ON BOUNDED ANALYTIC FUNCTIONS ON
TWO-SHEETED COVERING SURFACES

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In this note, we pose some problems which is related to the algebras of
bounded analytic functions on two-sheeted covering surfaces \((\tilde{R}, R, \pi)\), where
the base domain \(R\) is a Zalcman domain (or an \(L\)-domain in the terminology
of [5]). In [5], L. Zalcman showed some theorems related to the algebra
\(H^\infty(R)\) of bounded analytic functions on a domain \(R\) of infinite connectivity.
Especially, the distinguished homomorphism is of our interest. We summarize
Zalcman’s results in §1.

For the covering surface \((\tilde{R}, R, \pi)\), the point separation problem was studied in [2] and [3]. We review this problem in §2.

1 Zalcman’s results

Let \(\Delta\) be the open unit disc and \(\Delta_0 = \{0 < |z| < 1\}\) the punctured unit disc.
Let \(\{c_n\}\) and \(\{r_n\}\) be sequences satisfying:

\[
\begin{align*}
1 > c_1 > c_2 > \cdots > 0, & \quad \lim_{n \to \infty} c_n = 0, \\
1 > r_1 > r_2 > \cdots > 0, & \quad \lim_{n \to \infty} r_n = 0, \\
c_{n+1} + r_{n+1} < c_n - r_n, & \quad c_1 + r_1 < 1.
\end{align*}
\]

These conditions simply say that closed discs \(\{\overline{\Delta}_n\}\) are contained in \(\Delta_0\), are
mutually disjoint, and accumulate only at the origin. Consider the domain

\[
R = \Delta_0 \setminus \bigcup_{n=1}^{\infty} \overline{\Delta}(c_n, r_n),
\]

which is a simplest example of bounded infinitely connected domains in the
complex plane \(\mathbb{C}\). We call a domain \(R\) of the form (2) a Zalcman domain.

Each \(f \in H^\infty(R)\) has nontangential boundary values at almost every point
of \(\Gamma = \partial R\). And the Cauchy integral formula holds;

\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in R.
\]

Let \(\mathcal{M} = \mathcal{M}(R)\) be the maximal ideal space of \(H^\infty(R)\), the set of all
non-zero multiplicative linear functionals on \(H^\infty(R)\). The topology of \(\mathcal{M}\) is
the weak-* topology which it inherits from $\mathcal{H}^\infty(R)^*$. With this topology, we can regard the functions in $\mathcal{H}^\infty(R)$ as continuous functions on $\mathcal{M}$ by setting $f(\varphi) = \varphi(f)$ ($\varphi \in \mathcal{M}$, $f \in \mathcal{H}^\infty(R)$). In particular, the coordinate function $z$ can be regarded as a continuous function on $\mathcal{M}$. And we have $z(\mathcal{M}) = \tilde{R}$. The set $\mathcal{M}_\zeta = z^{-1}(\{\zeta\})$ is called the fiber over $\zeta$ ($\zeta \in \tilde{R}$).

For $\zeta \in R$, $\mathcal{M}_\zeta = \{\varphi_\zeta\}$, where $\varphi_\zeta$ is the point evaluation homomorphism ($\varphi_\zeta(f) = f(\zeta)$). And, for $\zeta \in \Gamma \setminus \{0\}$, $\mathcal{M}_\zeta$ is homeomorphic to $\mathcal{M}_1(\Delta)$. So, we are interested in the fiber $\mathcal{M}_0$.

Suppose that the sequences $\{c_n\}$ and $\{r_n\}$ satisfy the condition

$$\sum_{n=1}^{\infty} \frac{r_n}{c_n} < \infty \quad (3)$$

in addition to (1). Then $d\zeta/\zeta$ is a finite measure on $\Gamma$. By Lebesgue’s theorem, we have that $\lim_{x \nearrow 0} f(x)$ exists for all $f \in \mathcal{H}^\infty(R)$. Set $\varphi_0(f) = \lim_{x \nearrow 0} f(x)$. Then we have

(i) $\varphi_0 \in \mathcal{M}_0$,

(ii) $\varphi_0$ does not lie in the Shilov boundary of $\mathcal{H}^\infty(R)$,

(iii) $\varphi_0$ lies in the same Gleason part as $R$.

The homomorphism $\varphi_0$ is called the distinguished homomorphism.

2 Covering surfaces

Let $(\tilde{\Delta}_0, \Delta_0, \pi)$ be the unlimited two-sheeted covering surface whose branch points are those over $\{c_n\}$ (Fig. 1). In 1949, Myrberg pointed out that $\mathcal{H}^\infty(\tilde{\Delta}_0) = \mathcal{H}^\infty(\Delta_0) \circ \pi$. This means that for any point $z \in \Delta_0 \setminus \{c_n\}$, the points of the fiber $\pi^{-1}(z) = \{z_+, z_-\}$ can not be separated by $\mathcal{H}^\infty(\Delta_0)$.

Myrberg’s proof goes as follows. Let $F \in \mathcal{H}^\infty(\Delta_0)$, and consider the function $f$ on $\Delta_0$ defined by $f(z) = (F(z_+) - F(z_-))^2$. Then $f \in \mathcal{H}^\infty(\Delta_0)$ and, by Riemann’s theorem, $f \in \mathcal{H}^\infty(\Delta)$. Since $f(c_n) = 0$ and $c_n \to 0$, we have $f \equiv 0$.

Restricting the base domain $\Delta_0$ of the covering surface to $R$, and setting $\tilde{R} = \pi^{-1}(R)$, we obtain the two-sheeted smooth covering surface $(\tilde{R}, R, \pi)$ (Fig. 2). In spite of complete lack of branch points, it is shown in [2] and [3] that non-separating phenomenon may occur for $(\tilde{R}, R, \pi)$ depending on $\{c_n\}$ and $\{r_n\}$. Roughly speaking,

(i) if $r_n \to 0$ “rapidly”, then $\mathcal{H}^\infty(\tilde{R}) = \mathcal{H}^\infty(R) \circ \pi$, 
(ii) if $r_n \to 0$ "slowly", then $H^\infty(\widetilde{R}) \supsetneq H^\infty(R) \circ \pi$.

(Unfortunately, the necessary and sufficient condition for $H^\infty(\widetilde{R}) = H^\infty(R) \circ \pi$ is not known.)

3 Problems

The covering surface $(\widetilde{R}, R, \pi)$ induces the covering space $(\widetilde{\mathcal{M}}, \mathcal{M}, \tau)$, where $\widetilde{\mathcal{M}}$ is the maximal ideal space of $H^\infty(\widetilde{R})$ and the map $\tau$ is defined by

$$\tau(\varphi)(f) = \varphi(f \circ \pi), \quad \varphi \in \widetilde{\mathcal{M}}, \ f \in H^\infty.$$
Let $\iota: R \to \mathcal{M}$ and $\bar{\iota}: \bar{R} \to \bar{\mathcal{M}}$ be natural maps. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\bar{R} & \xrightarrow{\bar{\iota}} & \bar{\mathcal{M}} \\
\pi & \downarrow & \tau \\
R & \xrightarrow{\iota} & \mathcal{M}
\end{array}
$$

By Nakai's theorem ([4]), we see that the map $\tau$ is surjective and the fiber $\tau^{-1}(\varphi)$ over any point $\varphi \in \mathcal{M}$ consists of at most two points, i.e., the number $\#(\tau^{-1}(\varphi))$ of points of the fiber $\tau^{-1}(\varphi)$ is 1 or 2 for all $\varphi \in \mathcal{M}$.

Consider the problem to determine $\#(\tau^{-1}(\varphi))$. The following partial answer is trivial.

**Proposition.** (i) If $H^\infty(\bar{R}) = H^\infty(R) \circ \pi$, then $\#(\tau^{-1}(\varphi)) = 1$ for all $\varphi \in \mathcal{M}$.

(ii) If $H^\infty(\bar{R}) \supsetneq H^\infty(R) \circ \pi$, then $\#(\tau^{-1}(\varphi_z)) = 2$ for all $z \in R$.

Now we pose some problems related to the fiber over the distinguished homomorphism.

3.1. Suppose that $H^\infty(\bar{R}) \supset H^\infty(R) \circ \pi$. Determine $\#(\tau^{-1}(\varphi_0))$.

The distinguished homomorphism was defined by $\varphi_0(f) = \lim_{x \nearrow} f(x)$ for $f \in H^\infty(R)$. In view of this, the following problem is posed.

3.2. Does $\lim_{x \nearrow 0} F(x_+)$ (or $\lim_{x \nearrow 0} F(x_-)$) exist for all $F \in H^\infty(\bar{R})$?

Note that $\lim_{x \nearrow 0}(F(x_+) + F(x_-))$ exists for all $F \in H^\infty(\bar{R})$ because $F(z_+) + F(z_-) \in H^\infty(R)$. Therefore, the existence of one of the limits in the above problem implies the existence of the other.

Set $J = [-1/2, 0)$. Then Zalcman's result can be restated as $\bar{J} = J \cup \{\varphi_0\}$ in $\mathcal{M}$. Related to this statement, the following problem is posed.

3.3 Let $\pi^{-1}(J) = J^+ \cup J^-$. ($J^+ = \pi^{-1}(J) \cap \Delta_+$, $J^- = \pi^{-1}(J) \cap \Delta_-$.) Determine the closures $\bar{J}^+$, $\bar{J}^-$ and $\bar{J}^+ \cup \bar{J}^-$ in $\bar{\mathcal{M}}$.

**References**


