CYCLIC VECTORS IN FOCK-TYPE SPACES

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1. Introduction

Let $\mathbb{D}$ be the open unit disk of the complex plane $\mathbb{C}$. We denote the polynomial ring by $\mathcal{C}$, and the space of all entire functions by $Hol(\mathbb{C})$. Let $X$ be a complete semi-normed space of holomorphic functions on a domain $\Omega$ in $\mathbb{C}$. For a subset $E$ of $X$, let $\overline{E}$ be the closure of $E$ in $X$. A function $f$ is said to be cyclic in $X$ if $f^C \subset X$ and $\overline{f^C} = X$.

In the Hardy spaces $H^p(\mathbb{D})$ ($0 < p < \infty$), it is well known that a function is cyclic if and only if it is $H^p(\mathbb{D})$-outer (see [Gar]). Also in the Bergman spaces $L^p_a(\mathbb{D})$ ($0 < p < \infty$), it is known that a function is cyclic if and only if it is $L^p_a(\mathbb{D})$-outer (see [HKZ]). Recently the author has characterized the cyclic vectors in the classical Fock space. The classical Fock space $L^2_a(\mathbb{C})$ is

$$L^2_a(\mathbb{C}) = \left\{ f \in Hol(\mathbb{C}) : \|f\|_{L^2_a(\mathbb{C})} = \left\{ \int_{\mathbb{C}} |f(z)|^2 d\mu(z) \right\}^{1/2} < \infty \right\}$$

where

$$d\mu(z) = e^{-|z|^2/2} dA(z)/2\pi$$

is the Gaussian measure on $\mathbb{C}$ and $dA$ is the ordinary Lebesgue measure. In [Izu1], we have proved the following:

**Theorem A**. Let $h(z) \in Hol(\mathbb{C})$. Then the following are equivalent:

(i) $f(z)$ is a nonvanishing function in $L^2_a(\mathbb{C})$.

(ii) $f(z) = e^{h(z)}$, $h(z) = \alpha z^2 + \beta z + \gamma$ for $\alpha, \beta, \gamma \in \mathbb{C}$, $|\alpha| < \frac{1}{4}$.

(iii) $f(z)$ is cyclic in $L^2_a(\mathbb{C})$.

It is known that there are non-vanishing functions in $H^p(\mathbb{D})$ and $L^p_a(\mathbb{D})$ which are not cyclic in the respective spaces (see [Gar] and [HKZ]). In fact, Brown and Shields posed the following question [BS]:

**Question B**. Let $\Omega$ be bounded region in $\mathbb{C}$. Does there exist a polynomially dense Banach space $X$ of analytic functions in $\Omega$ with the two properties
(i) $zX \subset X$
(ii) for any $\lambda \in \Omega$, point evaluation functional for $\lambda$ is bounded,
in which a function $f(z)$ is cyclic if and only if $f(z) \neq 0$ for all $z \in \Omega$?

The above theorem is not the answer of this question. But it says
that there exists a polynomially dense Banach space in which every
non-vanishing function is cyclic.

In this paper, we consider the cyclic vectors in more generalized
spaces.

Let $0 < p < \infty$, $s > 0$ and $\alpha > 0$. Let $\phi$ be a positive function
on $[0, \infty)$. The space $L^p_\alpha(\mathbb{C}, \phi)$ consists of those entire functions whose
semi-norm

$$||f||_{L^p_\alpha(\mathbb{C}, \phi)} = \left\{ \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} dA(z) \right\}^{1/p}$$

is finite. This space is called Fock-type space. Throughout this paper,
we put $\phi(|z|) = \frac{\alpha}{p} |z|^s$. We study the cyclic vectors in $L^p_\alpha(\mathbb{C}, \phi)$.

This is a summary of the paper [Izu2].

2. Results

The following is our main result:

Theorem 1. Let $f$ be a function in $L^p_\alpha(\mathbb{C}, \phi)$ satisfying $fC \subset L^p_\alpha(\mathbb{C}, \phi)$. Then the following are equivalent:

(i) $f(z)$ is a non-vanishing function.
(ii) $f(z) = e^{h(z)}$ for $h(z) = \sum_{k=0}^\lceil s \rceil a_k z^k$, $a_k \in \mathbb{C}$, where $\lceil s \rceil$ is the
    largest integer with $\lceil s \rceil \leq s$, and in addition $|a_s| < \frac{\alpha}{p}$ if $s$ is an
    integer.
(iii) $f(z)$ is cyclic in $L^p_\alpha(\mathbb{C}, \phi)$.

We know that every non-vanishing function in the classical Fock
space $L^2(\mathbb{C})$ is cyclic. In our case, we notice that it is not valid for
some positive numbers; that is, if $s$ is not an integer or $s = 1, 2, 3, 4$,
then $L^p_\alpha(\mathbb{C}, \phi)$ has the same property as the one in $L^2(\mathbb{C})$, but if $s =
5, 6, 7, \cdots$, the situation is different. For example, although $f(z) = e^{\frac{\alpha}{p} z^s}$
is a non-vanishing function in $L^p_\alpha(\mathbb{C}, \phi)$, the function $f(z)$ does not satisfy
$fC \subset L^p_\alpha(\mathbb{C}, \phi)$. Obviously this function $f(z)$ is not cyclic. But if
we consider the non-vanishing functions just satisfying $fC \subset L^p_\alpha(\mathbb{C}, \phi)$,
then the situation is similar.
To prove Theorem 1, we introduce the space $F^p_\phi$ which is studied in [MMO]. The space is

$$F^p_\phi = \left\{ f \in Hol(\mathbb{C}) : \|f\|_{F^p_\phi}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} \rho^{-1} \Delta \phi dA(z) < \infty \right\}$$

where $\Delta \phi$ is the Laplacian of $\phi$ and $\rho^{-1} \Delta \phi$ is a regular version of $\Delta \phi$. If $p = 2$, then $F^2_\phi$ is a Hilbert space with inner product

$$\langle f, g \rangle_{F^2_\phi} = \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-2\phi(z)} \rho^{-1} \Delta \phi dA(z).$$

We denote the reproducing kernel of $F^2_\phi$ by $K_\lambda$, $\lambda \in \mathbb{C}$. The following lemma is proved by Marco, Massaneda and Ortega-Cerdà in [MMO, Lemma 21].

**Lemma 2.** There exists a positive number $C$ such that for any $\lambda \in \mathbb{C}$

$$C^{-1} e^{2\phi(\lambda)} \leq \|K_\lambda\|^2_{F^2_\phi} \leq Ce^{2\phi(\lambda)}.$$

In [CGH], Chen, Guo and Hou proved the following:

**Lemma 3.**

$$\lim_{|\lambda| \to \infty} \frac{\langle f, K_\lambda \rangle_{F^2_\phi}}{\|K_\lambda\|^2_{F^2_\phi}} = 0$$

for any $f \in F^2_\phi$.

By Lemma 2 and 3, we get the following:

**Lemma 4.** The following are equivalent:

(i) $f(z) \in L^p_a(\mathbb{C}, \phi)$ is a non-vanishing function satisfying $fC \subset L^p_\alpha(\mathbb{C}, \phi)$.

(ii) $f(z) = e^{h(z)}$ where $h(z) = \sum_{k=0}^{[s]} a_k z^k$ and in addition $|a_s| < \frac{s\alpha}{p}$ if $s$ is an integer.

By Lemma 4, (i) $\iff$ (ii) in Theorem 1 has been proved.

The following two lemmas are the generalizations of the results in [GW].

**Lemma 5.** Let $f \in L^p_\alpha(\mathbb{C}, \phi)$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then we have the following:

(i) There exists a constant $C_1 > 0$, which depends on $f$, satisfying

$$|c_n| \leq C_1 e^{2s - \frac{\alpha}{p_n}} \left( \frac{\frac{s\alpha e}{pn + 2 - s}}{n} \right)^n \|f\|_{L^p_\alpha(\mathbb{C}, \phi)}.$$
For large $n$,

\[
\|z^n\|_{L^p_\alpha(C, \phi)}^p = \frac{\alpha^{-\frac{pn+2}{s}}}{s} \Gamma\left(\frac{pn+2}{s}\right)
\sim \frac{1}{s\alpha} \left(2\pi\frac{pn+2-s}{s}\right)^{\frac{1}{2}} \left(\frac{pn+2-s}{s\alpha e}\right)^{\epsilon_{\frac{7\iota+2-s}{s}}},
\]

where \(\Gamma\) denotes the gamma function.

There is a constant \(C_2 > 0\), which depends on \(f\), satisfying

\[
\|c_n z^n\|_{L^p_\alpha(C, \phi)} \leq C_2 \left(\frac{pn+2-s}{s}\right)^{\frac{1}{2p}} \left(\frac{pn+2-s}{s\alpha}\right)^{\frac{2-s}{ps}} \|f\|_{L^p_\alpha(C, \phi)}.
\]

Using Lemma 5, we get the following lemma:

**Lemma 6.** The polynomial ring \(\mathcal{C}\) is dense in \(L^p_\alpha(C, \phi)\).

Finally we show (ii) \(\Leftrightarrow\) (iii) in Theorem 1. Since every cyclic vector is non-vanishing, (iii) \(\Rightarrow\) (ii) is trivial. The idea for proving the opposite direction is from [Izu1].

**References**


