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Hankel-type operators on the spaces of analytic functions

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Abstract. We will study Hankel-type operators on the spaces of analytic functions on the open unit disk. These operators are a natural generalization of the classical Hankel operator on the Hilbert Hardy space. They are related to tight uniform algebras, the Dunford-Pettis property, and Bourgain algebras.

1 Introduction

Let X be a Banach space and Y a closed subspace of X. For an element g such that $gY \subset X$, we define the operator $S_g : Y \to X/Y$ by

$$S_g f = gf + Y$$

for all $f \in Y$. The norm is considered as the quotient norm, that is,

$$\|S_g f\| = \|gf + Y\| = \inf\{\|gf + h\| : h \in Y\}$$

for all $f \in Y$. The quotient norm is the distance from gf to Y: $d(gf, Y) = \inf\{\|gf + h\| : h \in Y\}$. This operator is called a Hankel-type operator and is a natural generalization of the classical Hankel operator on the Hilbert Hardy space. Recall that S_g is said to be (*weakly*) *compact* if S_g maps every bounded set into a relatively (weakly) compact one, and that S_g is said to be *completely continuous* if S_g maps every weakly convergent sequence into a norm convergent one. In general, every compact operator is completely continuous. But the converse is not always true. We define the following sets of symbols;

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$Y_c = \{g : S_g \text{ is compact}\}$,

$Y_{wc} = \{g : S_g \text{ is weakly compact}\}$,

$Y_{cc} = \{g : S_g \text{ is completely continuous}\}$.

The conditions for S_g to be compact, weakly compact and completely continuous have been investigated in various function spaces. The problem of whether all S_g are weakly compact on a uniform algebra is related to a tight algebra [3] and the problem of complete continuity appears in the Dunford-Pettis property. The latter introduced a notion of Bourgain algebras which have been actively researched in analytic and harmonic function spaces on the open unit disk ([1], [2] and [10]). Recently, Dudziak, Gamelin, and Gorkin [5] studied Hankel-type operators on analytic function spaces and Izuchi and the author [9] investigated Hankel-type operators on the space of bounded harmonic functions on the unit disk. See [8] and [13] as surveys for convenience.

We here consider Hankel-type operators on the spaces of analytic functions on the open unit disk, explicitly, the disk algebra, Hardy and Bergman spaces.

Let $\mathbb{D}$ be the open unit disk in the complex plane and $\partial\mathbb{D}$ its boundary. Let $C(\partial\mathbb{D})$ and $C(\overline{\mathbb{D}})$ be the algebras of all continuous functions on $\partial\mathbb{D}$ and $\overline{\mathbb{D}}$ respectively. Let $A(\mathbb{D})$ be the disk algebra of all continuous functions on $\overline{\mathbb{D}}$ that are analytic on $\mathbb{D}$. Then $A(\mathbb{D})$ is the Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)|; z \in \overline{\mathbb{D}}\}.$$

For $1 \le p \le \infty$, let $L^p(\partial\mathbb{D})$ and $L^p(\mathbb{D})$ be the Lebesgue spaces on $\partial\mathbb{D}$ and $\mathbb{D}$ respectively. For $1 \le p < \infty$, we denote by H^p the classical Hardy space that is the Banach space of all analytic function f on $\mathbb{D}$ for which

$$\|f\|_{H^p} = \Big(\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p} < \infty,$$

and denote by L^p_a the Bergman space consisting of all analytic function f on $\mathbb{D}$ for which

$$\|f\|_{L^p_a} = \Big(\int_{\mathbb{D}} |f(z)|^p dA(z)\Big)^{1/p} < \infty$$

where dA is the normalized area measure on $\mathbb{D}$. Let H^∞ be the algebra of bounded analytic functions on $\mathbb{D}$. See [6], [7] and [15] for more information on the Hardy and Bergman spaces.

In the next section, we regard the disk algebra $A(\mathbb{D})$ as a closed subalgebra of $C(\partial\mathbb{D})$ or $C(\overline{\mathbb{D}})$ and H^∞ as a closed subalgebra of $L^\infty(\partial\mathbb{D})$ or $L^\infty(\mathbb{D})$ respectively. In section 3, we will consider the case of Hardy space H^p: for $g \in L^\infty(\partial\mathbb{D})$, we define the linear operator $S_g : H^p \longrightarrow L^p(\partial\mathbb{D})/H^p$ by $S_g f = gf + H^p$ for $f \in H^p$. Trivially $S_g : H^p \longrightarrow L^p(\partial\mathbb{D})/H^p$ is a bounded linear operator. When $p = 2$, let H_g be the classical Hankel operator on H^2; for $g \in L^\infty(\partial\mathbb{D})$, $H_g f = gf - P(gf)$, where P is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2. It is well known that H_g is compact if and only if $g \in H^\infty + C(\partial\mathbb{D})$. In section 3, Theorem 3.1 says that this equivalence holds on H^p for $1 < p < \infty$. When $p = 1$, Janson, Peetre and Semmes [11] studied the Hankel operator as form $H_b f = \overline{P}(bf)$ where f is analytic polynomial and $\overline{P}$ is the orthogonal projection of L^2 onto $\overline{H^2}$. So we will give the attention to Hankel operators on H^p from the another approach.

On the other hand, in the case of Bergman space, Leucking [12] characterized the compactness of Hankel operators on $L^p_a, 1 < p < \infty$. So we will note them in section 4. In section 5 we add the result on the space of bounded harmonic functions and in the last section we pose some open questions.

2 The disk algebra and H^∞

(1) The disk algebra $A(\mathbb{D})$

At first we regard the disk algebra $A = A(\mathbb{D})|_{\partial\mathbb{D}}$ as a closed subalgebra of $C(\partial\mathbb{D})$. For $g \in C(\partial\mathbb{D})$, we define the linear operator $S_g : A \longrightarrow C(\partial\mathbb{D})/A$ by

$$S_g f = gf + A \quad \text{for} \quad f \in A.$$

Then we would characterize the sets A_c, A_{wc} and A_{cc}. Each set is a closed subalgebra of $C(\partial\mathbb{D})$.

Theorem 2.1. *When we regard $A = A(\mathbb{D})|_{\partial\mathbb{D}}$ as a closed subalgebra of $C(\partial\mathbb{D})$, then*

$$A_c = A_{wc} = A_{cc} = C(\partial\mathbb{D}).$$

Proof. It is trivial that $A \subset A_c \subset A_{cc} \subset C(\partial\mathbb{D})$ and that $A_c \subset A_{wc}$. Let $f_n \in A$ with $\|f_n\|_\infty \leq 1$. Then $|f_n(0)| \leq 1$. Thus there exists a subsequence (which we do not relabel) of $\{f_n(0)\}$ such that $f_n(0) \to c$ for some constant c. Then, since $\overline{z}(f_n - f_n(0)) \in A$,

$$\begin{aligned}
&\|\overline{z}f_n - \overline{z}c + A\|_\infty \\
&\quad \leq \|\overline{z}f_n - \overline{z}c - \overline{z}(f_n - f_n(0))\|_\infty \\
&\quad = \|\overline{z}(c - f_n(0))\|_\infty \\
&\quad = |c - f_n(0)| \to 0.
\end{aligned}$$

So $S_{\overline{z}}$ is compact and $\overline{z} \in A_c$. Because A_c is a closed subalgebra of $C(\partial\mathbb{D})$,

$$A_c = A_{wc} = A_{cc} = C(\partial\mathbb{D}),$$

by the Stone-Weierstrass theorem. □

We here estimate the norm and the essential norm of S_g. Recall that the essential norm of a bounded linear operator T from Y to X/Y is defined as

$$\|T\|_e = \inf\{\|T + K\| : K \text{ is compact operator from } Y \text{ to } X/Y\}.$$

Using the basic duality relation ([7: Chapter IV]), we has the following.

Theorem 2.2. *For $g \in C(\partial\mathbb{D})$, then $\|S_g\| = d(g, A(\mathbb{D})|_{\partial\mathbb{D}})$ and $\|S_g\|_e = 0$.*

Secondly we regard $A = A(\mathbb{D})$ as a closed subalgebra of $C(\overline{\mathbb{D}})$. For $g \in C(\overline{\mathbb{D}})$, we define the linear operator $S_g : A \to C(\overline{\mathbb{D}})/A$ by

$$S_g f = gf + A \quad \text{for} \quad f \in A.$$

Then each set A_c, A_{wc} and A_{cc} is a closed subalgebra of $C(\overline{\mathbb{D}})$ and their equivalence was proved by Cole and Gamelin [3].

Theorem 2.3. *When we regard $A(\mathbb{D})$ as a closed subalgebra of $C(\overline{\mathbb{D}})$, then*

$$A_c = A_{wc} = A_{cc} = C(\overline{\mathbb{D}}).$$

In this case we also have the following using the basic duality relation.

Theorem 2.4. *For $g \in C(\overline{\mathbb{D}})$, then $\|S_g\| = d(g, A(\mathbb{D}))$ and $\|S_g\|_e = 0$.*

(2) H^∞

At first we regard H^∞ as a closed subalgebra of $L^\infty(\partial\mathbb{D})$. For $g \in L^\infty(\partial\mathbb{D})$, we define the linear operator $S_g : H^\infty \to L^\infty(\partial\mathbb{D})/H^\infty$ by

$$S_g f = gf + H^\infty \quad \text{for} \quad f \in H^\infty.$$

Then using the fact that H^∞ has the Dunford-Pettis property, Cima, Janson and Yale [1] and Gorkin [8] showed the following.

Theorem 2.5. *When we regard H^∞ as a closed subalgebra of $L^\infty(\partial\mathbb{D})$, then*

$$H^\infty_c = H^\infty_{wc} = H^\infty_{cc} = H^\infty + C(\partial\mathbb{D}).$$

The estimation of norms is the following.

Theorem 2.6. *For $g \in L^\infty(\partial\mathbb{D})$, then $\|S_g\| = d(g, H^\infty)$ and $\|S_g\|_e \leq d(g, H^\infty + C(\partial\mathbb{D}))$.*

Secondly we regard H^∞ as a closed subalgebra of $L^\infty(\mathbb{D})$. For $g \in L^\infty(\mathbb{D})$, we define the linear operator $S_g : H^\infty \to L^\infty(\mathbb{D})/H^\infty$ by

$$S_g f = gf + H^\infty \quad \text{for} \quad f \in H^\infty.$$

Then Cima, Stroethoff and Yale [2] obtained the following result.

Theorem 2.7. *When we regard H^∞ as a closed subalgebra of $L^\infty(\mathbb{D})$, then*

$$H^\infty_c = H^\infty_{wc} = H^\infty_{cc} = H^\infty + C(\overline{\mathbb{D}}) + V$$

where $V = \{g \in L^\infty(\mathbb{D}) : \|g\chi_{\mathbb{D}\setminus r\mathbb{D}}\| \to 0 \quad \text{as} \quad r \to 1^-\}$.

Furthermore the estimation of norms is the following.

Theorem 2.8. *For $g \in L^\infty(\mathbb{D})$, then $\|S_g\| = d(g, H^\infty)$ and $\|S_g\|_e \leq d(g, H^\infty + C(\overline{\mathbb{D}}) + V)$.*

3 The case of Hardy spaces H^p for $1 < p < \infty$

We here consider the case of Hardy spaces. Before starting our discussion, we recall results concerning the topology of Hardy spaces H^p.

Fact 1. ([4: Chap.20, Proposition 3.15]) If $1 < p < \infty$, $f \in H^p$, and f_n is a sequence in H^p, then the following are equivalent.

(a) $\{f_n\}$ converges weakly to $f \in H^p$.

(b) $\{f_n\}$ is bounded and $f_n \in H^p$ converges to f uniformly on every compact subset of $\mathbb{D}$.

(c) $\{f_n\}$ is bounded and $f_n(z)$ converges to $f(z)$ for all $z \in \mathbb{D}$.

(d) $\{f_n\}$ is bounded and $f_n^k(0)$ converges to $f^k(0)$ for all $k \geq 0$.

Fact 2. ([4: Chap.20, Proposition 3.16]) Put $S = \{f \in H^1 : \|f\|_{H^1} = 1\}$. Then S is weak*-compact and metrizable, but not weak compact.

For $f_n \in S$, the following are equivalent:

(i) f_n converges to f in the weak*-topology in H^1.

(ii) $f_n(z)$ converges to $f(z)$ for all $z \in \mathbb{D}$.

(iii) f_n converges to f uniformly on every compact subset of $\mathbb{D}$.

For $1 \leq p < \infty$ and $g \in L^\infty(\partial\mathbb{D})$, we define the linear operator S_g : $H^p \to L^p(\mathbb{D})/H^p$ by

$$S_g f = gf + H^p \quad \text{for} \quad f \in H^p.$$

Fact 3. For $1 < p < \infty$ and $g \in L^\infty(\partial\mathbb{D})$, the following are equivalent:

(i) $S_g : H^p \to L^p(\partial\mathbb{D})/H^p$ is compact (completely continuous).

(ii) If $\{f_n\}$ is bounded in H^p and converges to 0 uniformly on every compact subset of $\mathbb{D}$, then $\|S_g f_n\| \to 0$.

For $1 < p < \infty$, H^p is reflexive. So all completely continuous operator on H^p is compact and every bounded operator on H^p is always weakly compact. Thus $H^p_c = H^p_{cc}$ and $H^p_{wc} = L^\infty(\partial\mathbb{D})$.

For $g \in L^\infty(\partial\mathbb{D})$, let H_g be the classical Hankel operator on H^2 defined by $H_g f = gf - P(gf)$, where P is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2. Hartman's theorem says that $H_g : H^2 \to L^2(\partial\mathbb{D})$ is compact if and only if $g \in H^\infty + C(\partial\mathbb{D})$. Then for $1 < p < \infty$, P is bounded from

$L^p(\partial\mathbb{D})$ onto H^p and we can easily see the equivalence of compacteness of H_g and S_g. But the next result will give the characterization of Hankel operators on H^p from the another approach.

Theorem 3.1. *For $1 < p < \infty$, the following hold:*

$$H^p_c = H^p_{cc} = H^\infty + C(\partial\mathbb{D}) \quad \text{and} \quad H^p_{wc} = L^\infty(\partial\mathbb{D}).$$

Proof. First, we note that $H^\infty \subset H^p_c = H^p_{cc} \subset L^\infty(\partial\mathbb{D})$. Then $B := H^p_c = H^p_{cc}$ is a closed algebra and so a Douglas algebra.

Suppose that $H^\infty + C(\partial\mathbb{D}) \subsetneq B$. Thus there exists an interpolating Blaschke product $\psi \in H^\infty$ with $\overline{\psi} \in B$. That is, $S_{\overline{\psi}}$ is compact (completely continuous). Write $\psi(z) = e^{i\alpha}\prod_{n=1}^\infty b_n(z)$ where $b_n(z) = (z - z_n)/(1 - \overline{z_n}z)$. Put $f_k(z) = \prod_{n=k}^\infty b_n(z)$. Then $f_k \in H^p$, $\|f_k\|_{H^p} = 1$ and $f_k(z) \to 0$ for $z \in \mathbb{D}$ as $k \to \infty$. By Fact 1, $f_k(z) \to 0$ weakly in H^p. On the other hand, we have

$$\begin{aligned}
\|S_{\overline{\psi}} f_k\| &= \|\overline{\psi} f_k + H^p\| \\
&= \|\overline{e^{i\alpha} b_1 b_2 \cdots b_{k-1}} + H^p\| \\
&= \inf_{f \in H^p} \|1 + e^{i\alpha} b_1 b_2 \cdots b_{k-1} f\|_{H^p} \\
&\geq \inf_{f \in H^p} |1 + e^{i\alpha}(b_1 b_2 \cdots b_{k-1} f)(z)|(1 - |z|^2)^{1/p},
\end{aligned}$$

for $z \in \mathbb{D}$.

Put $z = z_1$, a zero of b_1. So

$$\|S_{\overline{\psi}} f_k\| \geq (1 - |z_1|^2)^{1/q} > 0.$$

As $S_{\overline{\psi}}$ is completely continuous,

$$\|S_{\overline{\psi}} f_k\| \to 0.$$

This contradicts. So $B = H^\infty + C(\partial\mathbb{D})$. □

Furthermore the estimation of norms is the following.

Theorem 3.2. *For $1 \leq p < \infty$ and $g \in L^\infty(\mathbb{D})$, then $\|S_g\| = d(g, H^\infty)$ and for $1 < p < \infty$, $\|S_g\|_e \leq d(g, H^\infty + C(\partial\mathbb{D}))$.*

4 The case of Bergman spaces L_a^p for $1 < p < \infty$

For $g \in L^\infty(\mathbb{D})$, let H_g be the classical Hankel operator defined by $H_g f = gf - P(gf)$, where P is the Bergman projection from L^p onto L_a^p.

Then we can easily see the equivalence of compacteness of H_g and S_g. On the other hand, Leucking [12] characterized the compactness of Hankel operators on $L_a^p, 1 < p < \infty$. And so we have the following.

Theorem 4.1. *For $1 < p < \infty$, then $g \in (L_a^p)_c = (L_a^p)_{cc}$ if and only if g admits a decomposition $g = g_1 + g_2$ so that*

$$\lim_{|z|\to 1} \frac{1}{|D(z)|} \int_{D(z)} |g_1|^2 dA = 0$$

and

$$g_2 \in C^1(\mathbb{D}), \quad \lim_{|z|\to 1} (1 - |z|)\bar{\partial} g_2(z) = 0$$

where $D(z)$ is the Bergman disk with center z.

Moreover it holds that $(L_a^p)_{wc} = L^\infty(\mathbb{D})$.

5 The space of bounded harmonic functions

We here consider Hankel-type operators on the space of bounded harmonic functions. Let $h^\infty := h^\infty(\mathbb{D})$ be the set of all bounded harmonic functions on $\mathbb{D}$. It follows that h^∞ is a closed subspace of $L^\infty(\mathbb{D})$. We can define $h_c^\infty, h_{wc}^\infty$ and h_{cc}^∞ as before. The Bourgain algebra h_{cc}^∞ is characterized by Izuchi, Stroethoff and Yale [10].

For a function $f \in L^\infty(\partial\mathbb{D})$, we denote by $\hat{f}$ the Poisson integral of f on $\mathbb{D}$, that is,

$$\hat{f}(z) = \int_0^{2\pi} f(e^{i\theta}) P_z(e^{i\theta}) d\theta / 2\pi,$$

where P_z is the Poisson kernel of $z \in \mathbb{D}$. Then $\hat{f} \in h^\infty$. For any nonempty subset B of $L^\infty(\partial\mathbb{D})$, we write $\hat{B} = \{\hat{f} : f \in B\}$. It is known that f in h^∞ has a boundary function f^* on $\partial\mathbb{D}$ and $\widehat{f^*} = f$ on $\mathbb{D}$, so that $h^\infty = \widehat{L^\infty(\partial\mathbb{D})}$. Let $H^\infty(\partial\mathbb{D})$ be the space of boundary functions of bounded analytic functions on $\mathbb{D}$. The algebra QC of bounded quasi-continuous functions on $\partial\mathbb{D}$ is given by

$$QC = (H^\infty(\partial\mathbb{D}) + C(\partial\mathbb{D})) \cap \overline{(H^\infty(\partial\mathbb{D}) + C(\partial\mathbb{D}))}.$$

Refer to [7] and [14] for more information.

The equality $h^\infty_{cc} = \widehat{QC} + V$ was given as Corollary 3 in [10], where V is the same set as in Theorem 2.7. Then Izuchi and the author [9] show the following result.

Theorem 5.1. $h^\infty_c = h^\infty_{cc} = \widehat{QC} + V$.

6 Problems

Problem 6.1. *Estimate the essential norms of Hankel-type operators in cases of Hardy and Bergman spaces.*

Problem 6.2. *How about the case $p = 1$? That is, what are H^1_c, H^1_{wc}, H^1_{cc}?*

Problem 6.3. *Let h^∞ be the space of bounded harmonic functions on $\mathbb{D}$. Then characterize h^∞_{wc}.*

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