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A SURVEY OF LENGTH SERIES IDENTITIES FOR SURFACES, 3-MANIFOLDS AND REPRESENTATION VARIETIES

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ABSTRACT. We survey some of our recent results on length series identities for hyperbolic (cone) surfaces, possibly with cusps and/or boundary geodesics; classical Schottky groups; representations/characters of the one-holed torus group to $\mathrm{SL}(2,\mathbf{C})$; and hyperbolic 3 manifolds obtained by hyperbolic Dehn surgery on punctured torus bundles over the circle. These can be regarded as generalizations and variations of McShane's identity for cusped hyperbolic surfaces, which has found some striking applications in the recent work of Mirzakhani. We discuss some of the methods and techniques used to obtain these identities.

1. Introduction

In his thesis [12], Greg McShane gave a remarkable series identity for the lengths of simple closed geodesics on a complete hyperbolic torus with one cusp. He generalized this later in [13] to an identity for a complete hyperbolic surface M with cusps. The identity he obtained is as follows:

Theorem 1.1. (McShane [13]) *In a finite area hyperbolic surface M with cusps and without boundary, let Δ_0 be a distinguished cusp of M. Then*

$$\sum \frac{1}{1+\exp\frac{1}{2}(|\alpha|+|\beta|)} = \frac{1}{2}, \tag{1}$$

where the sum is taken over all unordered pairs of simple closed geodesics α, β (where α or β might be a cusp treated as a simple closed geodesic of length 0) on M such that α, β and Δ_0 bound an embedded pair of pants on M, and $|\alpha|$ denotes the length of α.

In the case of the cusped torus, $\alpha = \beta$ for all pairs in the sum, and the sum is over all simple closed geodesics α on the torus, which was the original identity obtained in his thesis.

The proof of the identity was mostly geometric/topological. For simplicity, consider the case where the surface M has only one cusp Δ_0. Let $\mathcal{H}$ be the set of all geodesics emanating from Δ_0. The subset $\mathcal{S} \subset \mathcal{H}$ of *simple* geodesics (no self-intersection) emanating from Δ_0 turns out to be rather sparse, in fact, by the Birman-Series Theorem [3], this set $\mathcal{S}$ has zero measure in the set $\mathcal{H}$. Furthermore, apart from a countable set of isolated points corresponding to simple geodesics which also terminate at Δ_0, this set forms a Cantor subset of $\mathcal{H}$. Now identifying $\mathcal{H}$ with a horocycle of length one about the cusp, it turns out each gap formed by

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the complement of the Cantor set has end points corresponding to simple geodesics which spiral around *simple closed geodesics* α and β on the surface (with 'opposite' spiralling orientation), where the pair α and β bound together with Δ_0 an embedded pair of pants on the surface M. Conversely, to every such pair α, β, there are two gaps with end points corresponding to simple geodesics which spiral around α and β with opposite orientation, and they both have the same width. Furthermore, by a simple hyperbolic geometry calculation, the width of each gap depends only on $|\alpha|$ and $|\beta|$ and is given by the summand in the left hand side of (1). Theorem 1.1 then follows. It should be noted that the hyperbolic geometry needed to obtain the formula can be restricted to pairs of pants.

The isolated simple geodesics in $\mathcal{H}$ which start and end at Δ_0 also have an important geometric interpretation, each such geodesic δ defines uniquely a pair of geodesics α and β on M bounding with Δ_0 an embedded pair of pants in M such that the geodesic δ is embedded in the pair of pants. Furthermore, the two ends of δ lie in the corresponding two pairs of gaps.

In brief, the key ingredients in the proof of Theorem 1.1 are:

- the study of the set of simple geodesics on M emanating from the cusp Δ_0,
- the Birman-Series theorem, and
- some simple geometric identities for hyperbolic pairs of pants.

There have been several generalizations and variations of the identity:

- Bowditch gave an independent proof of the identity for the cusped torus in [4] and [6], with substantial generalizations to *type-preserving* representations of the punctured torus group to $\mathrm{SL}(2,\mathbf{C})$ satisfying certain conditions, and also a variation of the identity for complete hyperbolic 3-manifolds which are punctured torus bundles over the circle in [5].
- Akiyoshi, Miyachi and Sakuma gave variations of the identity for quasi-fuchsian punctured torus groups (in particular, to certain points on the boundary of quasi-fuchsian space) and hyperbolic punctured surface bundles over the circle in [1] and [2].
- McShane himself gave variations of the identity arising from a similar analysis of simple geodesics passing through the Weierstrass points of a cusped torus and a closed hyperbolic genus two surface in [14] and [15].
- More recently, Mirzakhani proved and used a version of the identity for bordered hyperbolic surfaces (surfaces with totally geodesic boundary) to obtain some striking applications and connections to the Weil-Petersson volume of the moduli space of bordered Riemann surfaces, the asymptotic behavior of the number of simple closed geodesics of length less than $L > 0$ on a closed hyperbolic surface and the Kontsevich-Witten formula on the intersection numbers of tautological classes on the moduli space of curves in [16], [17] and [18]. An important observation used in her papers is that the identity is independent of the hyperbolic structure on M (with fixed boundary lengths), that is, it holds for all points in the moduli space.

In a different direction, we have also given generalizations and variations of the identity to

- hyperbolic cone surfaces with cusps and/or geodesic boundary (with all cone angles bounded above by π), with applications to generalizations of

the Weierstrass identities for the one-hole/cone torus and closed genus two surface [19];
- classical Schottky groups with applications to hyperbolic surfaces with geodesic boundary in [20]; and
- general (not necessarily type-preserving) representations of the punctured torus groups to $SL(2, \mathbf{C})$ with applications to *closed* hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on hyperbolic punctured torus bundles over the circle [22], [21].

In this paper, we will give an exposition of some of the main results and ideas in [19], [20], [21] and [22], and also the connection with the works of McShane, Bowditch, Mirzakhani, and Akiyoshi-Miyachi-Sakuma.

The main point of departure from the works of McShane, Bowditch, and Akiyoshi-Miyachi-Sakuma in [19] is that we allow Δ_0 (more generally Δ_j) to vary, so that it is not necessarily a cusp, but may be a cone point or boundary geodesic. In particular, we consider geometric structures whose holonomy groups are not necessarily discrete (which represents also a departure from the point of view of Mirzakhani). From this point of view, it is natural to consider cone points to have purely imaginary length. Extending this idea further, more generally, we may consider representations of the surface group to $SL(2, \mathbf{C})$ so that the lengths, in particular, the boundary lengths, are not necessarily real or purely imaginary. From this, we obtain generalizations of the identity to classical Schottky groups in [20]. The main tools are analytic continuation, a lifting argument, an application of the Birman-Series argument, and some general comparison results for the combinatorial length and complex length of an element of the fundamental group corresponding to a simple closed curve on a marked surface. In particular, this produces some surprising new identities for hyperbolic surfaces with boundary, arising from non-standard markings, for example, we have a nontrivial series identity for the hyperbolic pair of pants.

Finally, in [22] and [21], we pick up on the powerful ideas and combinatorial techniques of Bowditch to show that a very general version of the identity can be proved to hold for general $SL(2, \mathbf{C})$ characters of a one-holed torus satisfying some simple conditions. Similarly, we also show that various relative and restricted versions of the identity hold. In particular, we are able to give necessary and sufficient conditions (extended Bowditch Q-conditions) for the identity to hold for a general $SL(2, \mathbf{C})$ character of the one-holed torus, [21], and furthermore, to give relative versions of the identity to characters which are stabilized by certain cyclic subgroups of the mapping class group generated either by an Anosov or reducible element, and which satisfy a relative version of the Bowditch Q-conditions [22]. These in turn have applications to complete and incomplete hyperbolic structures on punctured torus bundles over the circle, and in particular, give length series identities for "almost all" closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on a complete hyperbolic torus bundle over the circle.

A feature of these techniques is that we do not have to use analytic continuation to obtain the identities, and also, the identities can be proven for very general representations for which the geometric interpretation is not necessarily clear. Another interesting feature of this method is that it gives an independent proof of the Birman-Series result that the set of simple complete geodesics is sparse, the point is that in the proof of the series identity, one is able to prove not just the absolute convergence of the series, but also to show that a suitably interpreted error term approaches 0.

The rest of this paper is organized as follows. In §2, we discuss the identities for cone surfaces, and also applications via covering arguments to generalized Weierstrass identities for the one-holed torus and genus two surface. In §3, we discuss the identities for the classical Schottky groups and finally in §4, we discuss the identities for $\mathrm{SL}(2,\mathbf{C})$ characters of a one-holed torus.

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2. Hyperbolic cone surfaces

McShane's original identity (1) can be generalized to hyperbolic cone surfaces, possibly with cusps and/or totally geodesic boundary, where all cone points have cone angles less than or equal to π. For this purpose, it is convenient to consider the cone points, cusps and boundary geodesics as *geometric* boundary components of M and to define the complex length of a cone point as $i\theta$, where θ is the cone angle, the complex length of a cusp as 0, and the complex length of a boundary geodesic as just the usual hyperbolic length. We call such a surface M a *compact hyperbolic cone surface.* We also define a *generalized simple closed geodesic* as

(a) a simple closed geodesic in the geometric interior of M; or
(b) a geometric boundary component (cone point/cusp/geodesic boundary) of M, with the corresponding complex lengths as defined earlier; or
(c) the double (cover) of a simple geodesic segment joining two angle π cone points on M, with length twice the length of the geodesic segment.

The result is then stated as follows.

Theorem 2.1. (Theorem 1.16 [19]) *Let M be a compact hyperbolic cone surface with all cone angles in $(0,\pi]$, and geometric boundary components $\Delta_0,\Delta_1,\cdots,\Delta_N$ with complex lengths $L_0,L_1,\cdots,L_N$ respectively. Then*

$$\sum_{\alpha,\beta} 2\tanh^{-1}\left(\frac{\sinh\frac{L_0}{2}}{\cosh\frac{L_0}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right) + \sum_{j=1}^{N}\sum_{\beta}\tanh^{-1}\left(\frac{\sinh\frac{L_0}{2}\sinh\frac{L_j}{2}}{\cosh\frac{|\beta|}{2}+\cosh\frac{L_0}{2}\cosh\frac{L_j}{2}}\right)=\frac{L_0}{2}, \tag{2}$$

if Δ_0 is a cone point or a boundary geodesic; and

$$\sum_{\alpha,\beta}\frac{1}{1+\exp\frac{|\alpha|+|\beta|}{2}}+\sum_{j=1}^{N}\sum_{\beta}\frac{1}{2}\frac{\sinh\frac{L_j}{2}}{\cosh\frac{|\beta|}{2}+\cosh\frac{L_j}{2}}=\frac{1}{2}, \tag{3}$$

if Δ_0 is a cusp; where in either case the first sum is taken over all unordered pairs of generalized simple closed geodesics α,β on M which bound with Δ_0 an embedded pair of pants on M (note that one of α,β might be a geometric boundary component) and the sub-sum in the second sum is taken over all generalized simple *closed geodesics β which bounds with Δ_j and Δ_0 an embedded pair of pants on M. Furthermore, each series in* (2) *and* (3) *converges absolutely.*

The summands in the first sum correspond to main gaps and those in the second series correspond to side gaps. Note that McShane's identity (1) is a special case of (3) where Δ_0 is a cusp and all the summands in the second series are zero since none of Δ_j, $j = 1, \dots N$ are cone points or boundary geodesics. Also, the identity (3) can be derived from the first order infinitesimal terms of the identity (2).

For the purpose of generalizations to classical Schottky groups later, we define the functions $G(x,y,z)$ and $S(x,y,z)$ corresponding to the "main gaps" and the "side gaps" as follows, where the log function takes its principal branch, i.e., with imaginary part in $(-\pi,\pi]$, and the function $\tanh^{-1}$ is defined by

$$\tanh^{-1}(x) = \frac{1}{2}\log\frac{1+x}{1-x} \qquad \text{for } x \in \mathbf{C}\backslash\{\pm 1\},$$

and hence has imaginary part in $(-\pi/2, \pi/2)$.

Definition 2.2. For $x, y, z \in \mathbf{C}$, we define

$$G(x,y,z) := 2\tanh^{-1}\left(\frac{\sinh(x)}{\cosh(x)+\exp(y+z)}\right), \tag{4}$$

$$S(x,y,z) := \tanh^{-1}\left(\frac{\sinh(x)\sinh(y)}{\cosh(z)+\cosh(x)\cosh(y)}\right). \tag{5}$$

It can be shown that $G(x,y,z)$ and $S(x,y,z)$ can also be expressed as

$$G(x,y,z) = \log\frac{\exp(x)+\exp(y+z)}{\exp(-x)+\exp(y+z)}, \tag{6}$$

$$S(x,y,z) = \frac{1}{2}\log\frac{\cosh(z)+\cosh(x+y)}{\cosh(z)+\cosh(x-y)}, \tag{7}$$

as used by Mirzakhani in [16].

The basic idea of the proof of Theorem 2.1 is similar to that in [13], we pick a distinguished boundary component Δ_0 (which may be a cone point, cusp, or boundary geodesic), and consider the set of all geodesics $\mathcal{H}$ emanating normally from Δ_0 (one only needs to worry about "normally" when Δ_0 is a boundary geodesic). Topologically, this set is a circle; geometrically, we can put a natural measure on this circle as follows. In the case Δ_0 is a cusp, we identify $\mathcal{H}$ as before with the horocycle of length one around Δ_0; in the case Δ_0 is a boundary geodesic of length L_0, we identify $\mathcal{H}$ with Δ_0 itself with length L_0; and in the case where Δ_0 is a cone point with cone angle θ_0, we identify $\mathcal{H}$ with a circle about Δ_0 with the natural radian measure θ_0. We then consider the subset $\mathcal{S} \subset \mathcal{H}$ consisting of the *simple, complete* geodesics, by which, we mean the geodesics emanating normally from Δ_0 which are simple and do not terminate at a cone point, cusp, or boundary geodesic, hence are complete in the forward direction. By a slight variation of the Birman-Series Theorem, this set again has zero measure in $\mathcal{H}$, and, as in the previous case, is essentially a Cantor set (there may be a countable collection of isolated points). As before, the complement $\mathcal{H} \setminus \mathcal{S}$ consists of gaps bounded by end points which correspond to geodesics spiralling around simple closed curves. However, in this case, besides the main gaps which we had before, side gaps can occur, if some of the other boundary components Δ_k are boundary geodesics. In this case, a side gap is bounded by two points corresponding to simple geodesics that spiral around the same boundary component, but in opposite directions.

It turns out that the analysis is actually easier if we look at the set of geodesics in $\mathcal{H}$ which are not in $\mathcal{S}$, that is, are not simple and complete. In this case, the geodesic either intersects a boundary component, or has a self intersection, in the latter case, we consider the initial part of the geodesic up to the first point of self intersection. In either case, by considering a tubular neighborhood of the union of this geodesic (segment) δ with Δ_0, one obtains two curves α and β, unique up to homotopy which bound together with Δ_0 an embedded pair of pants P in M which contains δ. Now the condition that all cone angles are less than or equal to π ensures that α and β are realizable as generalized geodesics, so we have an embedded pair of pants with Δ_0, α and β as the boundary components. Now the geodesic δ lies in either a main gap or a side gap, see Figures 1 and 2, where the geodesics γ_α and γ_β in Figure 2 are geodesics which spiral around α and β with opposite orientations and bound a main gap. The computation of the width of the gaps proceeds as before but is somewhat more complicated because of the various cases that can occur depending on whether or not one of α, β is a boundary geodesic around a cone point of M. Note that if Δ_j is a cone point, we do not expect to have a side gap from this point of view, however, if we wish to interpret the gaps as analytic functions of the boundary lengths, then there should be a purely imaginary side gap if for example Δ_0 is a boundary geodesic and Δ_j is a cone point. Similarly, in this case, the main gap is no longer real or purely imaginary. In fact, the formula given in Theorem 2.1 takes this analytic point of view and is a complexified, unified version of all these different cases. There is a geometric interpretation of these (complexified) gaps by considering the picture in $\mathbb{H}^3$, see [19] for details.

In the case where there are no cone points, $G(x,y,z)$ and $S(x,y,z)$ are positive real for all summands, and the absolute convergence is trivial, but if there are some cone points, the summands in the formula are not necessarily real and positive, and the absolute convergence of the various series is no longer obvious and requires justification, hence the last statement given in the theorem. The absolute convergence is proven by using a modification of the Birman-Series argument.

It is important to note that for the above analysis to work, all essential simple closed curves should be realizable as generalized simple geodesics, that is, either geodesics or the double cover of a geodesic segment between two angle π cone points. However, for this to be true, we require all cone angles to be less than or equal to π, and our proof is by a convexity argument and a suitable application of the Arzela-Ascoli Theorem. It is not clear how this condition can be relaxed, hence, a McShane type identity for general closed hyperbolic surfaces without boundary remains elusive.

The above is closely related to the formula obtained by Mirzakhani for hyperbolic surfaces with geodesic boundary in [16]. In particular, her analysis works for the cone hyperbolic surfaces we consider and the same (recursive) formula for the Weil-Petersson volumes of the moduli space of bordered Riemann surfaces holds for cone Riemann surfaces (possibly with geodesic boundary), where the lengths of cone boundary components are given by $i\theta$, where θ is the cone angle. It also seems that the same analysis she uses to study the asymptotics of the lengths of simple closed geodesics on closed hyperbolic surfaces in [17] should carry over to the situation we study, namely, there should be a constant C_M depending only on the hyperbolic structure on the cone hyperbolic surface M (with all cone angles bounded above by π) such that the number of simple closed geodesics on M of length less than L is asymptotic to $C_M \cdot L^{6g-6+2N}$ where N is the number of geometric boundary components, and $6g-6+2N > 0$. As for the relation to the Kontsevich-Witten formula, and the recursion formula for the volumes of the moduli space in [18], there is also some recent work of Do and Norbury [7] generalizing Mirzakhani's work to cone surfaces.

We summarize the various points raised above:

- For a cone hyperbolic surface M possibly with cusps and/or geodesic boundary, if all cone angles are less than or equal to π, then all essential simple closed curves on M are realizable by (generalized) simple closed geodesics.
- The Birman-Series theorem generalizes to these cone surfaces. A modification of the argument used in the proof can also be used to prove the absolute convergence of the various series in the identity.
- The gaps formed by taking the complement of the simple complete geodesics emanating normally from a fixed boundary component can be calculated. Apart from the main gaps which occur in the cusped case, side gaps may also occur if there are other boundary components which are cone points or boundary geodesics.
- It is easier to study the set of geodesics which are not simple and complete, these either have self intersection or intersect the boundary of M, and give rise to pairs of pants embedded in the surface.
- The analysis of geodesics in M emanating from a boundary component can be restricted to just the analysis of geodesics in a pair of pants.

Theorem 2.1 together with the fact that a one-holed hyperbolic torus/(closed hyperbolic surface of genus two) admits a canonical elliptic/(hyperelliptic) involution and some general covering arguments can be used to deduce further identities for the one-holed hyperbolic torus/(genus two surface). These can be regarded as generalizations of the Weierstrass identities given by McShane in [14] and [15]. Here when we say a one-holed torus, we mean that the boundary may be a geodesic, cusp or cone point. We have:

Corollary 2.3. (Corollary 1.10 [19]) *Let T be either a hyperbolic one-cone torus where the single cone point has cone angle $\theta \in [0, 2\pi)$ or a hyperbolic one-holed torus where the single boundary geodesic has length $l \geq 0$. Then we have respectively*

$$\sum_{\gamma \in \mathcal{A}} \tan^{-1}\left(\frac{\cos\frac{\theta}{4}}{\sinh\frac{|\gamma|}{2}}\right) = \frac{\pi}{2}, \tag{8}$$

$$\sum_{\gamma \in \mathcal{A}} \tan^{-1}\left(\frac{\cosh\frac{l}{4}}{\sinh\frac{|\gamma|}{2}}\right) = \frac{\pi}{2}, \tag{9}$$

where the sum in either case is taken over all the simple closed geodesics γ in a given Weierstrass class $\mathcal{A}$.

Note that a cusp can be regarded either as a cone point of cone angle 0 or a geodesic of length 0 in either of the cases above.

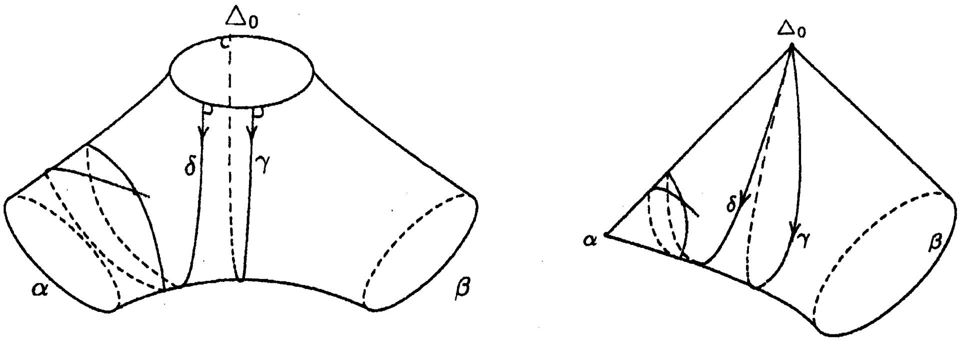

FIGURE 1.

Theorem 2.4. (Theorem 1.13, [19]) *Let M be a genus two closed hyperbolic surface. Then*

$$\sum \tan^{-1} \exp\left(-\frac{|\alpha|}{4} - \frac{|\beta|}{2}\right) = \frac{3\pi}{2}, \tag{10}$$

where the sum is taken over all ordered pairs (α, β) of disjoint simple closed geodesics on M such that α is separating and β is non-separating.

In fact, Corollary 2.3 can be extended to much more general representations of $\pi_1(T)$ to $\mathrm{SL}(2,\mathbf{C})$ (see [20]) and Theorem 2.4 can be extended to quasi-fuchsian representations of $\pi_1(M)$ to $\mathrm{PSL}(2,\mathbf{C})$ ([19] Addendum 1.15).

SKETCH OF PROOF OF COROLLARY 2.3 AND THEOREM 2.4. Let ι be the elliptic involution on T. Then T/ι is a sphere with four boundary components, three of which are cone points of angle π and the fourth a boundary component of length $l/2$ or a cone point of cone angle $\theta/2$ depending on whether T has a boundary geodesic of length l or a cone point of angle θ, respectively. Apply Theorem 2.1 to T/ι with one of the cone points of angle π as Δ_0. Then the sum is over all generalized simple closed geodesics on T/ι which are double covers of geodesic segments joining the other two cone points of angle π, these lift to geodesics on T which are in the Weierstrass class consisting of all geodesics which miss the lift of Δ_0 on T, giving Corollary 2.3. For Theorem 2.4, again consider the hyperelliptic involution ι on M. Then M/ι is a sphere with six cone points, all of cone angle π. Apply Theorem 2.1 to each of the six cone points. For each identity, the sum is now over all pairs of disjoint α' and β' on M/ι such that α' is a geodesic on M/ι which separates it to two pieces each containing three cone points, and β' is a double cover of a geodesic segment on the piece separated by α' containing Δ_0 which connects the other two cone points. Now take the sum over all the six identities and lift the result to M. Note that α' lifts to a separating geodesic on M and β' lifts to a disjoint non-separating geodesic on M and furthermore, all separating geodesics on M project to separating geodesics on M/ι which separate M/ι to two components each containing exactly three cone points while non-separating geodesics on M project to geodesic arcs on M/ι connecting exactly two of the cone points.

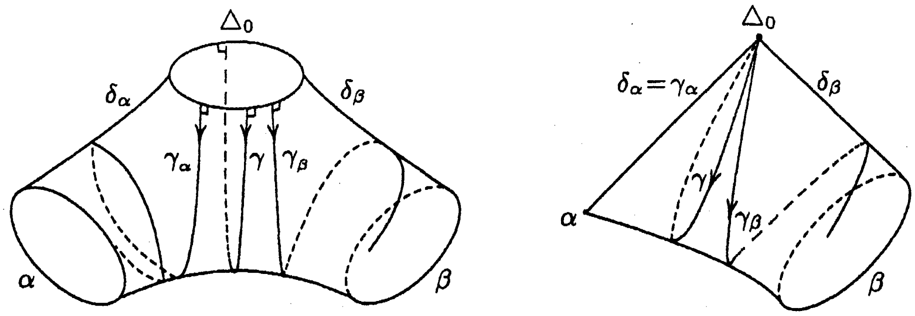

FIGURE 2.

3. Classical Schottky groups

We first note that if M is a hyperbolic surface with geodesic boundary components, then the holonomy group is in fact a fuchsian Schottky group. We next observe that in Theorem 2.1, the summands in the series are all analytic functions of the lengths (if we take the analytic continuation of the $\tanh^{-1}$ function). These are (real) analytic in the parameters of the Teichmüller space, which in turn is locally homeomorphic to the representation variety (modulo conjugation) of representations from $\pi_1(M)$ to $\mathrm{PSL}(2, \mathbf{R})$. It is natural to see if we can apply analytic continuation to obtain generalizations of the result to representations of $\pi_1(M)$ to $\mathrm{PSL}(2, \mathbf{C})$ or $\mathrm{SL}(2, \mathbf{C})$. The absolute convergence of the series in question and the connectedness of the deformation space are the two key issues. There are other important technicalities. It turns out we can do this and obtain series identities for classical Schottky groups which generalize McShane's identity. We summarize below some of the relevant points that crop up:

- Absolute convergence of the series in (2) for classical Schottky groups;
- Connectedness of the deformation space;
- Lifting of the representations from $\mathrm{PSL}(2, \mathbf{C})$ to $\mathrm{SL}(2, \mathbf{C})$;
- Determination of an explicit half-length for transformations in $\mathrm{SL}(2, \mathbf{C})$;
- Choice of a fuchsian marking that will determine how the summands in (2) are obtained.

To start with, we define classical Schottky space. Fix $n \geq 2$. This is the space of (marked) faithful representations from the free group F_n on n generators to $\mathrm{PSL}(2, \mathbf{C})$, up to conjugation, such that the image is a classical Schottky group. We keep track of the marking, as this makes the statement of the results clearer and more precise later.

Definition 3.1. A (*marked*) *classical Schottky group* (of rank n) is a discrete, faithful representation $\rho : F_n \to \mathrm{PSL}(2, \mathbf{C})$ such that there is a region $D \subset \mathbf{C}_\infty$, where D is bounded by $2n$ disjoint geometric circles $C_1, C_1', \cdots, C_n, C_n'$ in $\mathbf{C}_\infty$, so that, for $i = 1, \ldots, n$, $\rho(a_i)(C_i) = C_i'$, and $\rho(a_i)(D) \cap D = \emptyset$. It is *fuchsian* if the representation can be conjugated to a representation into $\mathrm{PSL}(2, \mathbf{R})$. Two representations are equivalent if they are conjugate by an element of $\mathrm{PSL}(2, \mathbf{C})$.

The space of equivalent classes of marked classical Schottky groups is the marked classical Schottky space, denoted by $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$. To simplify notation, we use ρ instead of $[\rho]$ to denote the elements of $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$. Note that every element of a classical Schottky group is loxodromic. One may associate a complex length $l(A)$ to each loxodromic element $A \in \mathrm{PSL}(2, \mathbf{C})$, where if we consider A as an orientation preserving isometry of $\mathbb{H}^3$, the real part of $l(A)$ is the (positive) translation distance of A along its axis, and the imaginary part is the rotation about the axis, where the orientation is naturally induced by the translation direction of A. The complex length $l(A)$ is related to the trace by the formula

$$l(A) = 2\cosh^{-1}\big(-\tfrac{1}{2}\mathrm{tr}\,(A)\big), \tag{11}$$

and is chosen to have positive real part (note that we could have done away with the minus sign inside the $\cosh^{-1}$ function since the trace is only defined up to $\pm$ sign, we add it here for consistency with the definition for the half length to be given later). Then $l(A)$ is defined up to multiples of $2\pi i$, and depend only on $\pm\mathrm{tr}\,(A)$ or $\mathrm{tr}^2\,(A)$. More explicitly, we have $l(A) = \cosh^{-1}(\frac{1}{2}\mathrm{tr}^2(A) - 1)$.

We may give a natural parametrization of $\mathcal{S}^{\mathrm{mc}}_{\mathrm{alg}}$ by the ideal fixed points, and the square of the traces or the complex lengths of $\rho(a_i)$, $i = 1, \dots, n$ as follows; here we use $\mathrm{Fix}^{\pm}\rho(a_i)$ to denote the attracting and repelling fixed points of $\rho(a_i)$.

We first normalize ρ by conjugation so that

$$\mathrm{Fix}^{-}\rho(a_1) = 0, \quad \mathrm{Fix}^{+}\rho(a_1) = \infty \quad \text{and} \quad \mathrm{Fix}^{-}\rho(a_2) = 1.$$

Then it is not difficult to see that we can parameterize ρ by

$$\begin{aligned}&(\mathrm{Fix}^{+}\rho(a_2), \mathrm{Fix}^{-}\rho(a_3), \mathrm{Fix}^{+}\rho(a_3), \cdots, \mathrm{Fix}^{+}\rho(a_n);\ \mathrm{tr}^2\,\rho(a_1), \cdots, \mathrm{tr}^2\,\rho(a_n))\\ &\in \mathbf{C}^{2n-3}_{\infty} \times \mathbf{C}^n,\end{aligned}$$

or, alternatively, by

$$\begin{aligned}&(\mathrm{Fix}^{+}\rho(a_2), \mathrm{Fix}^{-}\rho(a_3), \mathrm{Fix}^{+}\rho(a_3), \cdots, \mathrm{Fix}^{+}\rho(a_n);\ l(\rho(a_1)), \cdots, l(\rho(a_n)))\\ &\in \mathbf{C}^{2n-3}_{\infty} \times (\mathbf{C}/2\pi i\mathbf{Z})^n.\end{aligned}$$

With this normalized parametrization we have

Lemma 3.2. (Maskit [11]) *The marked classical Schottky space $\mathcal{S}^{\mathrm{mc}}_{\mathrm{alg}}$ is a path connected open subset of $\mathbf{C}^{2n-3}_{\infty} \times (\mathbf{C}/2\pi i\mathbf{Z})^n$.*

Definition 3.3. A fuchsian marking in $\mathcal{S}^{\mathrm{mc}}_{\mathrm{alg}}$ is a *fuchsian* representation $\rho_0 \in \mathcal{S}^{\mathrm{mc}}_{\mathrm{alg}}$.

For a fuchsian marking ρ_0, $\mathbb{H}^2/\rho_0(F_n)$ is a complete hyperbolic surface. Its convex core, M_0, is a hyperbolic surface with geodesic boundary, which we call the hyperbolic surface corresponding to the fuchsian marking. Let $\Delta_0, \Delta_1, \dots, \Delta_m$ be the boundary components of M_0. The image $\rho_0(F_n)$, and hence F_n (since ρ_0 is faithful), can be identified with $\pi_1(M_0)$, and if we define an equivalence relation $\sim$ on F_n by $g \sim h$ if g is conjugate to h or h^{-1}, then there is a bijection

$$\mathfrak{f} : F_n/\sim\ \to \mathcal{C}$$

from $F_n/\sim$ to the set $\mathcal{C}$ of free homotopy classes of closed curves on M_0. Note that there is a unique geodesic representative on M_0 for each nontrivial element of $\mathcal{C}$.

Definition 3.4. For a fixed fuchsian marking ρ_0, let M_0 be the corresponding hyperbolic surface. Let $\Delta_0, \Delta_1, \dots, \Delta_n$ be the boundary components of M_0, and let $[d_i] \in F_n/\sim$, $i = 0, \dots, m$ be the equivalence class corresponding to the boundary component Δ_i, that is $\mathfrak{f}[d_i] = \Delta_i$.

(a) We define $\mathcal{P}$ to be the set of all unordered pairs $\{[g],[h]\}$ of elements in $F_n/\sim$ such that $\mathfrak{f}[g]$ and $\mathfrak{f}[h]$ are free homotopy classes of simple closed curves which bound together with Δ_0 an embedded pair of pants in M_0 (note that it is possible that $\mathfrak{f}[g]=\Delta_k$, for some $1\le k\le m$).

(b) For $j=1,\dots,m$, we define $\mathcal{B}_j$ to be the set of elements $[g]\in F_n/\sim$ such that $\mathfrak{f}[g]$ bounds together with Δ_0 and Δ_j an embedded pair of pants in M_0.

We will also need to define the half lengths, for which we need representations into $\mathrm{SL}(2,\mathbf{C})$ instead of $\mathrm{PSL}(2,\mathbf{C})$. The main idea is that by choosing a lift of the representation to $\mathrm{SL}(2,\mathbf{C})$, one can have a consistent choice of the half length for elements of $\mathrm{SL}(2,\mathbf{C})$. This follows closely the approach of Fenchel in [8], and the reader is referred there for details.

Definition 3.5. If $\rho\in\mathcal{S}^{\mathrm{mc}}_{\mathrm{alg}}$ and $\tilde\rho$ is a lift of ρ to $\mathrm{SL}(2,\mathbf{C})$, then for an element $g\in F_n$, we define the specific half length $l(\tilde\rho(g))/2\in\mathbf{C}/2\pi i\mathbf{Z}$ of $\tilde\rho(g)$ by

$$\cosh\frac{l(\tilde\rho(g))}{2}=-\frac{\operatorname{tr}\tilde\rho(g)}{2},\tag{12}$$

with $\Re\, l(\tilde\rho(g))/2>0$.

Note that the real part of the half length is just half of the real part of the length, and both are positive, while the above choice fixes the imaginary part, up to multiples of $2\pi i$. The minus sign on the right-hand side of (12) is crucial.

Our main theorem for Schottky groups can then be stated as follows.

Theorem 3.6. *Let $\rho\in\mathcal{S}^{\mathrm{mc}}_{\mathrm{alg}}$, and let $\tilde\rho$ be any lift of ρ to* $\mathrm{SL}(2,\mathbf{C})$. *Suppose ρ_0 is a fuchsian marking, with corresponding hyperbolic surface M_0, and boundary components $\Delta_0,\dots,\Delta_m$. Let $\mathcal{P}$ and $\mathcal{B}_j$, $j=1,\dots,m$ be defined as in Definition 3.4, relative to M_0. Then*

$$\sum_{\{[g],[h]\}\in\mathcal{P}} G\Big(\frac{l(\tilde\rho(d_0))}{2},\frac{l(\tilde\rho(g))}{2},\frac{l(\tilde\rho(h))}{2}\Big)$$
$$+\sum_{j=1}^{m}\sum_{[g]\in\mathcal{B}_j} S\Big(\frac{l(\tilde\rho(d_0))}{2},\frac{l(\tilde\rho(d_j))}{2},\frac{l(\tilde\rho(g))}{2}\Big)=\frac{l(\tilde\rho(d_0))}{2}\mod \pi i.\tag{13}$$

Moreover, each series on the left-hand side of (13) *converges absolutely.*

Remark 3.7.

(a) In the case where $\rho=\rho_0$, the above is just a reformulation of Theorem 2.1 for the case of a hyperbolic surface with geodesic boundary components, and is true without the modulo condition. In fact, the lift can be chosen so that the right-hand side is real and positive.

(b) The identity (13) is true only modulo πi because we have fixed the choice of the $\tanh^{-1}$ function in the definition of the functions $G(x,y,z)$ and $S(x,y,z)$ (see Definition 2.2), which may differ from the values obtained by analytic continuation by some multiple of πi.

(c) The result is independent of the lift chosen. This is because if $\tilde\rho$ and $\bar\rho$ are two different lifts of ρ, then for each of the summands on the first series, either $\operatorname{tr}\tilde\rho(g)$, $\operatorname{tr}\tilde\rho(h)$ and $\operatorname{tr}\tilde\rho(d_0)$ are all equal to $\operatorname{tr}\bar\rho(g)$, $\operatorname{tr}\bar\rho(h)$ and $\operatorname{tr}\bar\rho(d_0)$ or exactly two of them differ by their signs (and similarly for the summands

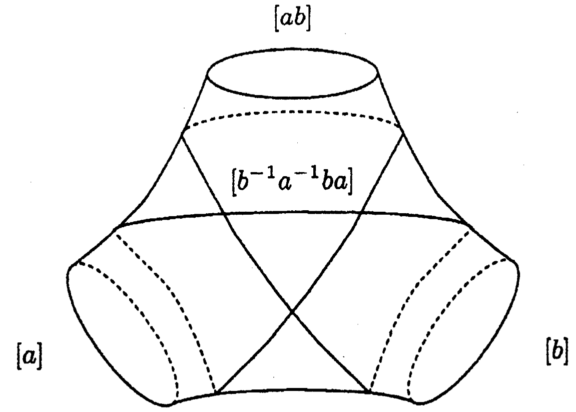

FIGURE 3. A commutator curve on the pair of pants

in the second series). In the latter case, two of the half lengths differ by πi, but it can be easily checked that both $G(x, y, z)$ and $S(x, y, z)$ remain the same if πi is added to two of the arguments.

(d) The choice of the half length functions given above is not arbitrary but arises from the computation of $G(x, y, z)$ and $S(x, y, z)$ as "gap" functions (this is based on the convention adopted by Fenchel in [8], see [19] and [26] for details). Roughly speaking, the relative positions of the axes for $\tilde{\rho}(g)$, $\tilde{\rho}(h)$ and $\tilde{\rho}(d_0)$ are completely determined by their traces. These axes form the non-adjacent sides of a right angled hexagon in $\mathbb{H}^3$ and the half lengths basically arise as the lengths of these sides of the hexagon.

We refer the reader to [20] for details of the proof. We mention here that to prove the absolute convergence of the series concerned, we use a combinatorial word length for elements of $\mathcal{P}$ and $\mathcal{B}_j$, and by adapting an argument from [3], we can show that there is a polynomial bound (in n) for the number of elements of $\mathcal{P}$ (respectively $\mathcal{B}_j$) with combinatorial length n. We then show that for $\rho \in \mathcal{S}_{\mathbf{alg}}^{\mathrm{mc}}$, the combinatorial lengths of the elements of $\mathcal{P}$ (respectively $\mathcal{B}_j$) are comparable to the real part of the complex lengths of their image under ρ in PSL(2, **C**) and use these two facts to prove the absolute convergence of the series in question.

Example 3.8. (A nontrivial identity for the hyperbolic pair of pants.) Theorem 3.6 can be applied to rank two classical Schottky groups to obtain some interesting nontrivial identities for the hyperbolic pair of pants with geodesic boundary. The idea here is that the fundamental group in this case is free on two generators and is isomorphic to the fundamental group of the one-holed torus. The holonomy ρ for the pair of pants is in $\mathcal{S}_{\mathbf{alg}}^{\mathrm{mc}}$, as is the holonomy ρ_0 for the one-holed torus. Using the identity obtained from ρ_0, one obtains a nontrivial identity for ρ via Theorem 3.6. There are interesting geometric interpretations for each of the terms in the identity, see §5 of [20] for details. Note that in this case, the commutator $aba^{-1}b^{-1}$ of a pair of generators is a non-simple closed curve on the pair of pants, as shown in Figure 3, and that its trace $\operatorname{tr} \rho(aba^{-1}b^{-1}) > 18$.

4. The SL(2, C) characters of a one-holed torus

The restriction of Theorem 2.1 to a torus with a cusp was the original identity obtained by McShane in his thesis. The identity (2) restricted to a one-cone or one-holed torus T can be regarded as generalizations of this original identity, and reinterpreted as an identity for representations (or more accurately, characters) of the one-holed torus group $\pi := \pi_1(T)$ to $\mathrm{PSL}(2, \mathbf{R})$, as we saw in the previous section. It is natural to ask how far this result can be extended to representations into $\mathrm{PSL}(2, \mathbf{C})$. Indeed one already has the extension to quasi-fuchsian representations by Bowditch, and the example given at the end of the previous section showed that we can extend the identity to the case of classical Schottky representations, including those arising from a hyperbolic pair of pants. These, however, required some special properties including the discreteness of the representation, which seemed unnecessarily restrictive. For example, given a representation arising as the holonomy of a one-cone hyperbolic torus with say cone angle θ (which is not necessarily discrete), one would expect that for sufficiently small perturbations of the representation into $\mathrm{PSL}(2, \mathbf{C})$, the identity would still hold. This turns out to be true, and in fact, one can give very comprehensive answers to the questions posed above. For example, we can obtain necessary and sufficient conditions for the generalized McShane's identity to hold for representations/characters of π into $\mathrm{PSL}(2, \mathbf{C})$. For this, it turns out that Bowditch's proof via algebraic/combinatiorial methods are extremely useful, and this is the approach we use and generalize in [22] and [21] to solve this problem. Another useful corollary of this method is that we are able to obtain various restricted and relative versions of the identity, the latter of which have geometric interpretations in terms of punctured torus bundles over the circle, and in particular, allows us to prove identities for certain complete hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on hyperbolic punctured torus bundles.

For the rest of this section, we will first start with some basic definitions, then give statements of some of the main results, and finally list some of the key techniques and issues involved in the proofs of the results. We should warn the reader that the proofs are somewhat technical in some parts, details can be found in [6], [22] and [21]. Note also that we shall be stating and proving results for representations(characters) into $\mathrm{SL}(2, \mathbf{C})$ instead of $\mathrm{PSL}(2, \mathbf{C})$. This makes no essential difference since all representations of π into $\mathrm{PSL}(2, \mathbf{C})$ can be lifted to $\mathrm{SL}(2, \mathbf{C})$ as π is free, and the identities obtained will be independent of the lift chosen, and hence can be stated as identities for $\mathrm{PSL}(2, \mathbf{C})$ characters.

4.1. Basic Definitions.

Let T be a one-holed torus and π its fundamental group which is freely generated by two elements X, Y corresponding to simple closed curves on T with geometric intersection number one.

Definition 4.1. The $\mathrm{SL}(2, \mathbf{C})$ character variety

$$\mathcal{X} := \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbf{C}))//\mathrm{SL}(2, \mathbf{C})$$

of T is the set of equivalence classes of representations $\rho : \pi \mapsto \mathrm{SL}(2, \mathbf{C})$, where the equivalence classes are obtained by taking the closure of the orbits under conjugation by $\mathrm{SL}(2, \mathbf{C})$.

The character variety stratifies into relative character varieties: for $\kappa \in \mathbf{C}$, the κ-relative character variety $\mathcal{X}_\kappa$ is the set of equivalence classes $[\rho]$ such that

$$\operatorname{tr}\rho(XYX^{-1}Y^{-1}) = \kappa$$

for one (and hence any) pair of generators X, Y of π. Note that the commutator $XYX^{-1}Y^{-1}$ represents a peripheral curve in T. By classical results of Fricke, we have the following identifications:

$$\mathcal{X} \cong \mathbf{C}^3,$$

$$\mathcal{X}_\kappa \cong \{(x,y,z) \in \mathbf{C}^3 \mid x^2 + y^2 + z^2 - xyz - 2 = \kappa\},$$

where the identification is given by

$$\iota : [\rho] \mapsto (x,y,z) := (\operatorname{tr}\rho(X), \operatorname{tr}\rho(Y), \operatorname{tr}\rho(XY)),$$

for a fixed pair of generators X, Y of π. The topology on $\mathcal{X}$ and $\mathcal{X}_\kappa$ will be that induced by the above identifications.

The outer automorphism group of π, $\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi) \cong \mathrm{GL}(2,\mathbb{Z})$ is isomorphic to the mapping class group $\Gamma := \pi_0(\mathrm{Homeo}(T))$ of T and acts on $\mathcal{X}$, preserving the trace of the commutator of a pair of generators, hence it also acts on $\mathcal{X}_\kappa$, the action is given by

$$\phi([\rho]) = [\rho \circ \phi^{-1}],$$

where $\phi \in \mathrm{Out}(\pi)$ and $[\rho] \in \mathcal{X}$ or $\mathcal{X}_\kappa$ respectively. It is often convenient to consider only the subgroup $\mathrm{Out}(\pi)^+$ of "orientation-preserving" automorphisms, corresponding to the orientation-preserving homeomorphisms Γ^+ of T, which is isomorphic to $\mathrm{SL}(2,\mathbf{Z})$. The action of $\mathrm{Out}(\pi)^+$ (respectively, $\mathrm{Out}(\pi)$) on $\mathcal{X}$ and $\mathcal{X}_\kappa$ is not effective, the kernel is $\{\pm I\}$, generated by the elliptic involution of T so that the effective action is by $\mathrm{PSL}(2,\mathbf{Z})$ (respectively, $\mathrm{PGL}(2,\mathbf{Z})$).

4.2. Simple curves; Pants graph.

Definition 4.2. We denote by $\mathscr{C}$ the set of free homotopy classes of nontrivial, non-peripheral, unoriented simple closed curves on T. Elements of $\mathscr{C}$ are usually denoted by X, Y, Z, W.

The elements of $\mathscr{C}$ correspond to certain elements of $\pi/\sim$, where the equivalence relation $\sim$ is that, for $g, h \in \pi$, $g \sim h$ if and only if g is conjugate to h or h^{-1}. We also denote the corresponding subset of $\pi/\sim$ by $\mathscr{C}$, there should be no confusion.

Definition 4.3. The pants graph $\mathscr{C}(T)$ of T, is defined to be the graph whose vertices are the elements of $\mathscr{C}$, and two vertices are joined by an edge if and only if the corresponding curves on T have geometric intersection number one.

The mapping class group Γ and $\mathrm{Out}(\pi)$ act on $\mathscr{C}$ (respectively $\mathscr{C}(T)$). We can realize $\mathscr{C}(T)$ as the Farey graph/triangulation of the upper half plane $\mathbb{H}^2$ so that $\mathscr{C}$ is identified with $\hat{\mathbf{Q}} := \mathbf{Q} \cup \{\infty\}$, the action of Γ is realized by the action of $\mathrm{PGL}(2,\mathbf{Z})$ on the Farey graph. The projective lamination space $\mathscr{PL}$ of T is then identified with $\hat{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$ and contains $\mathscr{C}$ as the (dense) subset of rational points.

4.3. **Bowditch Q-conditions (BQ-conditions).** We define a certain subspace of $\mathcal{X}$ which we will call the Bowditch space. First note that for $[\rho] \in \mathcal{X}$ and $X \in \mathscr{C}$, $\operatorname{tr}\rho(X)$ is well-defined.

Definition 4.4. The *Bowditch space* is the subset $\mathcal{X}_{BQ} \subset \mathcal{X}$ consisting of characters $[\rho]$ satisfying the following conditions (the *Bowditch Q-conditions*):

(1) $\operatorname{tr}\rho(X) \notin [-2,2]$ for all $X \in \mathscr{C}$;
(2) $|\operatorname{tr}\rho(X)| \leq 2$ for only finitely many (possibly no) $X \in \mathscr{C}$.

For a fixed $[\rho] \in \mathcal{X}$ and $U \subset \mathscr{C}$, we say that the BQ-conditions are satisfied on U for $[\rho]$ if conditions (1) and (2) above hold for all $X \in U$.

4.4. **Statement of results for $\mathrm{SL}(2,\mathbf{C})$ characters.** We have the following extension and generalization of Theorem 2.1 to characters in $\mathcal{X}$.

Theorem 4.5. (Theorems 2.2, 2.3 and Proposition 2.4 of [22])

(a) *Bowditch space $\mathcal{X}_{BQ}$ is open in the whole character space $\mathcal{X}$.*

(b) *The mapping class group Γ acts properly discontinuously on $\mathcal{X}_{BQ}$. Furthermore, $\mathcal{X}_{BQ}$ is the largest open subset of $\mathcal{X}$ for which this holds.*

(c) *For a character $[\rho] \in \mathcal{X}_{BQ} \cap \mathcal{X}_\kappa$,*

$$\sum_{X \in \mathscr{C}} \log \frac{e^{\nu} + e^{l(\rho(X))}}{e^{-\nu} + e^{l(\rho(X))}} = \nu \mod 2\pi i, \tag{14}$$

where $\nu = \cosh^{-1}(-\kappa/2)$, and the sum converges absolutely.

Remark 4.6.

(1) The (complex) length $l(\rho(X))$ is related to the trace as in equation (11).
(2) We are using the formula for $G(x,y,z)$ given in (6) for part (c), note that there are no $S(x,y,z)$ terms since there is only one boundary component.
(3) In the case when $\kappa = -2$, $\nu = 0$ and all the terms of (14) are identically zero. However, if we take the first order infinitesimals, or the formal derivative of (14) with respect to ν and evaluate at $\nu = 0$, we get

$$\sum_{X \in \mathscr{C}} \frac{1}{1 + e^{l(\rho(X))}} = \frac{1}{2}, \tag{15}$$

which is McShane's original identity in [12] for real type-preserving characters, and also Bowditch's generalization in [4] and [6] for type-preserving characters satisfying the BQ-conditions.
(4) When $\kappa = 2$, which corresponds to the reducible characters, the identity is also trivial. In this case, however, the Bowditch Q-conditions are never satisfied, see [23].
(5) Parts (a) and (b) of the above were originally stated in [22] in terms of the relative character varieties $\mathcal{X}_\kappa$.
(6) ν is a specific choice of half of the complex length of the peripheral curve on T, note that the minus sign is crucial for the identity to hold.

4.5. **Necessary and sufficient conditions.** Replacing condition (1) of the BQ-conditions by (1′) $\operatorname{tr}\rho(X) \notin (-2,2)$ for all $X \in \mathscr{C}$, we get the *extended Bowditch space* $\hat{\mathcal{X}}_{BQ}$, and we have the following result:

Theorem 4.7. (Theorem 1.5 of [21]) *For $[\rho] \in \mathcal{X}$, the identity* (14) *of Theorem* 4.5(c) *holds (with absolute convergence of the sum) if and only if $[\rho]$ lies in the extended Bowditch space $\hat{\mathcal{X}}_{BQ}$.*

The above result gives a complete answer to the question of when the generalized McShane's identity holds for $SL(2, \mathbf{C})$ characters of T.

4.6. **McShane-Bowditch identities for punctured torus bundles.** We next consider further variations of the McShane-Bowditch identities. Recall that $\theta \in \mathrm{Out}(\pi) \cong \Gamma$ acts on $\mathcal{X}$ where the action is given by

$$\theta([\rho]) = [\rho \circ \theta^{-1}].$$

Suppose that $[\rho] \in \mathcal{X}$ is stabilized by an Anosov element $\theta \in \Gamma^+$ (this corresponds to a hyperbolic element if we identify Γ^+ with $SL(2, \mathbf{Z})$), that is, $\theta([\rho]) = [\rho]$. We can associate to this a representation of $\pi_1(M)$ into $SL(2, \mathbf{C})$, where M is a punctured torus bundle over the circle, with monodromy θ. The restriction of the representation to the fibre is $[\rho]$. We can find a specific lift of θ to $\mathrm{Aut}(\pi)$ which corresponds to choosing a specific longitude of the boundary torus of M (see [5] or [22] for details). So fixing a representation ρ in the class $[\rho]$, there exists $A \in SL(2, \mathbf{C})$ such that for all $\alpha \in \pi$,

$$\theta(\rho)(\alpha) = A \cdot \rho(\alpha) \cdot A^{-1}.$$

Note that $\mathrm{tr}\, A$ is independent of the choice of ρ in the conjugacy class $[\rho]$. Note also that $\mathrm{tr}\, \rho(X)$ is well-defined on the equivalence classes $[X] \in \mathscr{C}/\langle\theta\rangle$. Suppose further that $[\rho]$ satisfies the *relative* Bowditch Q-conditions on $\mathscr{C}/\langle\theta\rangle$, that is,

(1) $\mathrm{tr}\, \rho(X) \notin [-2, 2]$ for all $[X] \in \mathscr{C}/\langle\theta\rangle$;
(2) $|\mathrm{tr}\, \rho(X)| \leq 2$ for only finitely many $[X] \in \mathscr{C}/\langle\theta\rangle$.

Using the identification of Γ^+ with $SL(2, \mathbf{Z})$ and $\mathscr{C}$ with $\hat{\mathbf{Q}} \subset \hat{\mathbf{R}} \cong \mathscr{PL}$ in §4.2, we get that the repelling and attracting fixed points of θ, $\mu_-, \mu_+ \in \mathscr{PL}$ partition $\mathscr{C}$ into two subsets $\mathscr{C}_L \sqcup \mathscr{C}_R$ which are invariant under the action of θ. We have the following generalizations of the McShane-Bowditch identities:

Theorem 4.8. (Theorems 5.6 and 5.9 of [22]) *Suppose that $[\rho]$ is stabilized by an Anosov element $\theta \in \Gamma^+$ and satisfies the relative Bowditch Q-conditions as stated above. Then*

$$\sum_{[X] \in \mathscr{C}/\langle\theta\rangle} \log \frac{e^{\nu} + e^{l(\rho(X))}}{e^{-\nu} + e^{l(\rho(X))}} = 0 \mod 2\pi i, \tag{16}$$

and

$$\sum_{[X] \in \mathscr{C}_L/\langle\theta\rangle} \log \frac{e^{\nu} + e^{l(\rho(X))}}{e^{-\nu} + e^{l(\rho(X))}} = \pm l(A) \mod 2\pi i, \tag{17}$$

where the sums converge absolutely; and $l(A)$ is the complex length of the conjugating element A corresponding to θ as described above, and the sign in (17) *depends only on our choice of orientations.*

Remark 4.9. For type-preserving characters ($\kappa = -2$), the result is due to Bowditch [5], where the summands of (16) and (17) should be replaced appropriately as in Remark 4.6(3) by the summands of McShane's original identity, and $l(A)$ in (17) should be replaced by λ, the modulus of the cusp of M with the complete, finite volume hyperbolic structure. There are also similar identities in the case where θ is reducible, that is, corresponds to a parabolic element of $\mathrm{SL}(2,\mathbf{Z})$, see [21].

The above result has applications to closed hyperbolic 3-manifolds. As before, let M be an orientable 3-manifold which fibers over the circle, with the fiber a once-punctured torus, T and suppose that the monodromy θ of M is Anosov. By results of Thurston, see [25] and [24], M has a complete finite-volume hyperbolic structure with a single cusp, which can in turn be deformed to incomplete hyperbolic structures, on which hyperbolic Dehn surgery can be performed to obtain complete hyperbolic manifolds without cusps. Restricting the holonomy representation to the fiber gives us characters which are stabilized by θ, and in the complete case, the relative Bowditch Q-conditions are satisfied (see [5]). For small deformations of the complete structure to incomplete structures, the relative BQ-conditions are still satisfied since these are open conditions (see [22]). The identities can be interpreted as series identities for these (in)complete structures, involving the complex lengths of certain geodesics corresponding to the homotopy classes of essential simple closed curves on the fiber. The quantity ν can be interpreted as half the complex length of the meridian of the boundary torus, and $l(A)$ as the complex length of a (suitably chosen) longitude of the boundary torus. In particular, the identity can be interpreted as an identity for the closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on the original complete manifold, if the Dehn surgery invariants are sufficiently close to ∞.

One question which arises is whether the identity holds for all closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on a hyperbolic punctured torus bundle over the circle. The openness of the relative BQ-conditions ensures that this is true for almost all such manifolds (except for possibly a finite number of exceptions). Another question arising is whether a similar result holds in the case of hyperbolic Dehn surgery on punctured surface bundles, as studied by Akiyoshi, Miyachi and Sakuma [2].

4.7. **Key points used in the proofs.** The combinatorial structure of $\mathscr{C}(T)$ as well as the Fricke trace relation which can be interpreted as an edge relation play fundamental roles which we sketch here.

Recall that $\mathscr{C}(T)$ has the structure of the Farey tessellation, and the set of vertices $\mathscr{C}$ can be identified with $\hat{\mathbf{Q}}$. The dual graph Σ to $\mathscr{C}(T)$ is a trivalent tree whose complementary regions can be identified with the vertices of $\mathscr{C}(T)$. Denote by $V(\Sigma)$, $E(\Sigma)$, $\vec{E}(\Sigma)$ and $\Omega(\Sigma)$ the sets of vertices, edges, directed edges and complementary regions of Σ respectively. Call $(X,Y) \in \mathscr{C} \times \mathscr{C}$ a generating pair if X and Y are connected by an edge in $\mathscr{C}(T)$, and $(X,Y,Z) \in \mathscr{C} \times \mathscr{C} \times \mathscr{C}$ a generating triple if X, Y and Z are the vertices of a triangle in $\mathscr{C}(T)$. Generating pairs correspond to edges of Σ and generating triples correspond to vertices of Σ. More specifically, to an edge e of Σ, we write $e = (X,Y;Z,Z')$ if (X,Y) corresponds to e and (X,Y,Z), (X,Y,Z') are generating triples. Similarly, we use $\vec{e} = (X,Y;Z \to Z')$ to indicate that the directed edge $\vec{e}$ points from Z to Z', see Figure 5, where we have drawn part of Σ, and used the identification of $\Omega(\Sigma)$ with $\mathscr{C}$. Denote by $-\vec{e}$ the directed edge with the opposite direction to $\vec{e}$. For $\vec{e} = (X,Y;Z \to Z')$, we define $\mathrm{Tail}(\vec{e})$, the tail of $\vec{e}$ to be the subset of $\mathscr{C}$ in the interval between X and Y (inclusive) which contains Z. In particular, $\mathrm{Tail}(\vec{e}) \cup \mathrm{Tail}(-\vec{e}) = \mathscr{C}$, and $\mathrm{Tail}(\vec{e}) \cap \mathrm{Tail}(-\vec{e}) = \{X,Y\}$.

For each character $[\rho] \in \mathcal{X}_\kappa$, by taking the trace function, we obtain a *trace map*

$$\phi : \mathscr{C} \to \mathbf{C} \quad \text{where} \quad \phi(X) = \mathrm{tr}\rho(X).$$

(We call it a generalized Markoff map in [22] following [6].)

Henceforth, for a fixed trace map ϕ, we adopt the convention of using the lower case letters to represent the values of ϕ, that is, $\phi(X) = x$, $\phi(Y) = y$, etc. Then ϕ satisfies the following vertex and edge relations, arising from the Fricke trace identities:

Vertex relation. For every generating triple (X, Y, Z),

$$x^2 + y^2 + z^2 - xyz - \kappa - 2 = 0. \tag{18}$$

Edge relation. For every edge $e = (X, Y; Z, Z')$,

$$z + z' = xy. \tag{19}$$

It turns out that the edge relation is more fundamental than the vertex relation. To start with, one can show easily that if the edge relation is satisfied for all edges, than the vertex relation propagates along the edges to cover the entire tree Σ. Secondly, ϕ is completely determined by its values on any generating triple (X, Y, Z) by successively applying the edge relation (19).

Each $[\rho] \in \mathcal{X}$ (equivalently, the induced trace map ϕ on $\mathscr{C}$) determines a map $f : E(\Sigma) \to \vec{E}(\Sigma)$, where each edge e is assigned a direction or flow from the larger absolute value to the smaller one, that is,

$$f(e) = \vec{e} = (X, Y; Z \to Z')$$

if $|z| \geq |z'|$. There is some ambiguity when $|z| = |z'|$ in which case we can assign either direction. This ambiguity does not affect the large scale behavior of $f(E(\Sigma))$, except in some very special trivial cases. We then have the following elementary but important results (see [22] for proofs). For the purposes of our discussion, we fix $[\rho] \in \mathcal{X}_\kappa$ where $\kappa \neq 2$ ($[\rho]$ is not reducible), with corresponding trace map ϕ.

Proposition 4.10. *If (X, Y, Z) is a generating triple corresponding to the vertex $v \in V(\Sigma)$ and $f(e)$ points away from v for at least two of the edges adjacent to v, then* $\min(|x|, |y|, |z|) \leq 2$.

Lemma 4.11. (Bowditch [6]) *For all $K \geq 2$, $\mathscr{C}(K) := \{X \in \mathscr{C} \mid \phi(X) \leq K\}$ is connected, that is, the subgraph of $\mathscr{C}(T)$ spanned by $\mathscr{C}(K)$ is connected. In particular, $\mathscr{C}(2)$ is connected.*

The above can be regarded as a quasi-convexity result, namely, for any $K \geq 2$, for any $X, Y \in \mathscr{C}(K)$, the geodesic in $\mathscr{C}(T)$ joining X to Y is a bounded distance from the subgraph in $\mathscr{C}(T)$ spanned by $\mathscr{C}(K)$.

Proposition 4.12. *Suppose that $X \in \mathscr{C}$ and Y_n, $n \in \mathbf{Z}$ are the neighbors of X, in cyclical order.*

(a) *If $x \notin [-2, 2] \cup \{\pm\sqrt{\kappa + 2}\}$, then $\lim_{n \to \pm\infty} |y_n| = \infty$ with exponential growth in $|n|$.*

(b) *If $x = \pm 2$ and $\kappa \neq 2$, then $\lim_{n \to \pm\infty} |y_n| = \infty$ with linear growth in $|n|$.*

The proof of Theorem 4.5 now proceeds as follows:

First we show that the BQ-conditions are open conditions. This is achieved by showing that the conditions are controlled by a finite subtree of Σ; the proof is essentially that given by Bowditch in [6], with some slight modifications.

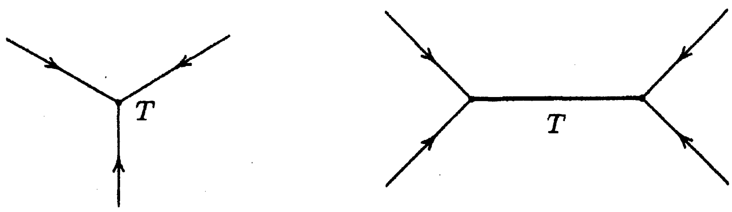

FIGURE 4. Two simple circular sets $C(T)$

Next, we show that if ϕ satisfies the BQ-conditions, then the function $\log^+|\phi| := \max\{0, \log|\phi|\}$ on $\mathscr{C}$ has lower Fibonacci growth. Roughly, this means that it is comparable to the combinatorial (word) length function on $\mathscr{C}$, that is, there exists some $k > 0$ such that $\log^+|\phi(X)| \geq k\|X\|_w$ for all but a finite number of $X \in \mathscr{C}$, where $\|X\|_w$ is the (cyclically reduced) word length of X with respect to any fixed pair of generators for π. This is achieved by showing that one can find a finite subtree T of Σ such that $f(e)$ is directed towards T for all $e \in \Sigma \setminus T$, by applying Proposition 4.10, Lemma 4.11 and Proposition 4.12. We can think of this subtree T as the union of sufficiently long boundary paths of the elements of $\mathscr{C}(2)$ in Σ (which by the BQ-conditions is finite). Define the circular set $C(T)$ of T to be the set of directed edges $\vec{e} \in \vec{E}(\Sigma)$ adjacent to T and directed towards T, see Figure 4 where examples of $C(T)$ are given for the two simplest cases of T. Then $C(T)$ is finite, and $\bigcup_{\vec{e}\in C(T)} \mathrm{Tail}(\vec{e}) = \mathscr{C}$. The lower bound is achieved by showing that the lower bound holds for each of the sets $\mathrm{Tail}(\vec{e})$ (with possibly different constants k), where $\vec{e} \in C(T)$, from which the general result on $\mathscr{C}$ follows since it is a finite union of these sets. Now Theorem 4.5(b) follows from the lower Fibonacci growth, and one deduces in a fairly straightforward manner that the series in (14) of Theorem 4.5(c) converges absolutely, recalling the relation between the traces and the complex lengths given in (11).

To complete the proof of Theorem 4.5(c) and show that the sum is indeed as stated, we need to modify the methods of Bowditch [6] slightly. First note that $\kappa = -2$ for all the characters $[\rho]$ considered in [6] (also called type-preserving). Bowditch used the edge-weight function (which depends on the character $[\rho]$, or, equivalently, the corresponding trace map ϕ) $\psi := \psi_\phi : \vec{E}(\Sigma) \to \mathbf{C}$ defined by

$$\psi(\vec{e}) = z/xy, \tag{20}$$

where $\vec{e} = (X, Y; Z' \to Z)$, and $x = \phi(X)$, $y = \phi(Y)$, $z = \phi(Z)$ in a very ingenious manner to prove the result, by taking sums of $\psi(\vec{e})$ over circular sets of larger and larger subtrees T of Σ which eventually exhaust Σ. The main properties of ψ which were used are the following:

Lemma 4.13. (Properties of the edge-weight function ψ in the case $\kappa = -2$.) *Suppose ϕ corresponds to a type-preserving character $[\rho]$ (that is, $\kappa = -2$) and $\phi(X) \neq 0$ for all $X \in \Omega$. If $\psi := \psi_\phi$ is the edge-weight function defined by* (20), *then*

(i) *for a directed edge* $\vec{e} \in \vec{E}$,

$$\psi(\vec{e}) + \psi(-\vec{e}) = 1; \tag{21}$$

(ii) *for a circular set* $C(T) \subset \vec{E}$,

$$\textstyle\sum_{\vec{e}\in C(T)} \psi(\vec{e}) = 1. \tag{22}$$

Note that (21) is just the edge relation (19), and (22) is just the vertex relation (18) in the case where $\kappa = -2$ and T a vertex. The case for general T follows easily by an inductive argument using (21).

Now for general $[\rho] \in \mathcal{X}_\kappa$ where $\kappa \neq \pm 2$, we need to find a corresponding edge weight function ψ (depending on $[\rho]$ and κ) such that a suitable generalization of Lemma 4.13 holds. It turns out we can define an edge-weight function ψ such that:

Lemma 4.14. (Properties of the general edge-weight function ψ for $\kappa \neq \pm 2$) *Suppose* $[\rho] \in \mathcal{X}_\kappa$, $\kappa \neq \pm 2$, *with corresponding trace map* ϕ, *and* $\phi(X) \neq 0, \pm\sqrt{\kappa+2}$ *for all* $X \in \mathscr{C}$. *Then*

(i) *for a directed edge* $\vec{e} \in \vec{E}$,

$$\psi(\vec{e}) + \psi(-\vec{e}) = \nu \mod 2\pi i; \tag{23}$$

(ii) *for a circular set* $C(T) \subset \vec{E}$,

$$\textstyle\sum_{\vec{e}\in C(T)} \psi(\vec{e}) = \nu \mod 2\pi i, \tag{24}$$

where $\nu = \cosh^{-1}(-\kappa/2)$ *and for* $\vec{e} = (X, Y; Z' \to Z)$

$$\psi(\vec{e}) := \log\left(\frac{1 + (e^\nu - 1)(z/xy)}{\sqrt{1-(\kappa+2)/x^2}\sqrt{1-(\kappa+2)/y^2}}\right). \tag{25}$$

The rest of the proof of Theorem 4.5(c) then proceeds along the same lines as that used in [6], with some extra technicalities required to prove that the error term approaches zero. It should be pointed out that the function ψ has a geometric interpretation in the case of a hyperbolic one-cone torus in terms of certain gaps formed by the simple geodesics emanating and terminating at the cone point (c.f. §1). We have a similar geometric interpretation for ψ for a hyperbolic one-holed torus with geodesic boundary. In these cases, the properties (23) and (24) are almost self-evident from the geometric interpretation. The general formula for ψ is then obtained from these geometric cases by analytic continuation.

For Theorem 4.6, one requires restricted and relative versions of Theorem 4.5. These can be obtained fairly easily now that we have found the edge-weight function ψ with the desired properties. The proofs follow essentially the same lines as Bowditch's in [5]. Details can be found in [22] or [21].

A challenge is to extend the methods described above to give similar algebraic/combinatorial proofs for characters of general punctured surfaces. It seems that a good understanding of the pants graph or curve complex for such surfaces is essential, although this was circumvented rather cleverly in the work of Akiyoshi, Miyachi and Sakuma in [2] when they studied punctured torus bundles over the circle.

For characters (of the one-holed torus) which do not satisfy the (extended) BQ-conditions, or the relative BQ-conditions, the dynamics of the action of the mapping class group is very interesting, and the general methods described above can be applied to study the question. For a character $[\rho] \in \mathcal{X}$, we say that $\lambda \in \mathscr{PL}$ is

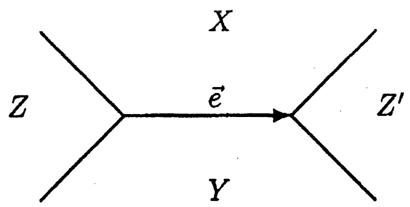

FIGURE 5. The directed edge $\vec{e} = (X, Y; Z \to Z')$

an *end invariant* of $[\rho]$ if there exists a sequence of distinct $X_k \in \mathscr{C}$ converging to λ such that $|\mathrm{tr}\rho(X_k)|$ is bounded. It is easy to see that the set of end invariants is empty if $[\rho] \in \hat{\mathscr{X}}_{BQ}$ by Theorem 4.7, and is equal $\{\mu_+, \mu_-\}$ if $[\rho]$ satisfies the conditions of Theorem 4.8. In particular, the conjecture is that the set of end invariants is a Cantor set if it contains at least three elements and is not the entire projective lamination space $\mathscr{PL}$. See [23], where the set of end invariants was studied in various cases, with supporting evidence for the conjecture.

REFERENCES

[1] Hirotaka Akiyoshi, Hideki Miyachi, and Makoto Sakuma, *A refinement of McShane's identity for quasifuchsian punctured torus groups*, In the Tradition of Ahlfors and Bers, III. The Ahlfors-Bers Colloquium, Oct. 2001, Univ. of Connecticut at Storrs, W. Abikoff, A. Haas (Eds.), Contemporary Mathematics **355**, 21–40, American Mathematical Society, 2004.

[2] Hirotaka Akiyoshi, Hideki Miyachi, and Makoto Sakuma, *Variations of McShane's identity for punctured surface groups*, London Mathematical Society Lecture Notes **329**, Y. Minsky, M. Sakuma & C. Series (Eds.), 151–185, Cambridge University Press, 2006.

[3] Joan S. Birman and Caroline M. Series, *Geodesics with bounded intersection number on surfaces are sparsely distributed*, Topology **24** (1985), no. 2, 217–225.

[4] Brian H. Bowditch, *A proof of McShane's identity via Markoff triples*, Bull. London Math. Soc. **28** (1996), no. 1, 73–78.

[5] Brian H. Bowditch, *A variation of McShane's identity for once-punctured torus bundles*, Topology **36** (1997), no. 2, 325–334.

[6] Brian H. Bowditch, *Markoff triples and quasi-Fuchsian groups*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 697–736.

[7] Norman Do and Paul Norbury, *Weil-Petersson volumes and cone surfaces*, arXiv:math.AG/0603406.

[8] Werner Fenchel, *Elementary Geometry in Hyperbolic Space*, Walter de Gruyter & Co., Berlin, 1989.

[9] William M. Goldman, *Ergodic theory on moduli spaces*, Ann. of Math. (2) **146** (1997), 1–33.

[10] William M. Goldman, *The modular group action on real* SL(2)*-characters of a one-holed torus*, Geom. Topol. **7** (2003), 443–486.

[11] Bernard Maskit, *On the topology of classical Schottky space*, Complex manifolds and hyperbolic geometry (Guanajuato, 2001), 305–311, Contemp. Math. **311**, Amer. Math. Soc., Providence, RI, 2002.

[12] Greg McShane, *A remarkable identity for lengths of curves*, Ph.D. Thesis, University of Warwick, 1991.

[13] Greg McShane, *Simple geodesics and a series constant over Teichmuller space*, Invent. Math. **132** (1998), no. 3, 607–632.

[14] Greg McShane, *Weierstrass points and simple geogesics*, Bull. London Math. Soc. **36** (2004), no. 2, 181–187.

[15] Greg McShane, *Simple geodesics on surfaces of genus 2*, Annales Academiae Scientiarum Fennicae Mathematica **31** (2006), 31-38.

[16] Maryam Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, preprint.
[17] Maryam Mirzakhani, *Growth of the number of simple closed geodesics on hyperbolic surfaces*, to appear, Ann. of Math.
[18] Maryam Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, to appear, J. Amer. Math. Soc.
[19] Ser Peow Tan, Yan Loi Wong, and Ying Zhang, *Generalizations of McShane's identity to hyperbolic cone-surfaces*, J. Differential Geom. **72** (2006), 73-112.
[20] Ser Peow Tan, Yan Loi Wong, and Ying Zhang, *McShane's identity for classical Schottky groups*, arXiv:math.GT/0411628.
[21] Ser Peow Tan, Yan Loi Wong, and Ying Zhang, *Necessary and sufficient conditions for McShane's identity and variations*, to appear, Geometriae Dedicata.
[22] Ser Peow Tan, Yan Loi Wong, and Ying Zhang, *Generalized Markoff maps and McShane's identity*, arXiv:math.GT/0502464.
[23] Ser Peow Tan, Yan Loi Wong, and Ying Zhang, *End Invariants of* $\mathrm{SL}(2,\mathbf{C})$ *characters of the one-holed torus*, arXiv:math.GT/0511621.
[24] William P. Thurston, *The Geometry and Topology of 3-Manifolds*, Lecture Notes, Princeton University, 1977/78.
[25] William P. Thurston, *Hyperbolic structures on 3-manifolds II: surface groups and 3-manifolds that fiber over the circle*, arXiv:math.GT/9801045.
[26] Ying Zhang, *Hyperbolic cone-surfaces, generalized Markoff maps, Schottky groups and McShane's identity*, Ph.D. Thesis, National University of Singapore, 2004.

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