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The composition operators on the weighted Bergman Spaces with closed range

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Abstract

We study the multiplication operators and the integration operators and the composition operators with closed range on the Bergman spaces by using the sampling property.

Key Words and Phrases : reverse Carleson measure, sampling set, integration operator, Bergman space, Hardy space, closed range, bounded below.

§0. Introduction

Let D be the open unit disk in complex plane C. For $z, w \in D$, $0 < r < 1$, let $\varphi_z(w) = \dfrac{z-w}{1-\bar{z}w}$ and let $\rho(z,w) = \left|\dfrac{z-w}{1-\bar{z}w}\right|$ and $D(w,r) = \{z \in D, \rho(w,z) < r\}$. Let $H(D)$ be the space of all analytic functions on D.

For $\alpha > 0$, the space $\mathcal{B}_\alpha$ of D is defined to be the space of analytic functions f on D such that

$$\| f \|_{\beta_\alpha} = |f(0)| + \| f \|_{\mathcal{B}_\alpha} < +\infty ,$$

where $\| f \|_{\mathcal{B}_\alpha} = \sup_{z\in D}(1-|z|^2)^\alpha |f'(z)|$. Note that $\mathcal{B}_1 = \mathcal{B}$ is the Bloch space .

The space $\mathcal{B}_{\alpha,0}$ of D is defined to be the space of analytic functions f on D such that

$$(1-|z|^2)^\alpha |f'(z)| \to 0 \qquad (|z| \to 1^-).$$

Note that $\mathcal{B}_{1,0} = \mathcal{B}_0$ is the little Bloch space .

The space $\mathcal{B}^\alpha$ of D is defined to be the space of analytic functions f on D such that $\sup_{z\in D}(1-|z|^2)^\alpha |f(z)| < +\infty$.

For $\alpha > -1$, the weighted Dirichret space D^α is defined to be the space of analytic functions f on D such that

$$\int_D |f'(z)|^2(\alpha+1)(1-|z|^2)^\alpha dA(z) < +\infty,$$

where $dA(z)$ denote the area measure on D. In the case of $\alpha = 1$, then $D^1 = H^2$ is the Hardy space. In the case of $\alpha = 2$, then $D^2 = L^2_a$ is the Bergman space. If $\alpha > 1$, then $\int_D |f'(z)|^2(1-|z|^2)^\alpha dA(z)$ is comparable to $\int_D |f(z)|^2(\alpha+1)(1-|z|^2)^{\alpha-2} dA(z)$.

Let X be Banach spaces and let T be a linear operator from X into X. Then T is called to

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be bounded below on X if $\| Tf \| \geq C \| f \|$ for all $f \in X$ and positive constants $C > 0$.

For g analytic on D, the operators I_g, J_g, M_g are defined by the following:

$$I_g(f)(z) = \int_0^z g(\zeta)f'(\zeta)d\zeta\ ,\ J_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta\ ,\ M_g(f)(z) = g(z)f(z).$$

If $g(z) = z$, then J_g is the integration operator. If $g(z) = \log \frac{1}{1-z}$, then J_g is the Cesáro operator. In [10] Ch.Pommerenke proved the result with respect to the operator J_g. In [1] A.Aleman and A.G.Siskakis proved the result with respect to the operator J_g : In [2] A.Aleman and A.G.Siskakis proved the result with respect to the operator J_g.

In [3] Paul S.Bourdon proved the following result with respect to the the multiplication operators:

Theorem 0.1.(Paul S.Bourdon) *Let $h \in H^\infty$. The operator $M_h : L^2_a \to L^2_a$ is bounded below if and only if $h = \varphi F$, where $F, 1/F \in H^\infty$ and where φ is a finite product of interpolating Blaschke products.*

In [7] D.Luecking proved the following result with respect to the reverse Carleson measure:

Theorem 0.2.(D.Luecking) *Let τ be a bounded nonnegative measurable function in D. Then there is a constant $k > 0$ such that*

$$\int_D |f'(z)|^2\tau(z)\log\frac{1}{|z|^2}dA(z) \geq k\int_D |f'(z)|^2\log\frac{1}{|z|^2}dA(z)$$

for all $f \in H^2$ if and only if there exists a constant $c > 0$ such that the set $G_c = \{z \in D : \tau(z) > c\}$ satisfies the condition:

$(*)$ There exists a constant $\delta > 0$ such that

$$dA(G_c \cap D(\zeta, r)) > \delta dA(D \cap D(\zeta, r))$$

for all $\zeta \in \partial D$ and $r > 0$, where $D(\zeta, r)$ is a disc with a center ζ and a radius r.

In [8] D.Luecking proved the following result:

Theorem 0.3.(D.Luecking) *Let $\alpha > -1$, and let μ be a finite positive Borel measure on D. In order that there exists a constant $C > 0$ such that*

$$\left(\int_D |f'(z)|^2 d\mu(z)\right)^{\frac{1}{2}} \leq C\left(\int_D |f(z)|^2(1-|z|^2)^\alpha dA(z)\right)^{\frac{1}{2}}$$

for all analytic functions f if and only if there exists a constant $C' > 0$ such that

$$\mu\left(\{z \in D, \rho(z,a) < \frac{1}{2}\}\right) \leq C'(1-|z|^2)^{4+\alpha}.$$

In [5] P.Ghatage and D.Zheng and Nina Zorboska determined the composition operators on the Bloch space that have a closed range using sampling set for $\mathcal{B}$. So we also study when the operators I_g, J_g, M_g and the composition operators are bounded below on the Bergman spaces and the (weighted) Bloch space using sampling set for weighted Bloch spaces. In particular, the fact that I_g have the closed range on the weighted Dirichlet space D^α is equivalent to "the reverse Carleson measure", i.e. the definition of I_g with the closed range on the weighted Dirichlet space D^α is the following:

$$\int_D |f'(z)|^2|g(z)|^2(\alpha+1)(1-|z|^2)^\alpha dA(z) \geq k\int_D |f'(z)|^2(\alpha+1)(1-|z|^2)^\alpha dA(z)$$

And it is exactly equal to the definition of the reverse Carleson measure. And we characterize the reverse Carleson measure by using new way completely that is different from Theorem 0.2(D.Luecking's result) in this paper(Theorem 1.8). And by characterizing the operator J_g with closed range, we also get the result that corresponds to Theorem 0.3(D.Luecking's result) in this paper(Theorem 2.3). Moreover we also characterize the multiplication operator with the closed range on the weighted Bergman spaces that corresponds to Theorem 0.1 in this paper(Theorem 2.6).

§1. The closed range operator I_g on the Bergman space and Luecking's inequalities

In this section, we study the closed range operator I_g on the Bergman space and Luecking's inequalities.

Definition 1.1. *Let $\alpha > 0$. A set Γ of the open unit disk D is called a sampling set for $\mathcal{B}^\alpha$ if there exists a positive constant $C > 0$ such that*

$$\sup_{z\in D}(1-|z|^2)^\alpha|f(z)| \leq C\sup_{z\in\Gamma}(1-|z|^2)^\alpha|f(z)|,$$

for all $f \in \mathcal{B}^\alpha$.

Definition 1.2. *Let $\alpha > 0$. A set Γ of the open unit disk D is called a sampling set for $\mathcal{B}_\alpha$ if there exists a positive constant $C > 0$ such that*

$$\sup_{z\in D}(1-|z|^2)^\alpha|f'(z)| \leq C\sup_{z\in\Gamma}(1-|z|^2)^\alpha|f'(z)|,$$

for all $f \in \mathcal{B}_\alpha$.

In [12] we also proved the following result :

Theorem R.1. *Let $\beta \geq \alpha > 0$. Then the operator $I_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ is bounded (compact) if and only if*

$$\sup_{z\in D}(1-|z|^2)^{\beta-\alpha}|g(z)| < +\infty \quad (\lim_{|z|\to 1^-}(1-|z|^2)^{\beta-\alpha}|g(z)| = 0).$$

By using a sampling set for $\mathcal{B}_\alpha$, we can prove the following result with respect to the operator I_g:

Theorem 1.3. *Let $\beta \geq \alpha > 0$ and $g \in H(D)$. Let the operator $I_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ be bounded (i.e. $\sup_{z\in D}(1-|z|^2)^{\beta-\alpha}|g(z)| < +\infty$). Then the operator $I_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ is bounded below if and only if there exists a positive constant $(1 >)\epsilon > 0$ such that $\{z \in D, (1-|z|^2)^{\beta-\alpha}|g(z)| \geq \epsilon\}$ is a sampling set for $\mathcal{B}_\alpha$.*

Definition 1.4. *The space BMOA is defined to be the space of $f \in H(D)$ such that $\sup_{a\in D} \int_D (1-|\varphi_a(z)|^2)|f'(z)|^2 dA(z) < +\infty$.*
In the case of $0 < \alpha < 1$, the space Q_α is defined to be the space of $f \in H(D)$ such that $\sup_{a\in D} \int_D (1-|\varphi_a(z)|^2)^\alpha |f'(z)|^2 dA(z) < +\infty$.

The following lemma is well-known (See [6] and [13]):

Lemma 1.5. *Let $f \in H(D)$. If $\alpha > 1$, then $f \in \mathcal{B}$ if and only if*

$$\sup_{a\in D} \int_D (1-|\varphi_a(z)|^2)^\alpha |f'(z)|^2 dA(z) < +\infty.$$

By using the following proposition, we can prove Theorem 1.8:

Proposition 1.6. *Let $g \in H^\infty$. If the operator $I_g : H^2 \to H^2$ is bounded below, then $I_g : BMOA \to BMOA$ is bounded below. If the operator $I_g : L^2_a \to L^2_a$ is bounded below, then $I_g : \mathcal{B} \to \mathcal{B}$ is bounded below. For $0 < \alpha < 1$, if the operator $I_g : D^\alpha \to D^\alpha$ is bounded below, then $I_g : Q_\alpha \to Q_\alpha$ is bounded below.*

In [7] D.Leucking proved the following result:

Theorem D. ([7]) *Let $\alpha > -1$. There is a constant $C > 0$ such that*

$$\int_D |f'(z)|^2(1-|z|^2)^\alpha dA(z) \leq C \int_G |f'(z)|^2(1-|z|^2)^\alpha dA(z)$$

for all $f \in D^\alpha_2$ if and only if a subset G of D satisfy the condition that there exist $\delta > 0$ and $r > 0$ such that $\delta|D(a,r)| \leq |D(a,r) \cap G|$, where $|D(a,r)|$ is the (normalized) area of $D(a,r)$.

Lemma 1.7. *The operator $I_g : L^2_a \to L^2_a$ is bounded if and only if*

$$\sup_{z\in D} |g(z)| < +\infty.$$

We determined the integration operators I_g on the Bergman spaces that have a closed range using sampling set for $\mathcal{B}$. And the following theorem corresponds to Theorem 0.2:

Theorem 1.8. *Suppose that the operator $I_g : L_a^2 \to L_a^2$ is bounded (i.e $g \in H^\infty$). Then the following are equivalent.*

(1) *There is a constant $k > 0$ such that*

$$\int_D |f'(z)|^2 |g(z)|^2 (1-|z|^2)^2 dA(z) \geq k \int_D |f'(z)|^2 (1-|z|^2)^2 dA(z)$$

for all $f \in L_a^2$

(2) *There exists a positive constant $\epsilon > 0$ such that $\{z \in D, |g(z)| \geq \epsilon\}$ is a sampling set for $\mathcal{B}$.*

(3) $\sup_{z\in D}(1-|z|^2)|g(z)\varphi_w'(z)| \geq k$ *for all $w \in D$.*

(4) *For any $\epsilon < k$, $\rho(\Gamma, w) \leq R < 1$ for all $w \in D$, R depending only on ϵ, where $\Gamma = \{z \in D, |g(z)| \geq \epsilon\}$.*

§2. The integration operators J_g and the multiplication operators M_g on the weighted Bergman spaces and Luecking's inequalities

In this section, we study the integration operators and the multiplication operators with closed range on the weighted Bergman space L_a^2 by using the sampling property.

The following lemma is well-known result:

Lemma C.([17]) *Let $\alpha > 1$. For $f \in \mathcal{B}_\alpha$, the norm*

$$|f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha |f'(z)|$$

is equivalent to the norm

$$\sup_{z\in D}(1-|z|^2)^{\alpha-1} |f(z)| .$$

i.e. for some constant $C_1 > 0$ (independent of $f \in \mathcal{B}_\alpha$),

$$\frac{1}{C_1} \sup_{z\in D}(1-|z|^2)^{\alpha-1} |f(z)| \leq |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha |f'(z)| \leq C_1 \sup_{z\in D}(1-|z|^2)^{\alpha-1} |f(z)| .$$

In [11] we also proved the following result :

Theorem R.2. *Let $\beta \geq 1$. Then the operator $J_g : \mathcal{B} \to \mathcal{B}_\beta$ is bounded (compact) if and only if*

$$\sup_{z\in D}(1-|z|^2)^\beta \left(\log \frac{1}{1-|z|^2}\right) |g'(z)| < +\infty \quad \left(\lim_{|z|\to 1^-}(1-|z|^2)^\beta \left(\log \frac{1}{1-|z|^2}\right) |g'(z)| = 0\right)$$

If $\beta \geq \alpha > 1$, then the operator $J_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ is bounded (compact) if and only if $g \in \mathcal{B}_{\beta-\alpha+1}$ ($g \in \mathcal{B}_{\beta-\alpha+1,0}$). If $0 < \alpha < 1$, and $\alpha \leq \beta$, then the operator $J_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ is bounded (compact) if and only if $g \in \mathcal{B}_\beta$ ($g \in \mathcal{B}_{\beta,0}$).

By using a sampling set for $\mathcal{B}^\alpha$ and Lemma C, we can prove the following result with respect to the operator J_g:

Theorem 2.1. *Let $\beta \geq \alpha > 1$ and $g \in H(D)$. Let the operator $J_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ be bounded (i.e. $g \in \mathcal{B}_{\beta-\alpha+1}$). Then the operator $J_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ is bounded below if and only if there exists a positive constant $\epsilon > 0$ such that $\{z \in D, (1-|z|^2)^{\beta-\alpha+1}|g'(z)| \geq \epsilon\}$ is a sampling set for $\mathcal{B}^{\alpha-1}$*

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In [15] R.Zhao proved the following lemma :

Lemma R.Z. *Let f be an analytic function on D. Then $f \in \mathcal{B}_2$ if and only if $\sup_{a\in D} \int_D (1-|z|^2)^2(1-|\varphi_a(z)|^2)^2|f'(z)|^2 dA(z) < +\infty$.*

To prove Theorem 2.3, we prove the following result at first:

Proposition 2.2. *Let $g \in \mathcal{B}$. If $J_g : D^4 \to D^4$ is bounded below, then $J_g : \mathcal{B}_2 \to \mathcal{B}_2$ is bounded below.*

In [7] D.Leucking proved the following result:

Theorem D'. ([7]) *Let $\alpha > -1$. There is a constant $C > 0$ such that*

$$\int_D |f(z)|^2(1-|z|^2)^\alpha dA(z) \leq C \int_G |f(z)|^2(1-|z|^2)^\alpha dA(z)$$

for all $f \in L^2_a((1-|z|^2)^\alpha dA(z))$ if and only if a subset G of D satisfy the condition that there exist $\delta > 0$ and $r > 0$ such that $\delta|D(a,r)| \leq |D(a,r) \cap G|$, where $|D(a,r)|$ is the (normalized) area of $D(a,r)$.

We determined the integration operators J_g on the weighted Bergman spaces that have a closed range using sampling set for $\mathcal{B}^1$. And the following theorem corresponds to Theorem 0.3:

Theorem 2.3. *Suppose that the operator $J_g : D^4 \to D^4$ is bounded (i.e $g \in \mathcal{B}$). Then the following are equivalent.*

(1) *There is a constant $k > 0$ such that*

$$\int_D |f(z)|^2|g'(z)|^2(1-|z|^2)^4 dA(z) \geq k \int_D |f'(z)|^2(1-|z|^2)^4 dA(z)$$

for all $f \in D^4$

(2) *There exists a positive constant $\epsilon > 0$ such that $\{z \in D, (1-|z|^2)|g'(z)| \geq \epsilon\}$ is a sampling set for $\mathcal{B}^1$.*

(3) $\sup_{z\in D}(1-|z|^2)^2|g'(z)\varphi_w'(z)| \geq k$ *for all $w \in D$.*

(4) *For any $\epsilon < k$, $\rho(\Gamma, w) \leq R < 1$ for all $w \in D$, R depending only on ϵ, where $\Gamma = \{z \in D, (1-|z|^2)|g'(z)| \geq \epsilon\}$.*

By using a sampling set for $\mathcal{B}^\alpha$, we can prove the following result with respect to the multiplication operator M_g:

Theorem 2.4. *Let $\beta \geq \alpha > 1$ and $g \in H(D)$. Let the operator $M_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ be bounded. Then the operator $M_g : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ is bounded below if and only if there exists a positive constant $\epsilon > 0$ such that $\{z \in D, (1-|z|^2)^{\beta-\alpha}|g(z)| \geq \epsilon\}$ is a sampling set for $\mathcal{B}^{\alpha-1}$.*

With respect to the multiplication operators, we can prove the following:

Proposition 2.5. *Let $g \in H^\infty$. If $M_g : D^4 \to D^4$ is bounded below, then $M_g : \mathcal{B}_2 \to \mathcal{B}_2$ is bounded below.*

We determined the multiplication operators M_g on the weighted Bergman spaces that have a closed range using sampling set for $\mathcal{B}^1$.

Theorem 2.6. *Suppose that the operator $M_g : D^4 \to D^4$ is bounded (i.e $g \in H^\infty$). Then the following are equivalent.*

(1) *There is a constant $k > 0$ such that*

$$\int_D |f(z)|^2|g(z)|^2(1-|z|^2)^2 dA(z) \geq k \int_D |f(z)|^2(1-|z|^2)^2 dA(z)$$

for all $f \in D^4$

(2) *There exists a positive constant $\epsilon > 0$ such that $\{z \in D, |g(z)| \geq \epsilon\}$ is a sampling set for $\mathcal{B}^1$.*

(3) $\sup_{z\in D}(1-|z|^2)|g(z)\varphi_w'(z)| \geq k$ *for all $w \in D$.*

(4) *For any* $\epsilon < k$, $\rho(\Gamma, w) \leq R < 1$ *for all* $w \in D$, R *depending only on* ϵ, *where* $\Gamma = \{z \in D, |g(z)| \geq \epsilon\}$.

Suppose that $g \in H^\infty$. *Then there is a constant* $k > 0$ *such that*

$$\int_D |f(z)|^2 |g(z)|^2 (1-|z|^2)^2 dA(z) \geq k \int_D |f(z)|^2 (1-|z|^2)^2 dA(z)$$

for all $f \in D^4$ *if and only if there exists a positive constant* $\epsilon > 0$ *such that* $\{z \in D, |g(z)| \geq \epsilon\}$ *is a sampling set for* $\mathcal{B}^1$.

§3. The composition operators with closed range

In this section, we study the composition operators with closed range on the space $BMOA$, the Bloch spaces, the Bergman spaces, and the Hardy space. We use the following consequence ([5] P.Ghatage and D.Zheng and Nina Zorboska's results) in our proof of Proposition 3.1.

Theorem GZN.1. ([5]) *Suppose* φ *is a univalent self-map of the open unit disk* D. *Then the following are equivalent.*

(1) C_φ *is bounded below on* $\mathcal{B}$.
(2) $\| \varphi_w \circ \varphi \|_{\mathcal{B}/\mathcal{C}} \geq k$ *for all* $w \in D$.
(3) *For any* $\epsilon < k$, $\rho(G_\epsilon, z) \leq r < 1$ *for all* $z \in D$, r *depending only on* ϵ
(4) *For any* $\epsilon < k$, *for some* r, G_ϵ *satisfying* $|G_\epsilon \cap D(w,r)| \geq C|D(w,r)|$ *for all* $w \in D$.

Theorem GZN.2.([5]) *The composition operator* C_φ *is bounded below on* $\mathcal{B}$ *if and only if there exists some* $\epsilon > 0$ *such that* G_ϵ *is a sampling set for* $\mathcal{B}$.

Theorem GZN.3.([5]) *If* φ *is univalent and* C_φ *is bounded below on* $BMOA$, *then it is bounded below on the Bloch space.*

Theorem Z.([18]) *Suppose* φ *is univalent self-map of the open unit disk* D. *Then* C_φ *is bounded below on* L^2_a *if and only if* C_φ *is bounded below on* H^2.

In [15] R.Zhao proved the following lemma :

Lemma R.Z. *Let* $\alpha \geq 1$. *Let* f *be an analytic function on* D. *Then* $f \in \mathcal{B}_\alpha$ *if and only if* $\sup_{a \in D} \int_D (1-|z|^2)^{2(\alpha-1)} (1-|\varphi_a(z)|^2)^2 |f'(z)|^2 dA(z) < +\infty$.

If $\varphi(0) = a$ and $\psi = \varphi_a \circ \varphi$, then C_φ is bounded below on $BMOA$ if and only if C_ψ is bounded below on $BMOA$. So we assume from now on that $\varphi(0) = 0$ and that C_φ is acting on the subspace of functions that vanish at the origin.

Proposition 3.1. *Suppose φ is a univalent self-map of the open unit disk D. Suppose that there exists a positive constant ϵ satisfying the condition of TheoremA such that $\sup_{\{z\in D, \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\geq\epsilon\}} |\varphi'(z)| < +\infty$. If $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded below, then $C_\varphi : H^2 \to H^2$ is bounded below.*

Theorem 3.2. *If the composition operator $C_\varphi : L^2_a \to L^2_a$ is bounded below, then $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded below.*

Theorem 3.3. *Let $\alpha \geq 0$. Suppose that C_φ is bounded on $D^{\alpha+2}$. If $C_\varphi : D^{\alpha+2} \to D^{\alpha+2}$ is bounded below, then $C_\varphi : \mathcal{B}_{\alpha+1} \to \mathcal{B}_{\alpha+1}$ is bounded below.*

In [3] P.S.Bourdon and J.A.Cima and A.L.Matheson have shown that compactness of C_φ on $BMOA$ implies its compactness on the Hardy space H^2. Since the operator C_φ is bounded on the Hardy space, we can prove the following result :

Theorem 3.4. *If the composition operator $C_\varphi : H^2 \to H^2$ is bounded below, then $C_\varphi : BMOA \to BMOA$ is bounded below.*

Using Theorem 3.4 and Theorem GZN.3, we see the following.

Corollary 3.5. *Suppose φ is univalent self-map of the open unit disk D. Then if the composition operator $C_\varphi : H^2 \to H^2$ is bounded below, then $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded below.*

The following example shows that *$C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded below* does not imply that *$C_\varphi : H^2 \to H^2$ is bounded below.*

Example 3.6. Let φ be the Riemann map onto $D\setminus[0,1)$. Then the composition operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded below ([5, Example 2]), but $C_\varphi : H^2 \to H^2$ is not bounded below ([18, Remark 2 and Example 3]). In this case, we see that there exists a sufficiently small positive constant $\epsilon(< k)$, where k satisfies the condition of Theorem A such that $\sup_{\{z\in D, \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\geq\epsilon\}} |\varphi'(z)| = +\infty$ ([18, Remark 2 and Example 3]). In fact, if $\sup_{\{z\in D, \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\geq\epsilon\}} |\varphi'(z)| < +\infty$, since $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded below, Proposition 1 implies that $C_\varphi : H^2 \to H^2$ is bounded below. This contradicts the above.

Considering Example 3.6, and using Proposition 3.1, Theorem 3.3 and Corollary 3.5, we have the following :

Proposition 3.7. *Suppose φ is a univalent self-map of the open unit disk D and there exists a sufficiently small positive constant $\epsilon(< k)$, where k satisfies the condition of TheoremA such that* $\sup_{\{z\in D, \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\geq\epsilon\}} |\varphi'(z)| < +\infty$. *Then the following statements are equivalent :*

(1) $C_\varphi : BMOA \to BMOA$ *is bounded below.*
(2) $C_\varphi : \mathcal{B} \to \mathcal{B}$ *is bounded below.*
(3) $C_\varphi : H^2 \to H^2$ *is bounded below.*
(4) $C_\varphi : L_a^2 \to L_a^2$ *is bounded below.*

The following is an example that does not satisfy the condition $\sup_{\{z\in D, \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\geq\epsilon\}} |\varphi'(z)| < +\infty$.

Example 3.8. The singular inner function

$$\varphi(z) = \exp\left(\frac{z+1}{z-1}\right)$$

is in H^∞ but not in the little Bloch spaces $\mathcal{B}_0$. And it satisfies $\sup_{\{z\in D, \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\geq\epsilon\}} |\varphi'(z)| = +\infty$. But $C_\varphi : H^2 \to H^2$ *is bounded below.*

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