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ON THE JOHNSON HOMOMORPHISM OF THE AUTOMORPHISM GROUP OF A FREE GROUP

佐藤隆夫 (Takao Satoh)
東京大学大学院数理科学研究科 (The University of Tokyo)

ABSTRACT. In this paper we construct new obstructions for the surjectivity of the Johnson homomorphism of the automorphism group of a free group. We also determine the structure of the cokernel of the Johnson homomorphism for degrees 2 and 3.

1. Introduction

Let F_n be a free group of rank $n \geq 2$ and $F_n = \Gamma_n(1), \Gamma_n(2), \ldots$ its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n which induce the identity on the quotient group $F_n/\Gamma_n(k+1)$. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$. This filtration was introduced in 1963 with a remarkable pioneer work by S. Andreadakis [1] who showed that $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, $\ldots$ is a descending central series of $\mathcal{A}_n(1)$ and each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) = \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. He [1] also computed that $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^k(\mathcal{A}_2)$ for all $k \geq 1$ and $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^2(\mathcal{A}_3)$, and asserted $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_3) = 44$. In Section 5, however, we show that $\mathrm{gr}^3(\mathcal{A}_3) = 43$. Moreover, by a recent remarkable work by A. Pettet [15] we have $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^2(\mathcal{A}_n) = \frac{1}{3}n^2(n^2-4) + \frac{1}{2}n(n-1)$ for all $n \geq 3$. However, it is difficult to compute the rank of $\mathrm{gr}^k(\mathcal{A}_n)$.

Let H be the abelianization of F_n and $H^* = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H. Let $\mathcal{L}_n = \bigoplus_{k\geq 1} \mathcal{L}_n(k)$ be the free graded Lie algebra generated by H. Then for each $k \geq 1$, a $GL(n, \mathbf{Z})$-equivariant injective homomorphim

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

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is defined. (For definition, see Section 2.) This is called the k-th Johnson homomorphism of $\mathrm{Aut}\,F_n$. The theory of the Johnson homomorphism of a mapping class group of a compact Riemann surface began in 1980 by D. Johnson [6] and has been developed by many authors. There is a broad range of remarkable results for the Johnson homomorphism of a mapping class group. (For example, see [5] and [13].) However, the properties of the Johnson homomorphism of $\mathrm{Aut}\,F_n$ are far from being well understood.

The main interest of this paper is to determine the structure of the cokernel of the Johnson homomorphism τ_k as a $GL(n,\mathbf{Z})$-module. For $k = 1$, it is a well known fact that the first Johnson homomorphism τ_1 is an isomorphism. (See [8].) For $k \geq 2$, the Johnson homomorphism τ_k is not surjective. In fact, a recent remarkable work by Shigeyuki Morita indicates that there is a symmetric product $S^k H_{\mathbf{Q}}$ of $H_{\mathbf{Q}} = H \otimes_{\mathbf{Z}} \mathbf{Q}$ in the cokernel of $\tau_{k,\mathbf{Q}} = \tau_k \otimes id_{\mathbf{Q}}$ for each $k \geq 2$. To show this, he introduced a homomorphism

$$\mathrm{Tr}_k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H,$$

called the trace map, and showed that Tr_k vanishes on the image of τ_k and is surjective after tensoring with $\mathbf{Q}$ for all $k \geq 2$.

The trace maps were introduced in the 1993 by Morita [12] for a Johnson homomorphism of a mapping class group of a surface. He called these maps traces because they were defined using the trace of some matrix representation. Morita's traces are very important to study the Lie algebra structure of the target $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n = \mathrm{Der}(\mathcal{L}_n)$ of the Johnson homomorphisms. Here $\mathrm{Der}(\mathcal{L}_n)$ denotes the graded Lie algebra of derivations of $\mathcal{L}_n$. Morita conjectured that for any $n \geq 3$, the abelianization of the Lie algebra $\mathrm{Der}(\mathcal{L}_n)$ is given by

$$H_1(\mathrm{Der}(\mathcal{L}_n^{\mathbf{Q}})) \simeq (H_{\mathbf{Q}}^* \otimes_{\mathbf{Z}} \Lambda^2 H_{\mathbf{Q}}) \oplus \left(\bigoplus_{k \geq 2}^{\infty} S^k H_{\mathbf{Q}}\right)$$

where $\mathcal{L}_n^{\mathbf{Q}} = \mathcal{L}_n \otimes_{\mathbf{Z}} \mathbf{Q}$ and the right hand side is understood to be an abelian Lie algebra. Recently, combining a work of Kassabov [7] with the concept of the traces, he [14] showed that the isomorphism above holds up to degree $n(n-1)$.

The subgroup $\mathcal{A}_n(1)$ is called the IA-automorphism group of F_n and denoted by IA_n. The group IA_n is the kernel of the natural map $\mathrm{Aut}\,F_n \to GL(n,\mathbf{Z})$ which is given by the action of $\mathrm{Aut}\,F_n$ on H. The structures of IA_n plays an important role in the study Aut F_n. W. Magnus [10] showed that IA_n is finitely generated for all $n \geq 3$. However, it is not known whether IA_n is finitely presented or not for any $n \geq 4$. For $n = 3$, by a remarkable work by S. Krstić and J. McCool [9], it is known that IA_3 is not finitely presented. On the other hand, the abelianization of IA_n is given by

$$IA_n^{\mathrm{ab}} \simeq H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

as a $GL(n, \mathbf{Z})$-module. (See [8].)

Now let $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, ... be the lower central series of $IA_n = \mathcal{A}_n(1)$ and $\mathrm{gr}^k(\mathcal{A}'_n)$ its graded quotient of it for each $k \geq 1$. In Section 2, we define a $GL(n, \mathbf{Z})$-equivariant homomorphism

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

which is also called the k-th Johnson homomorphism of Aut F_n. In this paper, we construct new obstructions of the surjectivity of the Johnson homomorphism τ'_k. Let us denote the tensor products with $\mathbf{Q}$ of a $\mathbf{Z}$-module by attaching a subscript $\mathbf{Q}$ to the original one. For example, $H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathcal{L}_n^{\mathbf{Q}}(k) := \mathcal{L}_n(k) \otimes_{\mathbf{Z}} \mathbf{Q}$. Similarly, for a $\mathbf{Z}$-linear map $f : A \to B$ we denote by $f_{\mathbf{Q}}$ the $\mathbf{Q}$-linear map $A_{\mathbf{Q}} \to B_{\mathbf{Q}}$ induced by f. It is conjectured that $\mathrm{Coker}\, \tau'_{k,\mathbf{Q}} = \mathrm{Coker}\, \tau_{k,\mathbf{Q}}$ for $k \geq 1$. It is true for $1 \leq k \leq 3$. In fact, $\mathcal{A}_n(1) = \mathcal{A}'_n(1)$ by definition. We have $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ from the result stated above. (See [8].) Moreover, Pettet [15] showed that $\mathcal{A}'_n(3)$ has a finite index in $\mathcal{A}_n(3)$. Hence, $\mathrm{Coker}\, \tau'_{k,\mathbf{Q}} = \mathrm{Coker}\, \tau_{k,\mathbf{Q}}$ for $1 \leq k \leq 3$. Our main result is

Theorem 1.

(1) $\Lambda^k H_{\mathbf{Q}} \subset \mathrm{Coker}\, \tau'_{k,\mathbf{Q}}$ *for odd* k *and* $3 \leq k \leq n$.

(2) $H_{\mathbf{Q}}^{[2,1^{k-2}]} \subset \mathrm{Coker}\, \tau'_{k,\mathbf{Q}}$ *for even* k *and* $4 \leq k \leq n-1$.

Here $\Lambda^k H_{\mathbf{Q}}$ *denotes the* k*-th exterior product of* $H_{\mathbf{Q}}$*, and* $H_{\mathbf{Q}}^{[2,1^{k-2}]}$ *denotes the Schur-Weyl module of* $H_{\mathbf{Q}}$ *corresponding to the partition* $[2, 1^{k-2}]$.

In order to prove this, in Section 3, we introduce homomorphisms defined by

$$\mathrm{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H,$$
$$\mathrm{Tr}_{[2,1^{k-2}]} := (id_H \otimes f_{[1^{k-1}]}) \circ \Phi_2^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H \otimes_{\mathbf{Z}} \Lambda^{k-1} H$$

and show that these maps vanish on the image of the Johnson homomorphism τ'_k. Since these maps are constructed in a way similar to that of Morita's trace Tr_k, we also call these maps traces.

In Section 5, we determine the $GL(n, \mathbf{Z})$-module structure of the cokernel of the Johnson homomorphism τ_k for 2 and 3. Our result is

Theorem 2. *We have $GL(n, \mathbf{Z})$-equivariant exact sequences*

$$0 \to \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_2} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to S^2 H \to 0$$

and

$$0 \to \mathrm{gr}^3_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{3,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}^{\mathbf{Q}}_n(4) \to S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}} \to 0$$

for $n \geq 3$.

Thus we have

Corollary 1. *For $n \geq 3$,*

$$\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8).$$

2. Preliminaries

In this section we review some basic facts. First, we note that the group $\mathrm{Aut}\, F_n$ acts on F_n on the right. For any $\sigma \in \mathrm{Aut}\, F_n$ and $x \in F_n$, the action of σ on x is denoted by x^σ.

2.1. Commutators of higher weight.

In this paper, we often use basic facts of commutator calculus. The reader is referred to [11] and [16], for example. Let G be a group. For any elements x and y of G, the element

$$xyx^{-1}y^{-1}$$

is called the commutator of x and y, and denoted by $[x, y]$. In general, a commutator of higher weight is recursively defined as follows. First, a commutator of weight 1 is an element of G. For $k > 1$, a commutator of weight k is an element of the type $C = [C_1, C_2]$ where C_j is a commutator of weight a_j ($j = 1, 2$) such that $a_1 + a_2 = k$. The weight of the commutator C is denoted by $\mathrm{wt}\,(C) = k$. The commutator which has elements $g_1, \ldots, g_t \in G$ in the bracket components is called the commutator among the components $g_1, \ldots, g_t$. For elements $g_1, \ldots, g_t \in G$, a commutator of weight k among the components $g_1, \ldots, g_t$ of the type

$$[[\cdots[[g_{i_1}, g_{i_2}], g_{i_3}], \cdots], g_{i_k}], \quad i_j \in \{1, \ldots, t\}$$

with all of its brackets to the left of all the elements occuring is called a simple k-fold commutator and is denoted by

$$[g_{i_1}, g_{i_2}, \cdots, g_{i_k}].$$

For each $k \geq 1$, the subgroups $\Gamma_G(k)$ of the lower central series of G are defined recursively by

$$\Gamma_G(1) = G, \quad \Gamma_G(k+1) = [\Gamma_G(k), G].$$

We use the following basic lemma in later sections.

Lemma 2.1. *If a group G is generated by $g_1, \ldots, g_t$, then each of the graded quotients $\Gamma_G(k)/\Gamma_G(k+1)$ for $k \geq 1$ is generated by the cosets of the simple k-fold commutators*

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}], \quad i_j \in \{1, \ldots, t\}.$$

Now, for each $k \geq 1$, let $\Gamma_n(k)$ be the k-th subgroup $\Gamma_{F_n}(k)$ of the lower central series of a free group F_n of rank n and $\mathrm{gr}^k(\Gamma_n)$ its graded quotient $\Gamma_n(k)/\Gamma_n(k+1)$. We denote by $\mathrm{gr}\,(\Gamma_n) = \bigoplus_{k\geq 1}\mathrm{gr}^k(\Gamma_n)$ the associated graded sum. Then the set $\mathrm{gr}\,(\Gamma_n)$ naturally has a structure of a graded Lie algebra over $\mathbf{Z}$ induced from the commtator bracket on F_n. Let H be the abelianization of F_n and $\mathcal{L}_n = \bigoplus_{k\geq 1}\mathcal{L}_n(k)$ the free graded Lie algebla generated by H. It is well known that the Lie algebra $\mathrm{gr}\,(\Gamma_n)$ is isomorphic to $\mathcal{L}_n$ as a graded Lie algebra over $\mathbf{Z}$. Thus, in this paper, we identify $\mathrm{gr}\,(\Gamma_n)$ with $\mathcal{L}_n$. For any element $x \in \Gamma_n(k)$, we also denote by x the coset class of x in $\mathcal{L}_n(k) = \Gamma_n(k)/\Gamma_n(k+1)$. Let $T(H)$ be the tensor algebra of H over $\mathbf{Z}$. Then the algebra $T(H)$ is the universal envelopping algebra of the free Lie algebra $\mathcal{L}_n$ and the natural map $\mathcal{L}_n \to T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for X, $Y \in \mathcal{L}_n$ is an injective Lie algebra homomorphism. Hence we also regard $\mathcal{L}_n(k)$ as a submodule of $H^{\otimes k}$ for each $k \geq 1$.

2.2. IA-automorphism group.

The kernel of the natural map $\mathrm{Aut}\,F_n \to GL(n, \mathbf{Z})$ which is given by the action of $\mathrm{Aut}\,F_n$ on H is called the IA-automorphism group of F_n and denoted by IA_n. Let $\{x_1, \ldots, x_n\}$ be a basis of a free group F_n. Magnus [10] showed that IA_n is finitely generated by automorphisms

$$K_{ab} : \begin{cases} x_a & \mapsto x_b^{-1} x_a x_b, \\ x_t & \mapsto x_t, \quad (t \neq a) \end{cases}$$

and

$$K_{abc} : \begin{cases} x_a & \mapsto x_a x_b x_c {x_b}^{-1} {x_c}^{-1}, \\ x_t & \mapsto x_t, \quad (t \neq a) \end{cases}$$

for any distinct a, b and $c \in \{1, 2, \ldots, n\}$. It is known that the abelianization IA_n^{ab} of the IA-automorphism group is free abelian group with generators K_{ab} for distinct a and b, and K_{abc} for distinct a, b, c and $b < c$. More precisely, if we denote by $H^* = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H, we have a $GL(n, \mathbf{Z})$-module isomorphism $IA_n^{\mathrm{ab}} \simeq H^* \otimes_{\mathbf{Z}} \Lambda^2 H$. (For details, see [8].)

2.3. The associated graded Lie algebra.

Here we consider two descending filtrations of IA_n. The first one is $\{\mathcal{A}_n(k)\}_{k \geq 1}$ defined as above. Since the series $\mathcal{A}_n(1), \mathcal{A}_n(2), \ldots$ is central, the associated graded sum $\mathrm{gr}(\mathcal{A}_n) = \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ naturally has a structure of a graded Lie algebla over $\mathbf{Z}$ induced from the commutator bracket on $\mathcal{A}_n(1)$. For each $k \geq 1$, the group $\mathcal{A}_n(0) = \mathrm{Aut}\, F_n$ naturally acts on $\mathcal{A}_n(k)$ by conjugation, hence on $\mathrm{gr}^k(\mathcal{A}_n)$. Since the group $\mathcal{A}_n(1) = IA_n$ trivially acts on $\mathrm{gr}^k(\mathcal{A}_n)$, we see that the group $GL(n, \mathbf{Z}) \simeq \mathcal{A}_n(0)/\mathcal{A}_n(1)$ naturally acts on $\mathrm{gr}^k(\mathcal{A}_n)$.

The other is the lower central series $\mathcal{A}'_n(1), \mathcal{A}'_n(2), \ldots$ of $A_n(1)$. Let $\mathrm{gr}^k(\mathcal{A}'_n) = \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$ be the graded quotient for each $k \geq 1$. Similarly the associated graded sum $\mathrm{gr}(\mathcal{A}'_n) = \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}'_n)$ has a structure of a graded Lie algebra structure on $\mathbf{Z}$. Moreover, each graded quotient $\mathrm{gr}^k(\mathcal{A}'_n)$ is a $GL(n, \mathbf{Z})$-module. It is clear that $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for every $k \geq 1$. In particular, we have $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for $1 \leq k \leq 2$ and Pettet [15] showed that $\mathcal{A}'_n(3)$ has finite index in $\mathcal{A}_n(3)$ as mentioned in section 1. From Lemma 2.1, for each $k \geq 1$, the graded quotient $\mathrm{gr}^k(\mathcal{A}'_n)$ is generated by (the cosets of) the simple k-fold commutators among the components K_{ab} and K_{abc}.

2.4. Johnson homomorphism.

Here we define the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. For each $k \geq 1$, let $\tau_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ be the map defined by

$$\sigma \mapsto (x \mapsto x^{-1} x^{\sigma}) \tag{1}$$

for $\sigma \in \mathcal{A}_n(k)$ and $x \in H$. Then the map τ_k is a homomorphism and the kernel of τ_k is just $\mathcal{A}_n(k+1)$. Hence, identifying $\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, we obtain an injective $GL(n, \mathbf{Z})$-equivariant homomorphism, also denoted by τ_k,

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

This homomorphism is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. Similarly, for each $k \geq 1$, we can define a homomorphism $\tau'_k : \mathcal{A}'_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ as (1). Since $\mathcal{A}'_n(k+1)$ is contained in the kernel of τ'_k, we obtain a $GL(n, \mathbf{Z})$-equivariant homomorphism, also denoted by τ'_k,

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

We also call the map τ'_k the Johnson homomorphism of $\mathrm{Aut}\, F_n$.

Let $\{x_1, \ldots, x_n\}$ be a basis of F_n. It defines a basis of H as a free abelian group, also denoted by $\{x_1, \ldots, x_n\}$. Let $\{x_1^*, \ldots, x_n^*\}$ be the dual basis of H^*. For any $\sigma \in \mathrm{A}'_n(k)$, if we set $s_i(\sigma) := x_i^{-1} x_i^{\sigma} \in \mathcal{L}_n(k+1)$ $(1 \leq i \leq n)$ then we have

$$\tau_k(\sigma) = \tau'_k(\sigma) = \sum_{i=1}^{n} x_i^* \otimes s_i(\sigma) \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

Let $\mathrm{Der}\,(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree k part of $\mathrm{Der}\,(\mathcal{L}_n)$ is expressed as $\mathrm{Der}\,(\mathcal{L}_n)(k) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k)$. Thus we sometimes identify $\mathrm{Der}\,(\mathcal{L}_n)$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$. Then the Johnson homomorphism $\tau = \bigoplus_{k \geq 1} \tau_k$ is a graded Lie algebra homomorphism. In fact, if we denote by $\partial\sigma$ the element of $\mathrm{Der}\,(\mathcal{L}_n)$ corresponding to an element $\sigma \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$ and write the action of $\partial\sigma$ on $X \in \mathcal{L}_n$ as $X^{\partial\sigma}$ then we have

$$\tau'_{k+l}([\sigma, \tau]) = \sum_{i=1}^{n} x_i^* \otimes (s_i(\sigma)^{\partial\tau} - s_i(\tau)^{\partial\sigma}). \tag{2}$$

for any $\sigma \in \mathrm{A}'_n(k)$ and $\tau \in \mathrm{A}'_n(l)$.

In general, each $s_i(\sigma) \in \mathcal{L}_n(k+1)$ cannot be uniquely written as a sum of commutators among the components $x_1, \ldots, x_n$. In this paper, each $s_i(\sigma)$ is recursively computed in the following way. First, for $\sigma = K_{abc}$, we can set

$$s_a(K_{abc}) = [x_b, x_c], \quad s_t(K_{abc}) = 0 \quad \text{if} \quad t \neq a.$$

For $\sigma = K_{ab}$, we see that

$$x_t^{-1} x_t^{\sigma} = \begin{cases} [x_a^{-1}, x_b^{-1}] & \text{if} \quad t = a, \\ 1 & \text{if} \quad t \neq a \end{cases}$$

in F_n. Since $[x_a^{-1}, x_b^{-1}] = [x_a, x_b]$ in $\mathcal{L}_n(2)$, so we can set

$$s_a(K_{ab}) = [x_a, x_b], \quad s_t(K_{ab}) = 0 \quad \text{if} \quad t \neq a.$$

Next, if $\sigma = [\tau, K_{ab}]$ for k-fold simple commutator τ, following from (2), we can set

$$s_i(\sigma) = s_i(\tau)^{\partial K_{ab}} - s_i(K_{ab})^{\partial \tau}$$

for each i. Furthermore, since a commutator bracket of weight l is considered as a l-fold multilinear map from the cartesian product of l copies of $\mathcal{L}_n(1)$ to $\mathcal{L}_n(l)$, we can also set

$$s_i(\sigma) = \sum_{p=1}^{\alpha(i)} (-1)^{e_{i,p}} C_{i,p}$$

where $e_{i,p} = 0$ or 1, and $C_{i,p}$ is a commutator of degree $k+1$ among the components $x_1, \ldots, x_n$. We compute $s_i([\tau, K_{abc}])$ for $\sigma = [\tau, K_{abc}]$ similarly. These computations are perhaps easiest explained with examples, so we give two here. For distinct a, b, c and d, we have

$$\begin{aligned}
\tau_2'([K_{ab}, K_{bac}]) &= x_a^* \otimes ([x_a, x_b])^{\partial K_{bac}} - x_b^* \otimes ([x_a, x_c])^{\partial K_{ab}}, \\
&= x_a^* \otimes [x_a, [x_a, x_c]] - x_b^* \otimes [[x_a, x_b], x_c]
\end{aligned}$$

and

$$\begin{aligned}
\tau_3'([K_{ab}, K_{bac}, K_{ad}]) &= x_a^* \otimes ([x_a, [x_a, x_c]])^{\partial K_{ad}} - x_b^* \otimes ([[x_a, x_b], x_c])^{\partial K_{ad}} \\
&\quad - x_a^* \otimes ([x_a, x_d])^{\partial [K_{ab}, K_{bac}]}, \\
&= x_a^* \otimes [[x_a, x_d], [x_a, x_c]] + x_a^* \otimes [x_a, [[x_a, x_d], x_c]] \\
&\quad - x_b^* \otimes [[[x_a, x_d], x_b], x_c] \\
&\quad - x_a^* \otimes [[x_a, [x_a, x_c]], x_d].
\end{aligned}$$

3. The contractions

For $k \geq 1$ and $1 \leq l \leq k+1$, let $\varphi_l^k : H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$ be the contraction map defined by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_l}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_{l-1}} \otimes x_{j_{l+1}} \otimes \cdots \otimes x_{j_{k+1}}.$$

For the natural embedding $\iota_n^{k+1} : \mathcal{L}_n(k+1) \to H^{\otimes(k+1)}$, we obtain a $GL(n, \mathbf{Z})$-equivariant homomorphism

$$\Phi_l^k = \varphi_l^k \circ (id_{H^*} \otimes \iota_n^{k+1}) : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We also call the map Φ_l^k contraction.

Here we introduce one of methods of the computation of $\Phi_l^k(x_i^* \otimes C)$ for a commutator $C \in \mathcal{L}_n(k+1)$ among the components $x_1, \ldots, x_n$. In this paper, whenever we compute $\Phi_l^k(x_i^* \otimes C)$, we use the following method. First, if x_i does not appear among the components of C, then $\Phi_l^k(x_i^* \otimes C) = 0$. On the other hand, if x_i appears among the components of C m times, then we distinguish them and write such x_i's as $x_{i_1}, \ldots, x_{i_m}$ in C. Then $\Phi_l^k(x_i^* \otimes C)$ is given by rewriting $x_{i_1}, \ldots, x_{i_m}$ as x_i in

$$\sum_{j=1}^{m} \Phi_l^k(x_{i_j}^* \otimes C).$$

Thus it suffices to compute $\Phi_l^k(x_i^* \otimes C)$ for a commutator C which has only one x_i in its components. Now, C is written as $[X, Y]$ for some commutators X and Y. Rewriting the commutator C as $-[Y, X]$ if x_i appears in Y, we may always consider $C = \pm[X, Y]$ such that x_i appears among the components of X. By a recursive argument, we have $C = \pm[x_i, C_1, \ldots, C_t]$ where each C_j $(1 \leq j \leq t)$ is a commutator of weight d_j and $d_1 + \cdots + d_t = k$.

Lemma 3.1. *For a commutator $[x_i, C_1, \ldots, C_t] \in \mathcal{L}_n(k+1)$ as above,*

$$\Phi_1^k(x_i^* \otimes [x_i, C_1, \ldots, C_t]) = C_1 \otimes \cdots \otimes C_t.$$

Lemma 3.2. *For a commutator $[x_i, C_1, \ldots, C_t] \in \mathcal{L}_n(k+1)$ as above,*

$$\Phi_2^k(x_i^* \otimes [x_i, C_1, \ldots, C_t]) = -\sum_{\mathrm{wt}\,(C_j)=1} C_j \otimes C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_t.$$

Let $T(H) = \bigoplus_{k\geq 1} H^{\otimes k}$ and $S(H) = \bigoplus_{k\geq 1} S^k H$ be the tensor algebra and the symmetric algebra on H respectively. Then the kernel of a natural map $T(H) \to S(H)$ is a graded ideal of $T(H)$, and denoted by $I(H) = \bigoplus_{k\geq 1} I^k(H)$. For each $k \geq 2$, let $\mathcal{U}_n(k)$ be the $GL(n, \mathbf{Z})$-submodule of $H^{\otimes k}$ generated by elements type of

$$[A, B] := A \otimes B - B \otimes A$$

for $A \in H^{\otimes a}$, $B \in H^{\otimes b}$ and $a + b = k$. If we put $\mathcal{U}_n = \bigoplus_{k\geq 1} \mathcal{U}_n(k)$, then $\mathcal{U}_n$ is the kernel of the abelianizaton $T(H) \to T(H)^{\mathrm{ab}}$ as a Lie algebra. We have

$$\mathcal{L}_n(k) \subset \mathcal{U}_n(k) \subset I^k(H) \subset H^{\otimes k}.$$

3.1. The image of $\Phi_l^k \circ \tau_k'$.

Considering the image of any simple k-fold commutator σ among the components K_{ab} and K_{abc}, we prove the following proposotions.

Proposition 3.1. *For $n \geq 3$ and $k \geq 2$,* $\mathrm{Im}\,(\Phi_1^k \circ \tau_k') \subset \mathcal{U}_n(k)$.

Proposition 3.2. *For $n \geq 3$ and $k \geq 3$,* $\mathrm{Im}\,(\Phi_2^k \circ \tau_k') \subset H \otimes_{\mathbf{Z}} \mathcal{U}_n(k-1)$.

4. The trace maps

In this section, using the contractions defined in Section 3, we define a homomorphisms called the trace map which vanishes on the image of the Johnson homomorphism. Here we use some basic facts of the representation theory of $GL(n, \mathbf{Z})$. The reader is referred to, for example, Fulton-Harris [4] and Fulton [3].

For any $k \geq 1$ and any partition λ of k, we denote by H^λ the Schur-Weyl module of H corresponding to the partition λ of k. Let $f_\lambda : H^{\otimes k} \to H^\lambda$ be a natural homomorphism. In this paper, we mainly consider the case for $\lambda = [k]$ or $[1^k]$. The modules $H^{[k]}$ and $H^{[1^k]}$ are the symmetric product $S^k H$ and the exterior product $\Lambda^k H$ respectively. Using the natural map $\iota_n^k : \mathcal{L}_n(k) \to H^{\otimes k}$, we denote $f_{[1^k]} \circ \iota_n^k(C)$ by $\widehat{C}$ for any $C \in \mathcal{L}_n(k)$.

Lemma 4.1. *For any commutator C of weight $k \geq 3$, $\widehat{C} = 0$ in $\Lambda^k H$*

Lemma 4.2. *For $1 \leq k \leq n-2$ and any commutator C of weight $k+1$ among the components $x_1, \ldots, x_n$ except for x_i, there exists an element $\sigma \in \mathcal{A}_n'(k)$ such that*

$$\tau_k'(\sigma) = x_i^* \otimes C \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

4.1. Morita's trace (Trace map for $S^k H$).

Here we consider the map

$$\mathrm{Tr}_{[k]} = f_{[k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H.$$

By definition, this map coincides with the Morita's trace Tr_k. For $n \geq 3$ and $k \geq 2$, Morita defined the trace map Tr_k using the Magnus representation of $\mathrm{Aut}\, F_n$ and showed that Tr_k vanishes on the image of τ_k. By a recent work, he showed that $\mathrm{Tr}_k^{\mathbf{Q}}$ is surjective. Hence we have

Theorem 4.1. (Morita) *For $n \geq 3$ and $k \geq 2$,*

$$S^k H_{\mathbf{Q}} \subset \mathrm{Coker}\, \tau_{k,\mathbf{Q}}.$$

Corollary 4.1. *For $n \geq 3$ and $k \geq 2$,*

$$\operatorname{rank}_{\mathbf{Z}}(\operatorname{Coker}(\tau_k)) \geq \binom{n+k-1}{k}.$$

4.2. **Trace map for $\Lambda^k H$.**

Here we consider the map

$$\operatorname{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H.$$

Theorem 4.2.

(1) *For $3 \leq k \leq n$, $\operatorname{Tr}_{[1^k]}$ is surjective,*
(2) $\operatorname{Im}(\operatorname{Tr}_{[1^k]} \circ \tau_k') = 0$ *if k is odd and $3 \leq k \leq n$,*
(3) $\operatorname{Im}(\operatorname{Tr}_{[1^k]} \circ \tau_k') = 2(\Lambda^k H) \subset \Lambda^k H$ *if k is even and $4 \leq k \leq n-2$.*

Corollary 4.2. *For an odd k and $3 \leq k \leq n$,*

$$\Lambda^k H_{\mathbf{Q}} \subset \operatorname{Coker} \tau_{k,\mathbf{Q}}'.$$

Corollary 4.3. *For an odd k and $3 \leq k \leq n$,*

$$\operatorname{rank}_{\mathbf{Z}}(\operatorname{Coker}(\tau_k')) \geq \binom{n}{k}.$$

4.3. **Trace map for $H^{[2,1^{k-2}]}$.**

Here we consider the map

$$\operatorname{Tr}_{[2,1^{k-2}]} := (id_H \otimes f_{[1^{k-1}]}^{k-1}) \circ \Phi_2^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H \otimes_{\mathbf{Z}} \Lambda^{k-1} H.$$

Let I be the $GL(n, \mathbf{Z})$-submodule of $H \otimes_{\mathbf{Z}} \Lambda^{k-1} H$ defined by

$$I = \langle x \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge y + y \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge x \mid x, y, z_t \in H \rangle.$$

Theorem 4.3. *For an even k and $4 \leq k \leq n-1$,*

(1) $\operatorname{Im}(\operatorname{Tr}_{[2,1^{k-1}]}^{\mathbf{Q}}) = I_{\mathbf{Q}}$,
(2) $\operatorname{Im}(\operatorname{Tr}_{[2,1^{k-1}]} \circ \tau_k') = 0$.

Now we have $H_{\mathbf{Q}} \otimes_{\mathbf{Z}} \Lambda^{k-1} H_{\mathbf{Q}} \simeq H_{\mathbf{Q}}^{[2,1^{k-2}]} \oplus \Lambda^k H_{\mathbf{Q}}$ from the representation theory of $GL(n, \mathbf{Z})$. For even k, since $I_{\mathbf{Q}}$ is contained in the kernel of a natural map $H_{\mathbf{Q}} \otimes_{\mathbf{Z}} \Lambda^{k-1} H_{\mathbf{Q}} \to \Lambda^k H_{\mathbf{Q}}$ defined by $x \otimes y_1 \wedge \cdots \wedge y_{k-1} \mapsto x \wedge y_1 \wedge \cdots \wedge y_{k-1}$, we have $I_{\mathbf{Q}} \simeq H_{\mathbf{Q}}^{[2,1^{k-2}]}$.

Corollary 4.4. *For an even k and $4 \leq k \leq n-1$,*

$$H_{\mathbf{Q}}^{[2,1^{k-2}]} \subset \operatorname{Coker} \tau_{k,\mathbf{Q}}'.$$

Corollary 4.5. *For an even k and $4 \leq k \leq n-1$,*

$$\operatorname{rank}_{\mathbf{Z}}(\operatorname{Coker}(\tau_k')) \geq (k-1)\binom{n+1}{k}.$$

5. The cokernel of the Johnson homomorphism τ_k for $k = 2$ and 3

5.1. The case $k = 2$.

In this subsection we consider the case where $n \geq 3$. From Theorem 4.1 and $\operatorname{rank}_{\mathbf{Z}}(\operatorname{Coker}(\tau_2)) = \binom{n+1}{2}$ by Pettet [15], we have a $GL(n, \mathbf{Z})$-equivariant exact sequence

$$0 \to \operatorname{gr}^2_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{2,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}^{\mathbf{Q}}_n(3) \to S^2 H_{\mathbf{Q}} \to 0.$$

We show that the exact sequence above holds before tensoring with $\mathbf{Q}$. Namely,

Theorem 5.1. *For $n \geq 3$,*

$$0 \to \operatorname{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_2} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to S^2 H \to 0$$

is a $GL(n, \mathbf{Z})$-equivariant exact sequence.

5.2. The case $k = 3$.

Next we compute the cokernel of the Johnson homomorphism $\tau_{3,\mathbf{Q}}$ for $n \geq 3$ using the fact that $\operatorname{Coker} \tau_{3,\mathbf{Q}} = \operatorname{Coker} \tau'_{3,\mathbf{Q}}$. We have

Theorem 5.2. *For $n \geq 3$,*

$$0 \to \operatorname{gr}^3_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{3,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}^{\mathbf{Q}}_n(4) \to S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}} \to 0$$

is a $GL(n, \mathbf{Z})$-equivariant exact sequence.

Corollary 5.1. *For $n \geq 3$,*

$$\operatorname{rank}_{\mathbf{Z}} \operatorname{gr}^3(\mathcal{A}_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8). \tag{3}$$

In particular, substituting $n = 3$ into (3), we have $\operatorname{rank}_{\mathbf{Z}} \operatorname{gr}^3(\mathcal{A}_3) = 43$.

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA MEGURO-KU TOKYO 153-0041, JAPAN

E-mail address: `takao@ms.u-tokyo.ac.jp`