A note on invariant Hilbert spaces of holomorphic functions on the unit ball in $\mathbb{C}^d$ (Analytic Function Spaces and Their Operators)

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A note on invariant Hilbert spaces of holomorphic functions on the unit ball in $\mathbb{C}^d$

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1 Introduction

Invariant Hilbert spaces of holomorphic functions on bounded symmetric domains have been extensively studied[19]. The study is motivated by the unitary representation of the automorphism group of the bounded symmetric domains.

Let $\Omega$ be a bounded symmetric domain, and $\text{Aut}(\Omega)$ denote the automorphism group of $\Omega$. Let $G$ denote the connected component of the identity in $\text{Aut}(\Omega)$. Then $G$ can be naturally represented on the Bergman space $L^2_\alpha(\Omega)$, the representation map $\pi$ is defined by

$$\pi(\varphi)f = f \circ \varphi \cdot J\varphi, \quad f \in L^2_\alpha(\Omega), \quad \varphi \in G,$$

where $J\varphi$ is the complex Jacobian of $\varphi$. Moreover, this representation is unitary, that is, for any $\varphi \in G$, the operator $\pi(\varphi)$ is unitary. For natural Hilbert space $H$ of holomorphic functions on $\Omega$, the similar action of $G$ on $H$ can also been defined. J. Arazy[19] shows that, with some mild assumptions, the only Hilbert space which makes $\pi$ be a unitary representation is the Bergman space. Of course, J. Arazy deals with a more complicated case. For detailed information, one can refer to [19].

In this note, we will mainly concern Hilbert spaces of holomorphic functions on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$. In this case, the automorphism group $\text{Aut}(\mathbb{B}_d)$

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can be written precisely. In fact, by [Ru, Theorem 2.2.5], $\text{Aut}(\mathbb{B}_d)$ is generated by the unitary group $U_d$ of $\mathbb{C}^d$ and $\{\varphi_\lambda | \lambda \in \mathbb{B}_d\}$, where, for any $\lambda \in \mathbb{B}_d$, $\varphi_\lambda$ is defined as follows. If $\lambda = 0$, $\varphi_\lambda(z) = -z$. If $\lambda \neq 0$,

$$
\varphi_\lambda = \frac{\lambda - P_\lambda z - \sqrt{1 - |\lambda|^2} P_\lambda^\perp z}{1 - \langle z, a \rangle},
$$

(1.1)

where $P_\lambda$ is the orthogonal projection from $\mathbb{C}^d$ onto the complex line $[\lambda]$ spanned in $\mathbb{C}^d$ by $\lambda$, and $P_\lambda^\perp = I - P_\lambda$. Therefore, one can only consider the automorphism with the expression (1.1). We rewrite the above representation $\pi(\varphi_\lambda)$ as $U_\lambda$ in short, that is

$$
U_\lambda f = f \circ \varphi_\lambda \cdot J \varphi_\lambda.
$$

After some calculation, it is not difficult to see that the complex Jacobian $J \varphi_\lambda = (-1)^d \frac{(1 - |\lambda|^2)^{\frac{d+1}{2}}}{(1 - \langle z, \lambda \rangle)^{d+1}}$ is just the normalized Bergman kernel on $\mathbb{B}_d$ multiplied by $(-1)^d$.

For many interesting unitary invariant reproducing Hilbert space $H$ on $\mathbb{B}_d$, one can define the similar action by $V_\lambda f = f \circ \varphi_\lambda \cdot k_\lambda$, where $k_\lambda$ is the normalized reproducing kernel of $H$. So, the question is, when $V_\lambda$ is unitary? In other word, to ensure that $V_\lambda$ is unitary, the complex Jacobian $J \varphi_\lambda$ can be replaced to what kind of 'good' functions.

In this note, with some mild assumptions, we will prove that if $V_\lambda$ is unitary, then there is a positive number $\mu$, such that $k_\lambda = ((-1)^d J \varphi_\lambda)^\mu$.

We organize this note as follows. In section 2, we will introduce some notations of unitary invariant reproducing kernel. In section 3, we prove the main theorem.

## 2 Preliminaries

From a general theory of reproducing kernels [Aro], one sees that a reproducing function space is uniquely determined by its kernel. In this paper, we will mainly concern unitary invariant reproducing function space of holomorphic functions on $\mathbb{B}_d$. A reproducing function space is called unitary invariant, if for any unitary operator $U$ on $\mathbb{C}^d$, $f \circ U \in H$ whenever $f \in H$, and for all $f, g \in H$,

$$
\langle f \circ U, g \circ U \rangle = \langle f, g \rangle.
$$
By [GHX], $H$ is unitary invariant if and only if for any unitary operator $U$ on $\mathbb{C}^d$

$$K_{U \lambda}(Uz) = K_{\lambda}(z);$$

and this holds if and only if there is a holomorphic function on the unit disk

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

with $a_n \geq 0$, such that

$$K_{\lambda}(z) = f((z, \lambda)).$$

Without loss of generality, we will consider the case that all the $a_n > 0$, and $a_0 = 1$. Hence, by [GHX, Proposition 4.1], $H$ has a canonical orthonormal basis $\{a_{|\alpha|}^{1/2}z^\alpha\}$, and $\|z^\alpha\| = [\alpha!^{1/2}(1-|a||\lambda|)]^{1/2}$. Particularly, $\|1\| = 1$.

**Example.** Let $H^2_{\mu}(\mathbb{B}_d)$ be the reproducing function space defined by the reproducing kernel $K^\mu_{\lambda} = \frac{1}{(1-\langle z, \lambda \rangle)^\mu}$ ($\mu > 0$). It is easy to verify that $H^2_{\mu}(\mathbb{B}_d)$ is unitary invariant. When $\mu = 1$, $H^2_{\mu}(\mathbb{B}_d)$ is the symmetric Fock space $H^2_d$, which is deeply studied by W. Arveson [Arv]. When $\mu = d$, $H^2_{\mu}(\mathbb{B}_d)$ is the Hardy space $H^2(\mathbb{B}_d)$. When $\mu > d$, $H^2_{\mu}(\mathbb{B}_d)$ is the weighted Bergman space $L^2_d[(1-|z|^2)^{\mu-d-1}dV]$, and in particular $H^2_{d+1}(\mathbb{B}_d)$ is the usual Bergman space.

By [Guo, Section 4], for a given $\mu > 0$, the operator

$$V_{\lambda}f = f \circ \varphi_{\lambda} \cdot \frac{(1-|\lambda|^2)^{\mu}}{(1-\langle \cdot, \lambda \rangle)^{\mu}}$$

is a unitary operator on $H^2_{\mu}(\mathbb{B}_d)$ (For the case $\mu = 1$, this is also proved by D. Greene [Gr, Theorem 3.3]). Notice that $\frac{(1-|\lambda|^2)^{\mu}}{(1-\langle \cdot, \lambda \rangle)^{\mu}}$ is the normalized reproducing kernel of $H^2_{\mu}(\mathbb{B}_d)$.

### 3 The proof of the main theorem

In this section, we will prove the main theorem. As in Section 2, let $H$ be a unitary invariant reproducing functions space with the reproducing kernel $K_{\lambda}$. For any $\lambda \in \mathbb{B}_d$, define an operator $V_{\lambda}$ on $H$ by $V_{\lambda}f = f \circ \varphi_{\lambda} \cdot k_{\lambda}$, where $k_{\lambda}$ is the normalized reproducing kernel. We have the following theorem.
Theorem 3.1. With the above notations, if $V_{\lambda}$ is a unitary operator on $H$, then there is a positive number $\mu$ such that,

$$k_{\lambda} = \frac{(1 - |\lambda|^{2})^{\frac{\mu}{2}}}{(1 - \langle \cdot, \lambda \rangle)^{\mu}}.$$ 

Proof. Below, we will prove that if $V_{\lambda}$ is unitary, then the reproducing kernel $K_{\lambda} = \sum_{n=0}^{\infty} a_{n} (z, \lambda)^{n}$ is uniquely determined by $a_{1}$, that is,

Claim. For $n > 1$, each $a_{n}$ can be uniquely expressed by $a_{1}$.

We will prove the claim by induction.

At first, we will calculate $a_{2}$. Taking $\lambda = (r, 0, \cdots, 0)$, we simply write $\varphi_{\lambda} = \varphi_{r}$ and $k_{\lambda} = k_{r}$. Since $z_{1} = z_{1} \circ \varphi_{r} \circ \varphi_{r}$, we have

$$||z_{1}k_{r}||^{2} = ||z_{1} \circ \varphi_{r}||^{2} \tag{3.1}$$

We first calculate the left side of (1). By [GHX, Proposition 4.1], $||z_{1}^{n}||^{2} = \frac{1}{a_{n}}$, and $\langle z_{1}^{n}, z_{1}^{m} \rangle = 0$ whenever $n \neq m$.

$$||z_{1}k_{r}(z)||^{2} = \frac{|| \sum_{n=0}^{\infty} a_{n} r^{n} z_{1}^{n+1} ||^{2}}{\sum_{n=0}^{\infty} a_{n} r^{2n}} = \frac{\sum_{n=0}^{\infty} a_{n}^{2} r^{2n} ||z_{1}^{n+1}||^{2}}{\sum_{n=0}^{\infty} a_{n} r^{2n}} = \frac{\sum_{n=0}^{\infty} a_{n}^{2} r^{2n}}{\sum_{n=0}^{\infty} a_{n} r^{2n}}.$$

And now we calculate the right side of (3.1),

$$||z_{1} \circ \varphi_{r}||^{2} = || (r - z_{1}) \sum_{n=0}^{\infty} (rz_{1})^{n} ||^{2}$$

$$= || \sum_{n=0}^{\infty} (r^{n+1}z_{1}^{n} - r^{n}z_{1}^{n+1}) ||^{2}$$

$$= || r + \sum_{n=1}^{\infty} (r^{n+1} - r^{n-1})z_{1}^{n} ||^{2}$$

$$= r^{2} + \sum_{n=1}^{\infty} \frac{r^{2n-2}(r^{4} - 2r^{2} + 1)}{a_{n}}.$$
Hence
\[
\sum_{n=0}^{\infty} a_{n+1} r^{2n} = (\sum_{m=0}^{\infty} a_{m} r^{2m}) (r^2 + \sum_{n=1}^{\infty} \frac{r^{2n-2}}{a_n} r^{4n-1}). \tag{3.2}
\]
Comparing the coefficients of \(r^2\) in both sides of (3.2) first, we have
\[
\frac{a_1^2}{a_2} = 1 - \frac{2}{a_1} + \frac{1}{a_2} + \frac{a_1}{a_1}.
\]
Therefore, when \(a_1 \neq 1\),
\[
a_2 = \frac{a_1(a_1+1)}{2}. \tag{3.3}
\]
When \(a_1 = 1\), to determine \(a_2\), we compare the coefficient of \(r^4\) in both sides of (3.2). After some simple computation, we have
\[
\frac{a_2^2}{a_3} = \frac{1}{a_3} - \frac{1}{a_2} + a_2. \tag{3.4}
\]
We also need the following equation.
\[
\| z_1^2 \circ \varphi_r \cdot k_r \|^2 = \| z_1^2 \|^2 = \frac{1}{a_2}. \tag{3.5}
\]
Thus,
\[
\| z_1^2 \circ \varphi_r \cdot K_r \|^2 = \frac{1}{a_2} \sum_{n=0}^{\infty} a_n r^{2n}. \tag{3.5}
\]
Now, let us calculate the left side of (3.5). A careful verification shows that
\[
\| z_1^2 \circ \varphi_r \cdot K_r \|^2 = \| (r - z_1)^2 K_r \|^2
\]
\[
= \| (r - z_1)^2 [\sum_{n=0}^{\infty} (n+1)(rz_1)^n][\sum_{m=0}^{\infty} a_m (rz_1)^m] \|^2
\]
\[
= \| r^2 + (r^2(2r + a_1 r) - 2r)z_1
\]
\[
+ \sum_{n=2}^{\infty} r^{n-2}(r^4 \sum_{j=1}^{n+1} j a_{n+1-j} - 2r^2 \sum_{j=1}^{n} j a_{n-j} + \sum_{j=1}^{n-1} j a_{n-1-j})z_1^n \|^2
\]
Now, set $b_n = \sum_{j=1}^{n-1} ja_{n-1-j}$, and the above equation can be simplified as follows.

\[
\|z_1^2 \circ \varphi_r \cdot K_r\|^2
\]

\[
= \|r^2 + (r^2(2r + a_1 r) - 2r)z_1 + \sum_{n=2}^{\infty} r^{n-2}(r^4b_{n+2} - 2r^2b_{n+1} + b_n)z_1^n\|^2
\]

\[
= r^4 + [r^2(2r + a_1 r) - 2r]^2 \frac{1}{a_1} + \sum_{n=2}^{\infty} [r^{n-2}(r^4b_{n+2} - 2r^2b_{n+1} + b_n)]^2 \frac{1}{a_n}
\]

\[
= r^4 + [r^3(2 + a_1) - 2r]^2 \frac{1}{a_1} + \sum_{n=2}^{\infty} r^{2n-4} [r^8b_{n+2}^2 - 4r^6b_{n+2}b_n + r^4(4b_{n+1}^2 + 2b_{n+2}b_n) - 4r^2b_{n+1}b_n + b_n^2] \frac{1}{a_n}
\]

\[
= \frac{b_2^2}{a_2} + r^2\left(\frac{4}{a_1} - \frac{4b_3b_2}{a_2} + \frac{b_3^2}{a_3}\right)
\]

\[
+ \sum_{n=2}^{\infty} r^{2n}\left[\frac{b_{n+2}}{a_{n+2}} + C(a_1, \cdots, a_{n+1}, b_2, \cdots, b_{n+2})\right],
\]

where $C(a_1, \cdots, a_{n+1}, b_1, \cdots, b_{n+2})$ can be uniquely expressed by \{a_i\}^{n+1}_{i=1} and \{b_i\}^{n+2}_{i=2}. Now comparing the coefficients of $r^2$ in both sides of (3.5), we have

\[
\frac{4}{a_1} - \frac{2\cdot 2(2+a_1)}{a_2} + \frac{(2+a_1)^2}{a_3} = \frac{1}{a_2}.
\]

(3.6)

When $a_1 = 1$, combining (3.4) with (3.6), we have

\[
a_2 = 1 = \frac{a_1(a_1+1)}{2}
\]

Hence, by (3.3) and (3.7), the equality $a_2 = \frac{a_1(a_1+1)}{2}$ is always true.

And now we assume that $a_j$ is uniquely expressed by $a_1$ for $1 < j \leq m$. To prove $a_{m+1}$ is uniquely expressed by $a_1$, we compare the coefficient of $r^{2(m-1)}$ in both sides of (3.5).

\[
\frac{a_{m-1}}{a_2} = \frac{b_{m+1}^2}{a_{m+1}} + C(a_1, \cdots, a_m, b_2, \cdots, b_{m+1}).
\]
By the definition of $b_i$, we know that $b_i$ is uniquely expressed by $\{a_j\}_{j=1}^{i-2}$. By the inductive assumption, both $a_{m-1}$ and $C(a_1, \cdots, a_m, b_2, \cdots, b_{m+1})$ are uniquely expressed by $a_1$, and so is $a_{m+1}$. Thus the claim is proved.

Set $\mu = a_1$. By section 2, if

$$K_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^\mu} = 1 + \mu(z, \lambda) + \sum_{n=2}^{\infty} \frac{\mu(\mu + 1) \cdots (\mu + n - 1)}{n!} \langle z, \lambda \rangle^n,$$

then $V_\lambda$ is unitary. The above reasoning thus shows that

$$a_n = \frac{\mu(\mu + 1) \cdots (\mu + n - 1)}{n!}.$$

This means $K_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^\mu}$, which implies that $k_\lambda = \frac{(1 - |\lambda|^2)\frac{\mu}{2}}{(1 - \langle \cdot, \lambda \rangle)^\mu}$. \qed

**Proposition 3.2.** Let $H$ and $H'$ be two unitary invariant reproducing function spaces on $B_d$ with the reproducing kernels $K_\lambda$ and $K'_\lambda$ relatively. If

$$\|f \circ \varphi \cdot k'_\lambda\| = \|f\| \quad \text{for } \forall f \in H,$$

then $H = H'$, and hence by Theorem 3.1 $H = H^2_\mu(B_d)$ for some $\mu > 0$.

**Proof.** Write $K_\lambda(z) = \sum_{n=0}^{\infty} a_n \langle z, \lambda \rangle^n$ and $K'_\lambda(z) = \sum_{n=0}^{\infty} b_n \langle z, \lambda \rangle^n$. Denote the inner product of $H$ by $\| \cdot \|$ and the inner product of $H'$ by $\| \cdot \|^\prime$. Since $\|1\| = 1$, we have

$$\|1 \circ \varphi \cdot k'_\lambda\|^2 = \frac{\|K_\lambda\|}{\|K'_\lambda\|^\prime} = 1.$$

On the one hand, since $\langle z^\alpha, z^\beta \rangle = 0$ whenever $\alpha \neq \beta$,

$$\|K'_\lambda\|^2 = \sum_{n=0}^{\infty} b_n \|\langle z, \lambda \rangle^n\|^2.$$

On the other hand

$$\|K'_\lambda\|^2 = \sum_{n=0}^{\infty} b_n |\lambda|^{2n}.$$
Hence
\[ \sum_{n=0}^{\infty} b_n \| (z, \lambda)^n \|^2 = \sum_{n=0}^{\infty} b_n |\lambda|^{2n}. \]

Taking \( \lambda = (r, 0 \cdots, 0) \), we know \( \| z_1^n \|^2 = \frac{1}{b_n} \). By [GHX, Proposition 4.1], \( \frac{1}{a_n} = \| z_1^n \|^2 = \frac{1}{b_n} \), and hence \( K_\lambda = K'_\lambda \), which implies \( H = H' \). \( \square \)

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**References**


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