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Hankel-type operators on the spaces of analytic functions

Abstract. We will study Hankel-type operators on the spaces of analytic functions on the open unit disk. These operators are a natural generalization of the classical Hankel operator on the Hilbert Hardy space. They are related to tight uniform algebras, the Dunford-Pettis property, and Bourgain algebras.

1 Introduction

Let $X$ be a Banach space and $Y$ a closed subspace of $X$. For an element $g$ such that $gY \subset X$, we define the operator $S_g : Y \to X/Y$ by

$$S_g f = gf + Y$$

for all $f \in Y$. The norm is considered as the quotient norm, that is,

$$\|S_g f\| = \|gf + Y\| = \inf\{\|gf + h\| : h \in Y\}$$

for all $f \in Y$. The quotient norm is the distance from $gf$ to $Y$: $d(gf, Y) = \inf\{\|gf + h\| : h \in Y\}$. This operator is called a Hankel-type operator and is a natural generalization of the classical Hankel operator on the Hilbert Hardy space. Recall that $S_g$ is said to be (weakly) compact if $S_g$ maps every bounded set into a relatively (weakly) compact one, and that $S_g$ is said to be completely continuous if $S_g$ maps every weakly convergent sequence into a norm convergent one. In general, every compact operator is completely continuous. But the converse is not always true. We define the following sets of symbols:

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\[ Y_c = \{ g : S_g \text{ is compact} \}, \]
\[ Y_{wc} = \{ g : S_g \text{ is weakly compact} \}, \]
\[ Y_{cc} = \{ g : S_g \text{ is completely continuous} \}. \]

The conditions for \( S_g \) to be compact, weakly compact and completely continuous have been investigated in various function spaces. The problem of whether all \( S_g \) are weakly compact on a uniform algebra is related to a tight algebra [3] and the problem of complete continuity appears in the Dunford-Pettis property. The latter introduced a notion of Bourgain algebras which have been actively researched in analytic and harmonic function spaces on the open unit disk ([1], [2] and [10]). Recently, Dudziak, Gamelin, and Gorkin [5] studied Hankel-type operators on analytic function spaces and Izuchi and the author [9] investigated Hankel-type operators on the space of bounded harmonic functions on the unit disk. See [8] and [13] as surveys for convenience.

We here consider Hankel-type operators on the spaces of analytic functions on the open unit disk, explicitly, the disk algebra, Hardy and Bergman spaces.

Let \( \mathbb{D} \) be the open unit disk in the complex plane and \( \partial \mathbb{D} \) its boundary. Let \( C(\partial \mathbb{D}) \) and \( C(\overline{\mathbb{D}}) \) be the algebras of all continuous functions on \( \partial \mathbb{D} \) and \( \overline{\mathbb{D}} \) respectively. Let \( A(\mathbb{D}) \) be the disk algebra of all continuous functions on \( \overline{\mathbb{D}} \) that are analytic on \( \mathbb{D} \). Then \( A(\mathbb{D}) \) is the Banach algebra with the supremum norm

\[
\| f \|_{\infty} = \sup \{|f(z)|; z \in \overline{\mathbb{D}}\}.
\]

For \( 1 \leq p < \infty \), let \( L^p(\partial \mathbb{D}) \) and \( L^p(\mathbb{D}) \) be the Lebesgue spaces on \( \partial \mathbb{D} \) and \( \mathbb{D} \) respectively. For \( 1 \leq p < \infty \), we denote by \( H^p \) the classical Hardy space that is the Banach space of all analytic function \( f \) on \( \mathbb{D} \) for which

\[
\| f \|_{H^p} = \left( \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty,
\]

and denote by \( L^p_a \) the Bergman space consisting of all analytic function \( f \) on \( \mathbb{D} \) for which

\[
\| f \|_{L^p_a} = \left( \int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{1/p} < \infty.
\]
where $dA$ is the normalized area measure on $\mathbb{D}$. Let $H^\infty$ be the algebra of bounded analytic functions on $\mathbb{D}$. See [6], [7] and [15] for more information on the Hardy and Bergman spaces.

In the next section, we regard the disk algebra $A(\mathbb{D})$ as a closed subalgebra of $C(\partial \mathbb{D})$ or $C(\overline{\mathbb{D}})$ and $H^\infty$ as a closed subalgebra of $L^\infty(\partial \mathbb{D})$ or $L^\infty(\mathbb{D})$ respectively. In section 3, we will consider the case of Hardy space $H^p$: for $g \in L^\infty(\partial \mathbb{D})$, we define the linear operator $S_g : H^p \rightarrow L^p(\partial \mathbb{D})/H^p$ by $S_g f = gf + H^p$ for $f \in H^p$. Trivially $S_g : H^p \rightarrow L^p(\partial \mathbb{D})/H^p$ is a bounded linear operator. When $p = 2$, let $H_g$ be the classical Hankel operator on $H^2$; for $g \in L^\infty(\partial \mathbb{D})$, $H_g f = gf - P(gf)$, where $P$ is the orthogonal projection from $L^2(\partial \mathbb{D})$ onto $H^2$. It is well known that $H_g$ is compact if and only if $g \in H^\infty + C(\partial \mathbb{D})$. In section 3, Theorem 3.1 says that this equivalence holds on $H^p$ for $1 < p < \infty$. When $p = 1$, Janson, Peetre and Semmes [11] studied the Hankel operator as form $H_g f = \overline{P}(bf)$ where $f$ is analytic polynomial and $\overline{P}$ is the orthogonal projection of $L^2$ onto $\overline{H^2}$. So we will give the attention to Hankel operators on $H^p$ from the another approach.

On the other hand, in the case of Bergman space, Leucking [12] characterized the compactness of Hankel operators on $L_a^p, 1 < p < \infty$. So we will note them in section 4. In section 5 we add the result on the space of bounded harmonic functions and in the last section we pose some open questions.

## 2 The disk algebra and $H^\infty$

(1) The disk algebra $A(\mathbb{D})$

At first we regard the disk algebra $A = A(\mathbb{D})|_{\partial D}$ as a closed subalgebra of $C(\partial \mathbb{D})$. For $g \in C(\partial \mathbb{D})$, we define the linear operator $S_g : A \rightarrow C(\partial \mathbb{D})/A$ by

$$S_g f = gf + A \quad \text{for} \quad f \in A.$$

Then we would characterize the sets $A_c, A_{wc}$ and $A_{cc}$. Each set is a closed subalgebra of $C(\partial \mathbb{D})$. 


Theorem 2.1. When we regard $A = A(\mathbb{D})|_{\partial \mathbb{D}}$ as a closed subalgebra of $C(\partial \mathbb{D})$, then
\[ A_c = A_{wc} = A_{cc} = C(\partial \mathbb{D}). \]

Proof. It is trivial that $A \subset A_c \subset A_{cc} \subset C(\partial \mathbb{D})$ and that $A_c \subset A_{wc}$. Let $f_n \in A$ with $\|f_n\|_\infty \leq 1$. Then $|f_n(0)| \leq 1$. Thus there exists a subsequence (which we do not relabel) of $\{f_n(0)\}$ such that $f_n(0) \rightarrow c$ for some constant $c$. Then, since $\overline{z}(f_n - f_n(0)) \in A$,
\[
\|\overline{z}f_n - \overline{z}c + A\|_\infty \\
\leq \|\overline{z}f_n - \overline{z}c - \overline{z}(f_n - f_n(0))\|_\infty \\
= \|\overline{z}(c - f_n(0))\|_\infty \\
= |c - f_n(0)| \rightarrow 0.
\]
So $S_\overline{z}$ is compact and $\overline{z} \in A_c$. Because $A_c$ is a closed subalgebra of $C(\partial \mathbb{D})$,
\[ A_c = A_{wc} = A_{cc} = C(\partial \mathbb{D}), \]
by the Stone-Weierstrass theorem. \hfill \Box

We here estimate the norm and the essential norm of $S_g$. Recall that the essential norm of a bounded linear operator $T$ from $Y$ to $X/Y$ is defined as
\[ \|T\|_e = \inf \{\|T + K\| : K \text{ is compact operator from } Y \text{ to } X/Y\}. \]

Using the basic duality relation ([7: Chapter IV]), we has the following.

Theorem 2.2. For $g \in C(\partial \mathbb{D})$, then $\|S_g\| = d(g, A(\mathbb{D})|_{\partial \mathbb{D}})$ and $\|S_g\|_e = 0$.

Secondly we regard $A = A(\mathbb{D})$ as a closed subalgebra of $C(\overline{\mathbb{D}})$. For $g \in C(\overline{\mathbb{D}})$, we define the linear operator $S_g : A \rightarrow C(\overline{\mathbb{D}})/A$ by
\[ S_g f = g f + A \hspace{1em} \text{for} \hspace{1em} f \in A. \]

Then each set $A_c, A_{wc}$ and $A_{cc}$ is a closed subalgebra of $C(\overline{\mathbb{D}})$ and their equivalence was proved by Cole and Gamelin [3].

Theorem 2.3. When we regard $A(\mathbb{D})$ as a closed subalgebra of $C(\overline{\mathbb{D}})$, then
\[ A_c = A_{wc} = A_{cc} = C(\overline{\mathbb{D}}). \]

In this case we also have the following using the basic duality relation.
Theorem 2.4. For \( g \in C(\mathbb{D}) \), then \( \|S_g\| = d(g, A(\mathbb{D})) \) and \( \|S_g\|_e = 0 \).

(2) \( H^\infty \)

At first we regard \( H^\infty \) as a closed subalgebra of \( L^\infty(\partial \mathbb{D}) \). For \( g \in L^\infty(\partial \mathbb{D}) \), we define the linear operator \( S_g : H^\infty \to L^\infty(\partial \mathbb{D})/H^\infty \) by

\[
S_g f = g f + H^\infty \quad \text{for} \quad f \in H^\infty.
\]

Then using the fact that \( H^\infty \) has the Dunford-Pettis property, Cima, Janson and Yale [1] and Gorkin [8] showed the following.

Theorem 2.5. When we regard \( H^\infty \) as a closed subalgebra of \( L^\infty(\partial \mathbb{D}) \), then

\[
H^\infty_c = H^\infty_{wc} = H^\infty_{cc} = H^\infty + C(\partial \mathbb{D}).
\]

The estimation of norms is the following.

Theorem 2.6. For \( g \in L^\infty(\partial \mathbb{D}) \), then \( \|S_g\| = d(g, H^\infty) \) and \( \|S_g\|_e \leq d(g, H^\infty + C(\partial \mathbb{D})) \).

Secondly we regard \( H^\infty \) as a closed subalgebra of \( L^\infty(\mathbb{D}) \). For \( g \in L^\infty(\mathbb{D}) \), we define the linear operator \( S_g : H^\infty \to L^\infty(\mathbb{D})/H^\infty \) by

\[
S_g f = g f + H^\infty \quad \text{for} \quad f \in H^\infty.
\]

Then Cima, Stroethoff and Yale [2] obtained the following result.

Theorem 2.7. When we regard \( H^\infty \) as a closed subalgebra of \( L^\infty(\mathbb{D}) \), then

\[
H^\infty_c = H^\infty_{wc} = H^\infty_{cc} = H^\infty + C(\mathbb{D}) + V
\]

where \( V = \{ g \in L^\infty(\mathbb{D}) : \|g\chi_{\mathbb{D}\setminus r\mathbb{D}}\| \to 0 \quad \text{as} \quad r \to 1^- \} \).

Furthermore the estimation of norms is the following.

Theorem 2.8. For \( g \in L^\infty(\mathbb{D}) \), then \( \|S_g\| = d(g, H^\infty) \) and \( \|S_g\|_e \leq d(g, H^\infty + C(\mathbb{D}) + V) \).
3 The case of Hardy spaces $H^p$ for $1 < p < \infty$

We here consider the case of Hardy spaces. Before starting our discussion, we recall results concerning the topology of Hardy spaces $H^p$.

**Fact 1.** ([4: Chap.20, Proposition 3.15]) If $1 < p < \infty$, $f \in H^p$, and $f_n$ is a sequence in $H^p$, then the following are equivalent.

(a) $\{f_n\}$ converges weakly to $f \in H^p$.

(b) $\{f_n\}$ is bounded and $f_n \in H^p$ converges to $f$ uniformly on every compact subset of $\mathbb{D}$.

(c) $\{f_n\}$ is bounded and $f_n(z)$ converges to $f(z)$ for all $z \in \mathbb{D}$.

(d) $\{f_n\}$ is bounded and $f_n^k(0)$ converges to $f^k(0)$ for all $k \geq 0$.

**Fact 2.** ([4: Chap.20, Proposition 3.16]) Put $S = \{f \in H^1 : \|f\|_{H^1} = 1\}$. Then $S$ is weak*-compact and metrizable, but not weak compact.

For $f_n \in S$, the following are equivalent:

(i) $f_n$ converges to $f$ in the weak*-topology in $H^1$.

(ii) $f_n(z)$ converges to $f(z)$ for all $z \in \mathbb{D}$.

(iii) $f_n$ converges to $f$ uniformly on every compact subset of $\mathbb{D}$.

For $1 \leq p < \infty$ and $g \in L^\infty(\partial \mathbb{D})$, we define the linear operator $S_g : H^p \to L^p(\mathbb{D})/H^p$ by

$$S_g f = gf + H^p \quad \text{for} \quad f \in H^p.$$

**Fact 3.** For $1 < p < \infty$ and $g \in L^\infty(\partial \mathbb{D})$, the following are equivalent:

(i) $S_g : H^p \to L^p(\partial \mathbb{D})/H^p$ is compact (completely continuous).

(ii) If $\{f_n\}$ is bounded in $H^p$ and converges to 0 uniformly on every compact subset of $\mathbb{D}$, then $\|S_g f_n\| \to 0$.

For $1 < p < \infty$, $H^p$ is reflexive. So all completely continuous operator on $H^p$ is compact and every bounded operator on $H^p$ is always weakly compact. Thus $H^p_c = H^p_{cc}$ and $H^p_{wc} = L^\infty(\partial \mathbb{D})$.

For $g \in L^\infty(\partial \mathbb{D})$, let $H_g$ be the classical Hankel operator on $H^2$ defined by $H_g f = gf - P(gf)$, where $P$ is the orthogonal projection from $L^2(\partial \mathbb{D})$ onto $H^2$. Hartman's theorem says that $H_g : H^2 \to L^2(\partial \mathbb{D})$ is compact if and only if $g \in H^\infty + C(\partial \mathbb{D})$. Then for $1 < p < \infty$, $P$ is bounded from
$L^p(\partial \mathbb{D})$ onto $H^p$ and we can easily see the equivalence of compactness of $H_g$ and $S_g$. But the next result will give the characterization of Hankel operators on $H^p$ from the another approach.

**Theorem 3.1.** For $1 < p < \infty$, the following hold:

$$H^p_c = H^p_{cc} = H^\infty + C(\partial \mathbb{D}) \quad \text{and} \quad H^p_{wc} = L^\infty(\partial \mathbb{D}).$$

**Proof.** First, we note that $H^\infty \subset H^p_c = H^p_{cc} \subset L^\infty(\partial \mathbb{D})$. Then $B := H^p_c = H^p_{cc}$ is a closed algebra and so a Douglas algebra.

Suppose that $H^\infty + C(\partial \mathbb{D}) \subset B$. Thus there exists an interpolating Blaschke product $\psi \in H^\infty$ with $\overline{\psi} \in B$. That is, $S_{\overline{\psi}}$ is compact (completely continuous). Write $\psi(z) = e^{i\alpha} \prod_{n=1}^\infty b_n(z)$ where $b_n(z) = (z - z_n)/(1 - \overline{z_n}z)$. Put $f_k(z) = \prod_{n=k}^\infty b_n(z)$. Then $f_k \in H^p$, $\|f_k\|_{H^p} = 1$ and $f_k(z) \to 0$ for $z \in \mathbb{D}$ as $k \to \infty$. By Fact 1, $f_k(z) \to 0$ weakly in $H^p$.

On the other hand, we have

$$\|S_{\overline{\psi}} f_k\| = \|\overline{\psi} f_k + H^p\| = \|e^{i\alpha} b_1 b_2 \cdots b_{k-1} + H^p\| = \inf_{f \in H^p} \|1 + e^{i\alpha} b_1 b_2 \cdots b_{k-1} f\|_{H^p} \geq \inf_{f \in H^p} \|1 + e^{i\alpha} (b_1 b_2 \cdots b_{k-1} f)(z)((1 - |z|^2)^{1/p},$$

for $z \in \mathbb{D}$.

Put $z = z_1$, a zero of $b_1$. So

$$\|S_{\overline{\psi}} f_k\| \geq (1 - |z_1|^2)^{1/q} > 0.$$ 

As $S_{\overline{\psi}}$ is completely continuous,

$$\|S_{\overline{\psi}} f_k\| \to 0.$$

This contradicts. So $B = H^\infty + C(\partial \mathbb{D})$. $\square$

Furthermore the estimation of norms is the following.

**Theorem 3.2.** For $1 \leq p < \infty$ and $g \in L^\infty(\mathbb{D})$, then $\|S_g\| = d(g, H^\infty)$ and for $1 < p < \infty$, $\|S_g\|_e \leq d(g, H^\infty + C(\partial \mathbb{D}))$. 

4 The case of Bergman spaces $L^p_a$ for $1 < p < \infty$

For $g \in L^\infty(D)$, let $H_g$ be the classical Hankel operator defined by $H_g f = g f - P(g f)$, where $P$ is the Bergman projection from $L^p$ onto $L^p_a$. Then we can easily see the equivalence of compactness of $H_g$ and $S_g$. On the other hand, Leucking [12] characterized the compactness of Hankel operators on $L^p_a$, $1 < p < \infty$. And so we have the following.

**Theorem 4.1.** For $1 < p < \infty$, then $g \in (L^p_a)_c = (L^p_a)_{cc}$ if and only if $g$ admits a decomposition $g = g_1 + g_2$ so that

$$\lim_{|z| \to 1} \frac{1}{|D(z)|} \int_{D(z)} |g_1|^2 dA = 0$$

and

$$g_2 \in C^1(D), \lim_{|z| \to 1} (1 - |z|) \overline{\partial} g_2 (z) = 0$$

where $D(z)$ is the Bergman disk with center $z$.

Moreover it holds that $(L^p_a)_{wc} = L^\infty(D)$.

5 The space of bounded harmonic functions

We here consider Hankel-type operators on the space of bounded harmonic functions. Let $h^\infty := h^\infty(D)$ be the set of all bounded harmonic functions on $D$. It follows that $h^\infty$ is a closed subspace of $L^\infty(D)$. We can define $h^\infty_c, h^\infty_{wc}$ and $h^\infty_{cc}$ as before. The Bourgain algebra $h^\infty_{cc}$ is characterized by Izuchi, Stroethoff and Yale [10].

For a function $f \in L^\infty(\partial D)$, we denote by $\hat{f}$ the Poisson integral of $f$ on $D$, that is,

$$\hat{f}(z) = \int_0^{2\pi} f(e^{i\theta}) P_z(e^{i\theta}) d\theta / 2\pi,$$

where $P_z$ is the Poisson kernel of $z \in D$. Then $\hat{f} \in h^\infty$. For any nonempty subset $B$ of $L^\infty(\partial D)$, we write $\hat{B} = \{ \hat{f} : f \in B \}$. It is known that $f$ in $h^\infty$ has a boundary function $f^*$ on $\partial D$ and $\hat{f}^* = f$ on $D$, so that $h^\infty = \overline{L^\infty(\partial D)}$. Let $H^\infty(\partial D)$ be the space of boundary functions of
bounded analytic functions on $\mathbb{D}$. The algebra $QC$ of bounded quasi-continuous functions on $\partial \mathbb{D}$ is given by

$$QC = (H^\infty(\partial \mathbb{D}) + C(\partial \mathbb{D})) \cap \overline{(H^\infty(\partial \mathbb{D}) + C(\partial \mathbb{D}))}.$$  

Refer to [7] and [14] for more information.

The equality $h^\infty_{cc} = \widehat{QC} + V$ was given as Corollary 3 in [10], where $V$ is the same set as in Theorem 2.7. Then Izuchi and the author [9] show the following result.

**Theorem 5.1.** $h^\infty_c = h^\infty_{cc} = \widehat{QC} + V.$

## 6 Problems

**Problem 6.1.** *Estimate the essential norms of Hankel-type operators in cases of Hardy and Bergman spaces.*

**Problem 6.2.** *How about the case $p = 1$? That is, what are $H^1_c, H^1_{wc}, H^1_{cc}$?*

**Problem 6.3.** *Let $h^\infty$ be the space of bounded harmonic functions on $\mathbb{D}$. Then characterize $h^\infty_{wc}$.***

## References


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