{x_1, \ldots, x_n\}$ be a basis of $F_n$. It defines a basis of $H$ as a free abelian group, also denoted by $\{x_1, \ldots, x_n\}$. Let $\{x_1^*, \ldots, x_n^*\}$ be the dual basis of $H^*$. For any $\sigma \in A'_n(k)$, if we set $s_i(\sigma) := x_i^{-1}x_i^\sigma \in \mathcal{L}_n(k + 1)$ $(1 \leq i \leq n)$ then we have  

$$\tau_k(\sigma) = \tau'_k(\sigma) = \sum_{i=1}^{n} x_i^* \otimes s_i(\sigma) \in H^* \otimes \mathcal{L}_n(k + 1).$$

Let $\text{Der}(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree $k$ part of $\text{Der}(\mathcal{L}_n)$ is expressed as $\text{Der}(\mathcal{L}_n)(k) = H^* \otimes \mathcal{L}_n(k)$. Thus we sometimes identify $\text{Der}(\mathcal{L}_n)$ with $H^* \otimes \mathcal{L}_n$. Then the Johnson homomorphism $\tau = \bigoplus_{k \geq 1} \tau_k$ is a graded Lie algebra homomorphism. In fact, if we denote by $\partial \sigma$ the element of $\text{Der}(\mathcal{L}_n)$ corresponding to an element $\sigma \in H^* \otimes \mathcal{L}_n$ and write the action of $\partial \sigma$ on $X \in \mathcal{L}_n$ as $X^{\partial \sigma}$ then we have  

$$(2) \quad \tau'_{k+l}([\sigma, \tau]) = \sum_{i=1}^{n} x_i^* \otimes (s_i(\sigma)^{\partial \tau} - s_i(\tau)^{\partial \sigma}).$$

for any $\sigma \in A'_n(k)$ and $\tau \in A'_n(l)$.

In general, each $s_i(\sigma) \in \mathcal{L}_n(k + 1)$ cannot be uniquely written as a sum of commutators among the components $x_1, \ldots, x_n$. In this paper, each $s_i(\sigma)$ is recursively computed in the following way. First, for $\sigma = K_{abc}$, we can set  

$$s_a(K_{abc}) = [x_b, x_c], \quad s_t(K_{abc}) = 0 \quad \text{if} \quad t \neq a.$$  

For $\sigma = K_{ab}$, we see that  

$$x_t^{-1} x_t^\sigma = \begin{cases} 
    [x_a^{-1}, x_b^{-1}] & \text{if} \quad t = a, \\
    1 & \text{if} \quad t \neq a
\end{cases}$$
in $F_n$. Since $[x_a^{-1}, x_b^{-1}] = [x_a, x_b]$ in $L_n(2)$, so we can set
\[ s_a(K_{ab}) = [x_a, x_b], \quad s_t(K_{ab}) = 0 \quad \text{if} \quad t \neq a. \]
Next, if $\sigma = [\tau, K_{ab}]$ for $k$-fold simple commutator $\tau$, following from (2), we can set
\[ s_i(\sigma) = s_i(\tau)^{\partial K_{ab}} - s_i(K_{ab})^{\partial \tau} \]
for each $i$. Furthermore, since a commutator bracket of weight $l$ is considered as a $l$-fold multilinear map from the cartesian product of $l$ copies of $L_n(1)$ to $L_n(l)$, we can also set
\[ s_i(\sigma) = \sum_{p=1}^{a(i)}(-1)^{e_{i,p}}C_{i,p} \]
where $e_{i,p} = 0$ or 1, and $C_{i,p}$ is a commutator of degree $k + 1$ among the components $x_1, \ldots, x_n$. We compute $s_i([\tau, K_{abc}])$ for $\sigma = [\tau, K_{abc}]$ similarly. These computations are perhaps easiest explained with examples, so we give two here. For distinct $a, b, c$ and $d$, we have
\[ \tau_2'(\,[K_{ab}, K_{bac}] = x_a^* \otimes (\,[x_a, x_b]\,)^{\partial K_{bac}} - x_b^* \otimes (\,[x_a, x_c]\,)^{\partial K_{ab}}, \]
and
\[ \tau_3'(\,[K_{ab}, K_{abc}, K_{ad}] = x_a^* \otimes (\,[x_a, [x_a, x_c]\,]\,)^{\partial K_{ad}} - x_b^* \otimes (\,[x_a, x_b]\,)^{\partial K_{ad}} \]
\[ \quad - x_a^* \otimes ([x_a, x_d]\,)^{\partial [K_{ab}, K_{bac}]}, \]
\[ = x_a^* \otimes ([x_a, x_d]\,), [x_a, x_c]\, + x_a^* \otimes [x_a, [x_a, x_d]\,], x_c]\]
\[ - x_a^* \otimes (\,[x_a, x_d]\,), x_b]\, , \quad x_c]\]
\[ - x_a^* \otimes (\,[x_a, x_c]\,), x_d]. \]

3. The contractions

For $k \geq 1$ and $1 \leq l \leq k + 1$, let $\varphi^k_l : H^* \otimes_Z H^{\otimes (k+1)} \to H^{\otimes k}$ be the contraction map defined by
\[ x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_1}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_{l-1}} \otimes x_{j_{l+1}} \otimes \cdots \otimes x_{j_{k+1}}. \]
For the natural embedding $\iota_n^{k+1} : L_n(k+1) \to H^{\otimes (k+1)}$, we obtain a $GL(n, Z)$-equivariant homomorphism
\[ \Phi^k_l = \varphi^k_l \circ (id_H \otimes \iota_n^{k+1}) : H^* \otimes_Z L_n(k+1) \to H^{\otimes k}. \]
We also call the map $\Phi^k_l$ contraction.
Here we introduce one of methods of the computation of $\Phi_{l}^{k}(x_{i}^{*} \otimes C)$ for a commutator $C \in \mathcal{L}_{n}(k + 1)$ among the components $x_{1}, \ldots, x_{n}$. In this paper, whenever we compute $\Phi_{l}^{k}(x_{i}^{*} \otimes C)$, we use the following method. First, if $x_{i}$ does not appear among the components of $C$, then $\Phi_{l}^{k}(x_{i}^{*} \otimes C) = 0$. On the other hand, if $x_{i}$ appears among the components of $C$ $m$ times, then we distinguish them and write such $x_{i}$'s as $x_{i_{1}}, \ldots, x_{i_{m}}$ in $C$. Then $\Phi_{l}^{k}(x_{i}^{*} \otimes C)$ is given by rewriting $x_{i_{1}}, \ldots, x_{i_{m}}$ as $x_{i}$ in

$$\sum_{j=1}^{m} \Phi_{l}^{k}(x_{i_{j}}^{*} \otimes C).$$

Thus it suffices to compute $\Phi_{l}^{k}(x_{i}^{*} \otimes C)$ for a commutator $C$ which has only one $x_{i}$ in its components. Now, $C$ is written as $[X, Y]$ for some commutators $X$ and $Y$. Rewriting the commutator $C$ as $-[Y, X]$ if $x_{i}$ appears in $Y$, we may always consider $C = \pm[X, Y]$ such that $x_{i}$ appears among the components of $X$. By a recursive argument, we have $C = \pm[x_{i}, C_{1}, \ldots, C_{t}]$ where each $C_{j}$ $(1 \leq j \leq t)$ is a commutator of weight $d_{j}$ and $d_{1} + \cdots + d_{t} = k$.

**Lemma 3.1.** For a commutator $[x_{i}, C_{1}, \ldots, C_{t}] \in \mathcal{L}_{n}(k + 1)$ as above,

$$\Phi_{1}^{k}(x_{i}^{*} \otimes [x_{i}, C_{1}, \ldots, C_{t}]) = C_{1} \otimes \cdots \otimes C_{t}.$$

**Lemma 3.2.** For a commutator $[x_{i}, C_{1}, \ldots, C_{t}] \in \mathcal{L}_{n}(k + 1)$ as above,

$$\Phi_{2}^{k}(x_{i}^{*} \otimes [x_{i}, C_{1}, \ldots, C_{t}]) = -\sum_{\text{wt}(C_{j})=1} C_{j} \otimes C_{1} \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_{t}.$$

Let $T(H) = \bigoplus_{k \geq 1} H^{\otimes k}$ and $S(H) = \bigoplus_{k \geq 1} S^{k}H$ be the tensor algebra and the symmetric algebra on $H$ respectively. Then the kernel of a natural map $T(H) \to S(H)$ is a graded ideal of $T(H)$, and denoted by $I(H) = \bigoplus_{k \geq 1} I^{k}(H)$. For each $k \geq 2$, let $\mathcal{U}_{n}(k)$ be the $GL(n, \mathbb{Z})$-submodule of $H^{\otimes k}$ generated by elements type of

$$[A, B] := A \otimes B - B \otimes A$$

for $A \in H^{\otimes a}$, $B \in H^{\otimes b}$ and $a + b = k$. If we put $\mathcal{U}_{n} = \bigoplus_{k \geq 1} \mathcal{U}_{n}(k)$, then $\mathcal{U}_{n}$ is the kernel of the abelianization $T(H) \to T(H)^{ab}$ as a Lie algebra. We have

$$\mathcal{L}_{n}(k) \subset \mathcal{U}_{n}(k) \subset I^{k}(H) \subset H^{\otimes k}.$$
3.1. The image of $\Phi^k_i \circ \tau'_k$.

Considering the image of any simple $k$-fold commutator $\sigma$ among the components $K_{ab}$ and $K_{abc}$, we prove the following propositions.

**Proposition 3.1.** For $n \geq 3$ and $k \geq 2$, $\text{Im}(\Phi^k_1 \circ \tau'_k) \subset \mathcal{U}_n(k)$.

**Proposition 3.2.** For $n \geq 3$ and $k \geq 3$, $\text{Im}(\Phi^k_2 \circ \tau'_k) \subset H \otimes \mathbb{Z}\mathcal{U}_n(k-1)$.

4. The trace maps

In this section, using the contractions defined in Section 3, we define a homomorphisms called the trace map which vanishes on the image of the Johnson homomorphism. Here we use some basic facts of the representation theory of $GL(n, \mathbb{Z})$. The reader is referred to, for example, Fulton-Harris [4] and Fulton [3].

For any $k \geq 1$ and any partition $\lambda$ of $k$, we denote by $H^\lambda$ the Schur-Weyl module of $H$ corresponding to the partition $\lambda$ of $k$. Let $f_{\lambda} : H^{\otimes k} \rightarrow H^\lambda$ be a natural homomorphism. In this paper, we mainly consider the case for $\lambda = [k]$ or $[1^k]$. The modules $H^{[k]}$ and $H^{[1^k]}$ are the symmetric product $S^kH$ and the exterior product $\Lambda^kH$ respectively. Using the natural map $\iota^k_n : \mathcal{L}_n(k) \rightarrow H^{\otimes k}$, we denote $f_{[1^k]} \circ \iota^k_n(C)$ by $\hat{C}$ for any $C \in \mathcal{L}_n(k)$.

**Lemma 4.1.** For any commutator $C$ of weight $k \geq 3$, $\hat{C} = 0$ in $\Lambda^kH$

**Lemma 4.2.** For $1 \leq k \leq n-2$ and any commutator $C$ of weight $k+1$ among the components $x_1, \ldots, x_n$ except for $x_i$, there exists an element $\sigma \in \mathcal{A}'_n(k)$ such that

$$\tau'_k(\sigma) = x_i^* \otimes C \in H^* \otimes \mathbb{Z}\mathcal{L}_n(k+1).$$

4.1. Morita’s trace (Trace map for $S^kH$).

Here we consider the map

$$\text{Tr}_{[k]} = f_{[k]} \circ \Phi^k_1 : H^* \otimes \mathbb{Z}\mathcal{L}_n(k+1) \rightarrow S^kH.$$  

By definition, this map coincides with the Morita’s trace $\text{Tr}_k$. For $n \geq 3$ and $k \geq 2$, Morita defined the trace map $\text{Tr}_k$ using the Magnus representation of $\text{Aut} F_n$ and showed that $\text{Tr}_k$ vanishes on the image of $\tau_k$. By a recent work, he showed that $\text{Tr}^Q_k$ is surjective. Hence we have

**Theorem 4.1.** (Morita) For $n \geq 3$ and $k \geq 2$,

$$S^kH_Q \subset \text{Coker} \tau_{k,Q}.$$
Corollary 4.1. For \( n \geq 3 \) and \( k \geq 2 \),
\[
\text{rank}_Z(\text{Coker}(\tau_k)) \geq \binom{n+k-1}{k}.
\]

4.2. Trace map for \( \Lambda^kH \).
Here we consider the map
\[
\text{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes_Z \mathcal{L}_n(k+1) \to \Lambda^kH.
\]

Theorem 4.2.
(1) For \( 3 \leq k \leq n \), \( \text{Tr}_{[1^k]} \) is surjective,
(2) \( \text{Im}(\text{Tr}_{[1^k]} \circ \tau'_k) = 0 \) if \( k \) is odd and \( 3 \leq k \leq n \),
(3) \( \text{Im}(\text{Tr}_{[1^k]} \circ \tau'_k) = 2(\Lambda^kH) \subset \Lambda^kH \) if \( k \) is even and \( 4 \leq k \leq n-2 \).

Corollary 4.2. For an odd \( k \) and \( 3 \leq k \leq n \),
\[
\Lambda^kH_Q \subset \text{Coker} \tau'_{k,Q}.
\]

Corollary 4.3. For an odd \( k \) and \( 3 \leq k \leq n \),
\[
\text{rank}_Z(\text{Coker}(\tau'_k)) \geq \binom{n}{k}.
\]

4.3. Trace map for \( H^{[2,1^{k-1}]} \).
Here we consider the map
\[
\text{Tr}_{[2,1^{k-1}]} := (\text{id}_H \otimes f_{[1^{k-1}]}^{-1}) \circ \Phi_2^k : H^* \otimes_Z \mathcal{L}_n(k+1) \to H \otimes Z \Lambda^{k-1}H.
\]

Let \( I \) be the \( GL(n, Z) \)-submodule of \( H \otimes Z \Lambda^{k-1}H \) defined by
\[
I = \langle x \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge y + y \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge x \mid x, y, z_i \in H \rangle.
\]

Theorem 4.3. For an even \( k \) and \( 4 \leq k \leq n-1 \),
(1) \( \text{Im}(\text{Tr}_{[2,1^{k-1}]}^Q) = I_Q \),
(2) \( \text{Im}(\text{Tr}_{[2,1^{k-1}]}^Q \circ \tau'_k) = 0 \).

Now we have \( H_Q \otimes_Z \Lambda^{k-1}H_Q \simeq H_Q^{[2,1^{k-2}]} + \Lambda^kH_Q \) from the representation theory of \( GL(n, Z) \). For even \( k \), since \( I_Q \) is contained in the kernel of a natural map \( H_Q \otimes_Z \Lambda^{k-1}H_Q \to \Lambda^kH_Q \) defined by \( x \otimes y_1 \wedge \cdots \wedge y_{k-1} \mapsto x \wedge y_1 \wedge \cdots \wedge y_{k-1} \), we have \( I_Q \simeq H_Q^{[2,1^{k-2}]} \).

Corollary 4.4. For an even \( k \) and \( 4 \leq k \leq n-1 \),
\[
H_Q^{[2,1^{k-2}]} \subset \text{Coker} \tau'_k,Q.
\]
Corollary 4.5. For an even $k$ and $4 \leq k \leq n-1$,
\[ \text{rank}_Z(\text{Coker } (\tau'_k)) \geq (k-1) \binom{n+1}{k}. \]

5. The cokernel of the Johnson homomorphism $\tau_k$ for $k = 2$ and 3

5.1. The case $k = 2$.
In this subsection we consider the case where $n \geq 3$. From Theorem 4.1 and rank$_Z(\text{Coker } (\tau_2)) = \binom{n+1}{2}$ by Pettet [15], we have a $GL(n, Z)$-equivariant exact sequence
\[ 0 \to \text{gr}^2_Q(A_n) \stackrel{\tau_2}{\to} H^*_Q \otimes Z \mathcal{L}_n^Q(3) \to S^2 H_Q \to 0. \]
We show that the exact sequence above holds before tensoring with $Q$. Namely,

Theorem 5.1. For $n \geq 3$,
\[ 0 \to \text{gr}^2(A_n) \stackrel{\tau_2}{\to} H^* \otimes Z \mathcal{L}_n(3) \to S^2 H \to 0 \]
is a $GL(n, Z)$-equivariant exact sequence.

5.2. The case $k = 3$.
Next we compute the cokernel of the Johnson homomorphism $\tau_{3,Q}$ for $n \geq 3$ using the fact that Coker $\tau_{3,Q} = \text{Coker } (\tau'_3)$. We have

Theorem 5.2. For $n \geq 3$,
\[ 0 \to \text{gr}^3_Q(A_n) \stackrel{\tau_3}{\to} H^*_Q \otimes Z \mathcal{L}_n^Q(4) \to S^3 H_Q \oplus \Lambda^3 H_Q \to 0 \]
is a $GL(n, Z)$-equivariant exact sequence.

Corollary 5.1. For $n \geq 3$,
\[ (3) \quad \text{rank}_Z \text{gr}^3(A_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8). \]
In particular, substituting $n = 3$ into (3), we have rank$_Z \text{gr}^3(A_3) = 43$.

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