REPRESENTATION PROPERTY OF WEIGHTED HARMONIC BERGMAN FUNCTIONS ON THE UPPER HALF-SPACES

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1. INTRODUCTION

Let $\mathbf{H}$ denote the upper half space $\mathbb{R}^{n-1} \times \mathbb{R}_+$ where $\mathbb{R}_+$ denotes the set of all positive real numbers. We will write points $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbb{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$ and $1 \leq p < \infty$, let $b^p_{\alpha} = b^p_{\alpha}(\mathbf{H})$ denote the weighted harmonic Bergman space consisting of all real-valued harmonic functions $u$ on $\mathbf{H}$ such that

$$
\|u\|_{L^p_{\alpha}} := \left( \int_{\mathbf{H}} |u(z)|^p \, dV_{\alpha}(z) \right)^{1/p} < \infty
$$

where $dV_{\alpha}(z) = z_n^\alpha \, dz$ and $dz$ is the Lebesque measure on $\mathbb{R}^n$. Then we can see easily that the space $b^p_{\alpha}$ is a Banach space. In particular, $b^2_{\alpha}$ is a Hilbert space. Hence, there is a unique Hilbert space orthogonal projection $\Pi_{\alpha}$ of $L^2_{\alpha}$ onto $b^2_{\alpha}$ which is called the weighted harmonic Bergman projection. It is known that this weighted harmonic Bergman projection can be realized as an integral operator against the weighted harmonic Bergman kernel $R_{\alpha}(z, w)$. See section 2.

The purpose of this paper is to survey [8] concerning the representation property of $b^p_{\alpha}$-functions and the interpolation by $b^p_{\alpha}$-functions.

In the holomorphic case representation and interpolation properties of Bergman functions have been studied in [5] and [11]. In [5], the representation properties of harmonic Bergman functions, as well as harmonic Bloch functions, were also proved on the unit ball in $\mathbb{R}^n$. See [2] for the interpolation properties of holomorphic (little) Bloch functions. On the setting of the half-space of $\mathbb{R}^n$, Choe and Yi [6] have studied these two properties of harmonic Bergman spaces. In [6], the harmonic (little) Bloch spaces are also considered as limiting spaces of $b^p$.

2. PRELIMINARIES

First, we introduce the fractional derivative. Let $D$ denote the differentiation with respect to the last component and let $u \in b^p_{\alpha}$. Then the mean value
property, Jensen's inequality and Cauchy's estimate yield
\begin{equation}
|D^{k}u(z)| \leq cz_{n}^{-(n+\alpha)/p-k}
\end{equation}
for each $z \in \mathbf{H}$ and for every nonnegative integer $k$.

Let $\mathcal{F}_{\beta}$ be the collection of all functions $v$ on $\mathbf{H}$ satisfying $|v(z)| \leq cz_{n}^{-\beta}$ for $\beta > 0$ and let $\mathcal{F} = \bigcup_{\beta > 0} \mathcal{F}_{\beta}$. If $u \in \mathcal{F}$, then $v \in \mathcal{F}_{\beta}$ for some $\beta > 0$. In this case, we define the fractional derivative of $u$ of order $-s$ by
\begin{equation}
D^{-s}u(z) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} u(z', z_{n}+t) dt
\end{equation}
for the range $0 < s < \beta$. (Here, $\Gamma$ is the Gamma function.)

If $u \in b_{\alpha}^{p}$, then for every nonnegative integer $k$, $D^{k}u \in \mathcal{F}$ by (2.1). Thus for $s > 0$, we define the fractional derivative of $u$ of order $s$ by
\begin{equation}
D^{s}u = D^{-(\lceil s \rceil - s)} D^{\lceil s \rceil} u.
\end{equation}
Here, $\lceil s \rceil$ is the smallest integer greater than or equal to $s$ and $D^{0} = D^{0}$ is the identity operator. If $s > 0$ is not an integer, then $-1 < \lceil s \rceil - s - 1 < 0$ and $\lceil s \rceil \geq 1$. Thus we know from (2.1) that, for each $z \in \mathbf{H}$ and for every $u \in b_{\alpha}^{p}$, the integral
\begin{equation}
D^{s}u(z) = \frac{1}{\Gamma(\lceil s \rceil - s)} \int_{0}^{\infty} t^{\lceil s \rceil - s - 1} D^{\lceil s \rceil} u(z', z_{n}+t) dt
\end{equation}
always makes sense.

Let $P(z, w)$ be the extended Poisson kernel on $\mathbf{H}$ and put $P_{z} = P(z, \cdot)$. More explicitly,
\begin{equation}
P_{z}(w) = P(z, w) = \frac{2}{nV(B)} \frac{z_{n}+w_{n}}{|z-\overline{w}|^{n}}
\end{equation}
where $z, w \in \mathbf{H}$ and $\overline{w} = (w', -w_{n})$ and $B$ is the open unit ball in $\mathbb{R}^{n}$. It is known that the weighted harmonic Bergman projection $\Pi_{\alpha}$ of $L^{2}_{\alpha}$ onto $b_{\alpha}^{2}$ is given by
\begin{equation}
\Pi_{\alpha}f(z) = \int_{\mathbf{H}} f(w) R_{\alpha}(z, w) dV_{\alpha}(w)
\end{equation}
for all $f \in L^{2}_{\alpha}$. Here $R_{\alpha}(z, w)$ denotes the weighted harmonic Bergman kernel whose explicit formula is given by
\begin{equation}
R_{\alpha}(z, w) = C_{\alpha} D^{\alpha+1} P_{z}(w)
\end{equation}
where $C_{\alpha} = (-1)^{[\alpha]+1} 2^{\alpha+1}/\Gamma(\alpha+1)$. Also, it is known that
\begin{equation}
|D_{z_{n}}^{\beta} R_{\alpha}(z, w)| \leq \frac{C}{|z-\overline{w}|^{n+\alpha+\beta}}
\end{equation}
for all $z, w \in \mathbf{H}$. Here, $\beta > -n - \alpha$ and the constant $C$ is dependent only on $n, \alpha$ and $\beta$. Using (2.5), we know $R_{\alpha}(z, \cdot) \in b_{\alpha}^{q}$ for all $1 < q \leq \infty$. Thus, $\Pi_{\alpha}$
is well defined whenever \( f \in L_p^\alpha \) for \( 1 \leq p < \infty \). Also, for \( 1 \leq p < \infty \), \( u \in b_p^\alpha \), \( z \in \mathbb{H} \), we have the reproducing formula

\[
(2.6) \quad u(z) = \int_{\mathbb{H}} u(w) R_\beta(z, w) \, dV_\beta(w)
\]

whenever \( \beta \geq \alpha \). Furthermore, we have a useful norm equivalence. If \( \alpha > -1 \), \( 1 \leq p < \infty \) and \( (1+\alpha)/p + \gamma > 0 \), then

\[
(2.7) \quad \|u\|_{L_p^\alpha} \approx \|w_n^\gamma D^\gamma u\|_{L_p^\rho}
\]

as \( u \) ranges over \( b_p^\alpha \).

Set \( z_0 = (0,1) \). A harmonic function \( u \) on \( \mathbb{H} \) is called a Bloch function if

\[
\|u\|_{\mathcal{B}} = \sup_{w \in \mathbb{H}} w_n |\nabla u(w)| < \infty,
\]

where \( \nabla u \) denotes the gradient of \( u \). We let \( \mathcal{B} \) denote the set of Bloch functions on \( \mathbb{H} \) and let \( \tilde{\mathcal{B}} \) denote the subspace of functions in \( \mathcal{B} \) that vanish at \( z_0 \). Then the space \( \tilde{\mathcal{B}} \) is a Banach space under the Bloch norm \( \| \cdot \|_{\mathcal{B}} \).

A function \( u \in \tilde{\mathcal{B}} \) is called a harmonic little Bloch function if it has the following vanishing condition

\[
\lim_{z \to \partial^\infty \mathbb{H}} z_n |\nabla u(z)| = 0
\]

where \( \partial^\infty \mathbb{H} \) denotes the union of \( \partial \mathbb{H} \) and \( \{\infty\} \). Let \( \tilde{\mathcal{B}}_0 \) denote the set of all harmonic little Bloch functions on \( \mathbb{H} \). It is not hard to verify that \( \tilde{\mathcal{B}}_0 \) is a closed subspace of \( \tilde{\mathcal{B}} \). Let \( \mathcal{C}_0 \) denote the set of all continuous functions on \( \mathbb{H} \) vanishing at \( \infty \).

Because \( R_\alpha(z, \cdot) \) is not in \( L_1^\alpha \), \( \Pi_\alpha f \) is not well defined for \( f \in L^\infty \). So we need the following modified Bergman kernel. For \( z, w \in \mathbb{H} \), define

\[
\tilde{R}_\alpha(z, w) = R_\alpha(z, w) - R_\alpha(z_0, w).
\]

Then, there is a constant \( C = C(n, \alpha) \) such that

\[
(2.8) \quad |\tilde{R}_\alpha(z, w)| \leq C \left( \frac{|z - z_0|}{|z - w|^{n+\alpha}|z_0 - w|} + \frac{|z - z_0|}{|z - w||z_0 - w|^{n+\alpha}} \right)
\]

for all \( z, w \in \mathbb{H} \). Thus, (2.8) implies that \( \tilde{R}_\alpha(z, \cdot) \in L_1^\alpha \) for each fixed \( z \in \mathbb{H} \) and thus we can define \( \tilde{\Pi}_\alpha \) on \( L^\infty \) by

\[
\tilde{\Pi}_\alpha f(z) = \int_{\mathbb{H}} f(w) \tilde{R}_\alpha(z, w) \, dV_\alpha(w)
\]

for \( f \in L^\infty \). It turns out that \( \tilde{\Pi}_\alpha \) is a bounded linear map from \( L^\infty \) onto \( \tilde{\mathcal{B}} \). Also, \( \tilde{\Pi}_\alpha \) has the following property: If \( \gamma > 0 \) and \( v \in \tilde{\mathcal{B}} \) then

\[
(2.9) \quad \tilde{\Pi}_\alpha (w_n^\gamma D^\gamma v)(z) = Cv(z)
\]
where \( C = C(\alpha, \gamma) \). The Bloch norm is also equivalent to the normal derivative norm: If \( \gamma > 0 \), then
\[
(2.10) \quad \| u \|_B \approx \| w_n^\gamma \mathcal{D}^\gamma u \|_\infty
\]
as \( u \) ranges over \( \tilde{B} \). (See [7] for details.)

3. TECHNICAL LEMMAS

We first introduce a distance function on \( \mathbf{H} \) which is useful for our purposes. The pseudohyperbolic distance between \( z, w \in \mathbf{H} \) is defined by
\[
\rho(z, w) = \frac{|z - w|}{|z - \bar{w}|}.
\]
This \( \rho \) is an actual distance. (See [6].) Note that \( \rho \) is horizontal translation invariant and dilation invariant. In particular,
\[
(3.1) \quad \rho(z, w) = \rho(\phi_a(z), \phi_a(w))
\]
for \( z, w \in \mathbf{H} \) where \( \phi_a(a \in \mathbf{H}) \) denotes the function defined by
\[
\phi_a(z) = \left( \frac{z' - a'}{a_n}, \frac{z_n}{a_n} \right)
\]
for \( z \in \mathbf{H} \). Note that the Jacobian of \( \phi_a^{-1} \) is \( a_n^{-1} \). For \( z \in \mathbf{H} \) and \( 0 < \delta < 1 \), let \( E_\delta(z) \) denote the pseudohyperbolic ball centered at \( z \) with radius \( \delta \). Note that \( \phi_z(E_\delta(z)) = E_\delta(z_0) \) by the invariance property (3.1). Also, simple calculation shows that
\[
(3.2) \quad E_\delta(z) = B\left( \left( z', \frac{1 + \delta^2}{1 - \delta^2} z_n \right), \frac{2\delta}{1 - \delta^2} z_n \right)
\]
so that \( B(z, \delta z_n) \subset E_\delta(z) \subset B(z, 2\delta(1 - \delta)^{-1} z_n) \) where \( B(z, r) \) denotes the Euclidean ball centered at \( z \) with radius \( r \). From (3.2), we have two lemmas. For proofs of the following lemmas, see [6].

Lemma 3.1. Let \( z, w \in \mathbf{H} \). Then
\[
\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{z_n}{w_n} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.
\]
This lemma implies the following lemma.

Lemma 3.2. Let \( z, w \in \mathbf{H} \). Then
\[
\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{|z - \bar{s}|}{|w - \bar{s}|} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}
\]
for all \( s \in \mathbf{H} \).

The following lemma is used to prove the representation theorem. If \( \alpha \) is a nonnegative integer, then it is proved in [6].
Lemma 3.3. Let $\alpha > -1$ and $\beta$ be real. Then
\[
|z^\beta R_\alpha(s, z) - w^\beta R_\alpha(s, w)| \leq C \rho(z, w) \frac{z^\beta}{|z - s|^{n+\alpha}}
\]
whenever $\rho(z, w) < 1/2$ and $s \in \mathbf{H}$.

Let $\alpha > -1$ and let $1 \leq p < \infty$. Define $\Pi_\beta$ on the weighted Lebesque space $L^p_\alpha$ by
\[
\Pi_\beta f(z) = \int_\mathbf{H} f(w) R_\beta(z, w) dV_\beta(w)
\]
for $f \in L^p_\alpha$ and $z \in \mathbf{H}$. Then we have the following two lemmas from [7].

Lemma 3.4. Suppose $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Then $\Pi_\beta$ is bounded projection of $L^p_\alpha$ onto $b^p_\alpha$.

Lemma 3.5. For $b < 0, -1 < a + b$, there exists a constant $C = C(a, b)$ such that
\[
\int_\mathbf{H} \frac{w^{a+b}}{|z - w|^{n+a}} dw \leq C z^b
\]
for every $z, w \in \mathbf{H}$.

Lemma 3.6. Let $\alpha > -1$, $1 \leq p < \infty$ and let $(1 + \alpha)/p + \gamma > 0$. Suppose $0 < \delta < 1$. Then
\[
z^{n+p\gamma} |D^\gamma u(z)|^p \leq C \frac{d(E, \partial \Omega)^{n+p}}{\delta^{n+pk}} \int_{E_\delta(z)} |u(w)|^p dw
\]
for all $z \in \mathbf{H}$ and for every $u$ harmonic on $\mathbf{H}$ where $k = [\gamma]$ if $\gamma > -1$ and $k = 0$ if $\gamma \leq -1$. The constant $C = C(n, p, \gamma)$ is independent of $\delta$.

If $\gamma$ satisfies the condition of Lemma 3.6, we can show $D^\gamma u$ is harmonic on $\mathbf{H}$. If $\gamma$ is a nonnegative integer, then $D^\gamma u$ is harmonic on $\mathbf{H}$, because it is a partial derivative of a harmonic function. If $\gamma$ is not a nonnegative integer, we see also $D^\gamma u$ is harmonic on $\mathbf{H}$ by passing the Laplacian through the integral.

The notation $|E|$ denotes the Lebesque measure of a Borel subset $E$ of $\mathbf{H}$. Let $|E|_\alpha$ denote $V_\alpha(E)$. The following lemma is proved by using the mean value property and Cauchy's estimates. The notation $d(E, F)$ denotes the euclidean distance between two sets $E$ and $F$.

Lemma 3.7. Suppose $u$ is harmonic on some proper open subset $\Omega$ of $\mathbf{R}^n$. Let $\alpha > -1$ and let $1 \leq p < \infty$. Then, for a given open ball $E \subset \Omega$,
\[
\int_E |u(z) - u(a)|^p dV_\alpha(z) \leq C \frac{|E|^{p/n} |E|_\alpha}{d(E, \partial \Omega)^{n+p}} \int_\Omega |u(w)|^p dw
\]
for all $a \in E$. The constant $C$ depends only on $n, \alpha$ and $p$.  

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4. Representation theory

Let \( \{z_m\} \) be a sequence in \( \mathbf{H} \) and let \( 0 < \delta < 1 \). We say that \( \{z_m\} \) is \( \delta \)-separated if the balls \( E_\delta(z_m) \) are pairwise disjoint or simply say that \( \{z_m\} \) is separated if it is \( \delta \)-separated for some \( \delta \). Also, we say that \( \{z_m\} \) is a \( \delta \)-lattice if it is \( \delta/2 \)-separated and \( \mathbf{H} = \bigcup E_\delta(z_m) \). Note that any "maximal" \( \delta/2 \)-separated sequence is a \( \delta \)-lattice.

From [4] and [6], we have the following three lemmas.

**Lemma 4.1.** Fix a \( 1/2 \)-lattice \( \{a_m\} \) and let \( 0 < \delta < 1/8 \). If \( \{z_m\} \) is a \( \delta \)-lattice, then we can find a rearrangement \( \{z_{ij} : i = 1, 2, \ldots, j = 1, 2, \ldots, N_i\} \) of \( \{z_m\} \) and a pairwise disjoint covering \( \{D_{ij}\} \) of \( \mathbf{H} \) with the following properties:

(a) \( E_{\delta/2}(z_{ij}) \subset D_{ij} \subset E_\delta(z_{ij}) \)
(b) \( E_{1/4}(a_i) \subset \bigcup_{j=1}^{N_i} D_{ij} \subset E_{1/8}(a_i) \)
(c) \( z_{ij} \in E_{1/8}(a_i) \)

for all \( i = 1, 2, \ldots \) and \( j = 1, 2, \ldots, N_i \).

**Lemma 4.2.** Let \( \tau > 0 \) and let \( 0 < \tau \eta < 1 \). If \( \{z_m\} \) is an \( \eta \)-separated sequence, then there is a constant \( M = M(n, \tau, \eta) \) such that more than \( M \) of the balls \( E_{\tau \eta}(z_m) \) contain no point in common.

**Lemma 4.3.** Let \( N_i \) be the sequence defined in Lemma 4.1. Then

\[
\sup_i N_i \leq C\delta^{-n}
\]

for some constant \( C \) depending only on \( n \).

Analysis similar to that for the proof of Lemma 3.4 shows the following lemma which will be used in the proof of Proposition 4.5.

**Lemma 4.4.** Let \( \alpha > -1, 1 \leq p < \infty \) and \( \alpha + 1 < (\beta + 1)p \). For \( f \in L^p_\alpha \), define

\[
\Phi_\beta f(z) = \int_{\mathbf{H}} f(w) \frac{w^\beta}{|z-w|^{n+\beta}} \, dw
\]

for \( z \in \mathbf{H} \). Then, \( \Phi_\beta : L^p_\alpha \rightarrow L^p_\alpha \) is bounded.

Let \( \{z_m\} \) be a sequence in \( \mathbf{H} \). Let \( \alpha > -1, 1 \leq p < \infty \) and \( \alpha + 1 < (\beta + 1)p \). For \( (\lambda_m) \in l^p \), let \( Q_\beta(\lambda_m) \) denote the series defined by

\[
Q_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{(n+\beta)(1-1/p)+(\beta-\alpha)/p} R_\beta(z, z_m)
\]

for \( z \in \mathbf{H} \). For a sequence \( \{z_m\} \) good enough, \( Q_\beta(\lambda_m) \) will be harmonic on \( \mathbf{H} \). We say that \( \{z_m\} \) is a \( b_\alpha^p \)-representing sequence of order \( \beta \) if \( Q_\beta(l^p) = b_\alpha^p \).

Lemma 4.4 implies the following proposition which shows \( Q_\beta(l^p) \subset b_\alpha^p \) if the underlying sequence is separated.

**Proposition 4.5.** Let \( \alpha > -1, 1 \leq p < \infty \) and \( \alpha + 1 < (\beta + 1)p \). Suppose \( \{z_m\} \) is a \( \delta \)-separated sequence. Then \( Q_\beta : l^p \rightarrow b_\alpha^p \) is bounded.
The following theorem is the $b^p_\alpha$-representation result under the lattice density condition.

**Theorem 4.6.** Let $\alpha > -1, 1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Then there exists $\delta_0 > 0$ with the following property: Let $\{z_m\}$ be a $\delta$-lattice with $\delta < \delta_0$ and let $Q_{\beta} : l^p \to b^p_\alpha$ be the associated linear operator as in (4.1). Then there is a bounded linear operator $P_{\beta} : b^p_\alpha \to l^p$ such that $Q_{\beta} P_{\beta}$ is the identity on $b^p_\alpha$. In particular, $\{z_m\}$ is a $b^p_\alpha$-representing sequence of order $\beta$.

Since $D^\gamma u$ is harmonic and we have (2.7), we can have similar result with Proposition 4.8 of [6].

**Proposition 4.7.** Let $\alpha > -1, 1 \leq p < \infty$ and let $(1 + \alpha)/p + \gamma > 0$. If $\{z_m\}$ is a $\delta$-lattice with $\delta$ sufficiently small, then

$$||u||_{L^p_{\alpha}^p} \approx \sum |z_m^\alpha + \beta \gamma |D^\gamma u(z_m)|^p$$

as $u$ ranges over $b^p_\alpha$.

Let $\{z_m\}$ be a sequence in $H$ and let $\beta > -1$. For $(\lambda_m) \in l^\infty$, let

(4.2) $$\tilde{Q}_{\beta} (\lambda_m)(z) = \sum \lambda_m z_m^\alpha \beta R_{\beta}(z, z_m)$$

for $z \in H$. We say that $\{z_m\}$ is a $\tilde{B}$-representing sequence of order $\beta$ if $\tilde{Q}_{\beta}(l^\infty) = \tilde{B}$. We also say that $\{z_m\}$ is a $\tilde{B}_0$-representing sequence of order $\beta$ if $\tilde{Q}_{\beta}(C_0) = \tilde{B}_0$. Then we have the result which shows that a separated sequence represents a part of the whole space.

**Proposition 4.8.** Let $\beta > -1$ and suppose $\{z_m\}$ is a $\delta$-separated sequence. Then, $\tilde{Q}_{\beta} : l^\infty \to \tilde{B}$ is bounded. In addition, $\tilde{Q}_{\beta}$ maps $C_0$ into $\tilde{B}_0$.

If $\gamma$ is a positive integer, then the following lemma is proved in [6].

**Lemma 4.9.** Let $\gamma > 0$. Then

$$|z^n\gamma D^\gamma u(z) - w^n_\gamma D^\gamma u(w)| \leq C \rho(z, w) ||u||_B$$

for all $z, w \in H$ and $u \in \tilde{B}$.

The following theorem is the limiting version of the $b^p_\alpha$-representation theorem.

**Theorem 4.10.** Let $\beta > -1$. Then there exists a positive number $\delta_0$ with the following property: Let $\{z_m\}$ be a $\delta$-lattice with $\delta < \delta_0$ and let $\tilde{Q}_{\beta} : l^\infty \to \tilde{B}$ be the associated linear operator as in (4.2). Then there exists a bounded linear operator $\tilde{P}_{\beta} : \tilde{B} \to l^\infty$ such that $\tilde{Q}_{\beta} \tilde{P}_{\beta}$ is the identity on $\tilde{B}$. Moreover, $\tilde{P}_{\beta}$ maps $\tilde{B}_0$ into $C_0$. In particular, $\{z_m\}$ is a both $\tilde{B}$-representing and $\tilde{B}_0$-representing sequence of order $\beta$.

Lemma 4.9 yields the following result for $\tilde{B}$ analogous to Proposition 4.7.
Proposition 4.11. Let $\gamma > 0$. Let $\{z_m\}$ be a $\delta$-lattice with $\delta$ sufficiently small. Then
\[ \|u\|_B \approx \sup_m z_m^{\gamma} |D^\gamma u(z_m)| \]
as $u$ ranges over $\tilde{B}$.

5. INTERPOLATION THEORY

Let $\{z_m\}$ be a sequence on $\mathbf{H}$. Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. For $u \in b^p_\alpha$, let $T_\gamma u$ denote the sequence of complex numbers defined by
\[ T_\gamma u = (z_m^{(n+\alpha)/p+\gamma} D^\gamma u(z_m)) \]
If $T_\gamma (b^p_\alpha) = l^p$, we say that $\{z_m\}$ is a $b^p_\alpha$-interpolating sequence of order $\gamma$.

The following two lemmas are used to prove that separation is necessary for $b^p_\alpha$-interpolation.

Lemma 5.1. Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. Let $\{z_m\}$ be a $b^p_\alpha$-interpolating sequence of order $\gamma$. Then $T_\gamma : b^p_\alpha \to l^p$ is bounded.

The following lemma is a $b^p_\alpha$-version of Lemma 4.9 concerning $\tilde{B}$-functions. If $\gamma$ is a nonnegative integer, then the following lemma is proved in [6].

Lemma 5.2. Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. Then,
\[ \left| z_n^{(n+\alpha)/p+\gamma} D^\gamma u(z) - w_n^{(n+\alpha)/p+\gamma} D^\gamma u(w) \right| \leq C \rho(z, w) \|u\|_{L^p_\alpha} \]
for all $z, w \in \mathbf{H}$ and $u \in b^p_\alpha$.

Proposition 5.3. Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. Every $b^p_\alpha$-interpolating sequence of order $\gamma$ is separated.

For interpolation, we need the sufficient separation condition.

Theorem 5.4. Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. Then there exists a positive number $\delta_0$ with the following property: Let $\{z_m\}$ be a $\delta$-separated sequence with $\delta > \delta_0$ and let $T_\gamma : b^p_\alpha \to l^p$ be the associated linear operator as in (5.1). Then there is a bounded linear operator $S_\gamma : l^p \to b^p_\alpha$ such that $T_\gamma S_\gamma$ is the identity on $l^p$. In particular, $\{z_m\}$ is a $b^p_\alpha$-interpolating sequence of order $\gamma$.

Let $\gamma > 0$ and let $\{z_m\}$ be a sequence in $\mathbf{H}$. For $u \in \tilde{B}$, define
\[ \tilde{T}_\gamma u = (z_m^{\gamma} D^\gamma u(z_m)). \]
Then (2.10) implies the operator
\[ \tilde{T}_\gamma : \tilde{B} \to l^\infty \]
is bounded. If $\tilde{T}_\gamma (\tilde{B}) = l^\infty$, $\{z_m\}$ is called a $\tilde{B}$-interpolating sequence of order $\gamma$. Also, if $\tilde{T}_\gamma (\tilde{B}_0) = C_0$, $\{z_m\}$ is called a $\tilde{B}_0$-interpolating sequence of order $\gamma$. 

The following proposition shows that separation is also necessary for $\tilde{B}_0$ interpolation. Since we have Lemma 4.9, the proof of the following proposition is the same as that of Proposition 5.6 in [6].

**Proposition 5.5.** Let $\gamma > 0$. Every $\tilde{B}$-interpolating sequence of order $\gamma$ is separated. Also, every $\tilde{B}_0$-interpolating sequence of order $\gamma$ is separated.

**Theorem 5.6.** Let $\gamma > 0$. Then there exists a positive number $\delta_0$ with the following property: Let $\{z_m\}$ be a $\delta$-separated sequence with $\delta > \delta_0$ and let $\tilde{T}_\gamma : \tilde{B} \to l^\infty$ be the associated linear operator as in (5.2). Then there exists a bounded linear operator $\tilde{S}_\gamma : l^\infty \to \tilde{B}$ such that $\tilde{T}_\gamma \tilde{S}_\gamma$ is the identity on $l^\infty$. Moreover, $\tilde{S}_\gamma$ maps $C_0$ into $\tilde{B}_0$. In particular, $\{z_m\}$ is a both $\tilde{B}$-interpolating and $\tilde{B}_0$-interpolating sequence of order $\gamma$.

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