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Extreme points of the unit ball of the algebra generated by composition operators

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Abstract

We study the extreme points of the unit ball of the algebra generated by composition operators on the disk algebra.

1 Introduction

Let $\mathbb{D}$ be the open unit disk. We denote by $\overline{\mathbb{D}}$ its closure and by $\partial \mathbb{D}$ its boundary. Let $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ be the set of all analytic self-map of $\mathbb{D}$. Every analytic self-map $\varphi \in S(\mathbb{D})$ the composition operator $C_{\varphi}$ on $H(\mathbb{D})$ defined by

$$C_{\varphi}f(z) = f(\varphi(z)).$$

Let $H^{\infty}$ be the set of all bounded analytic functions on $\mathbb{D}$. Then $H^{\infty}$ is a Banach algebra with the supremum norm,

$$\|f\|_{\infty} = \sup_{z \in \mathbb{D}} |f(z)|.$$

Every composition operator is bounded on $H^{\infty}$ and $\|C_{\varphi}\| = 1$. It is known that $C_{\varphi}$ is compact on $H^{\infty}$ if and only if $\|\varphi\|_{\infty} < 1$.

Recall that the disk algebra $A$ is the Banach algebra of all functions analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ with the supremum norm. To define $C_{\varphi}$ on $A$, we need the condition $C_{\varphi}z = \varphi \in A$. Denote by $S(\overline{\mathbb{D}})$ the closed unit ball of $A$. Then every $\varphi \in S(\overline{\mathbb{D}})$ induces $C_{\varphi}$ which acts on $A$. If $\varphi$ is a constant function with value $\omega \in \partial \mathbb{D}$, then $\varphi$ is not in $S(\mathbb{D})$ but in $S(\overline{\mathbb{D}})$. We denote that $T = \{\varphi \equiv \omega \in \partial \mathbb{D}\}$. By the maximum modulus principle, it is shown that $S(\overline{\mathbb{D}}) \setminus T = S(\mathbb{D}) \cap A$. Similarly to the case of $H^{\infty}$, we can see that $\|C_{\varphi}\|_{A} = 1$ for every $\varphi \in S(\overline{\mathbb{D}})$ and $C_{\varphi}$ is compact on $A$ if and only if $\|\varphi\|_{\infty} < 1$ or $\varphi \equiv e^{i\theta}$. 
Let $\mathcal{X}$ be an analytic functional Banach space on $\mathbb{D}$, that is, each element is analytic on $\mathbb{D}$ and the evaluation at each point of $\mathbb{D}$ is a non-zero bounded linear functional on $\mathcal{X}$. Let $C(\mathcal{X})$ be the collection of all bounded composition operators on $\mathcal{X}$, endowed with the operator norm topology. Originally this topic was posed for the case of $C(H^2)$ by Shapiro and Sundberg in [7]. They raised the following three problems: (i) Characterize the path components of $C(H^2)$. (ii) Which composition operators are isolated in $C(H^2)$? (iii) Which differences of composition operators are compact on $H^2$? These problems are still open. In [6], MacCluer, Ohno and Zhao solved (i) and (ii) of the problems above for $C(H^\infty)$.

Their results was descrived by the terms of the pseudo-hyperbolic distance on $\mathbb{D}$. For $p \in \mathbb{D}$, let $\alpha_p$ be the automorphism of $\mathbb{D}$ exchanging 0 for $p$. Then $\alpha_p$ has the following form:

$$\alpha_p(z) = \frac{p - z}{1 - \overline{p}z}.$$ 

The pseudo-hyperbolic distance $\rho(z, w)$ between $z$ and $w$ in $\mathbb{D}$ is defined by

$$\rho(z, w) = |\alpha_z(w)| = \left| \frac{z - w}{1 - \overline{z}w} \right|.$$ 

Here we define the induced distance $d_\rho$ on $S(\mathbb{D})$, that is, 

$$d_\rho(\varphi, \psi) = \sup_{z \in \mathbb{D}} \rho(\varphi(z), \psi(z))$$

for $\varphi$ and $\psi$ in $S(\mathbb{D})$. In [6] the operator norms of the differences of composition operators on $H^\infty$ are estimated as following;

$$\|C_\varphi - C_\psi\| = \frac{2 - 2\sqrt{1 - d_\rho(\varphi, \psi)^2}}{d_\rho(\varphi, \psi)}. \quad (1)$$

Hence $C(H^\infty)$ can be identified with the space $S(\mathbb{D}, d_\rho)$. We denote $C_\varphi \sim_{\mathcal{X}} C_\psi$ if they are in the same component of $C(\mathcal{X})$. In [6], it is proved that $C_\varphi \sim_{H^\infty} C_\psi$ if and only if $d_\rho(\varphi, \psi) < 1$.

Let $\mathcal{Y}$ be a convex subset of a locally convex space. We recall that an element $y$ of $\mathcal{Y}$ is called an extreme point of $\mathcal{Y}$ if the conditions $0 < r < 1$, $y_1, y_2 \in \mathcal{Y}$ and $y = (1-r)y_1 + ry_2$, implies that $y_1 = y_2 = y$. For a normed space $\mathcal{Z}$, we denote by $U_\mathcal{Z}$ the cloed unit ball of $\mathcal{Z}$. By Rudin-de Leeuw's Theorem([4, Ch.9]), $\varphi$ is an extreme point of $U_{H^\infty}$ if and only if

$$\int_0^{2\pi} \log(1 - |\varphi(e^{i\theta})|)d\theta = -\infty. \quad (2)$$
MacCluer, Ohno and Zhao proved that if $C_\varphi$ is isolated in $\mathcal{C}(H^\infty)$, then $\varphi$ is an extreme point of $U_{H^\infty}$. In [5], the converse was proved. We remark that the connected components of $\mathcal{C}(H^\infty)$ are characterized by a equivalence relation which is in the similar form of the Gleason parts of the maximal ideal space of $H^\infty$. In this sense, the isolated points of $\mathcal{C}(H^\infty)$ corresponds to the single Gleason parts.

The topological structure of $\mathcal{C}(A)$ is similar to that of $\mathcal{C}(H^\infty)$. To introduce such results, we extend the pseudo-hyperbolic distance to $\overline{\mathbb{D}}$ as following; For $z \in \partial \mathbb{D}$ and $w \in \overline{\mathbb{D}}$ such that $z \neq w$, define that $\rho(z, z) = 0$ and $\rho(z, w) = 1$. Hence the induced distance $d_\rho$ is defined on $S(\overline{\mathbb{D}})$. We remark that $\varphi$ is extreme point of the closed unit ball $S(\overline{\mathbb{D}})$ of $A$ if and only if the condition (2) holds (see [4, p. 139]). We denote that $\mathcal{K} = \{C_\varphi \text{ is compact on } A\}$ and $\Delta = \{C_\varphi \in \mathcal{C}(A) : \varphi \equiv \omega \in \partial \mathbb{D}\}$. Now the results on the topological structure of $\mathcal{C}(H^\infty)$ can be applied on $\mathcal{C}(A)$ by the similar proof in [5] and [6].

**Theorem 1.1** Let $C_\varphi, C_\psi$ be in $\mathcal{C}(A)$. Then

(i) $\|C_\varphi - C_\psi\|_A = \frac{2 - 2\sqrt{1 - d_\rho(\varphi, \psi)^2}}{d_\rho(\varphi, \psi)}$.

(ii) $C_\varphi \sim_A C_\psi$ if and only if $\|C_\varphi - C_\psi\|_A < 2$.

(iii) The following are equivalent:

(a) $C_\varphi$ is isolated in $\mathcal{C}(A)$.

(b) For all $C_\psi \neq C_\varphi$, $\|C_\varphi - C_\psi\|_A = 2$.

(c) $\varphi$ is an extreme point of the closed unit ball of $A$.

(d) $\int_0^{2\pi} \log(1 - |\varphi(e^{i\theta})|) \, d\theta = -\infty$.

(iv) Every $C_\varphi \in \Delta$ is compact on $A$ and isolated in $\mathcal{C}(A)$.

(v) $\mathcal{K} \setminus \Delta$ is a component of $\mathcal{C}(A)$.

Denote by $\text{Comp}_\mathcal{X}(\varphi)$ the path component of $\mathcal{C}(\mathcal{X})$ which contains $C_\varphi$. Then we can immediately get the following corollary, which mentions the relation between the topological structure of $\mathcal{C}(A)$ and that of $\mathcal{C}(H^\infty)$.
Corollary 1.2 Let $C_\varphi$ and $C_\psi$ be in $C(A) \setminus \Delta$. Then we have the following.

(i) $\text{Comp}_A(\varphi) = \text{Comp}_{H^\infty}(\varphi) \cap C(A)$.

(ii) $C_\varphi \sim C_\psi$ in $C(A)$ if and only if $C_\varphi \sim C_\psi$ in $C(H^\infty)$.

(iii) $C_\varphi$ is isolated in $C(A)$ if and only if $C_\varphi$ is isolated in $C(H^\infty)$.

In general, $C(\mathcal{X})$ is a semigroup with respect to the products, but the finite linear combinations of composition operators are not in $C(\mathcal{X})$. We denote by $\langle C(\mathcal{X}) \rangle$ the collection of all finite linear combinations of composition operators on $\mathcal{X}$. Let $\mathcal{L}(\mathcal{X})$ denote the operator norm closure of $\langle C(\mathcal{X}) \rangle$. In the next section, we investigate the relation between the isolated points of $C(A)$ and the extreme points of $U_{\mathcal{L}(A)}$. Our main result states that $C_\varphi$ is a extreme point of $\mathcal{L}(A)$ if and only if $C_\varphi$ is a isolated point of $C(A)$.

2 Extreme point of $U_{\mathcal{L}(A)}$

At first, we observe that composition operators are linearly independent each other in $\langle C(\mathcal{X}) \rangle$.

Proposition 2.1 Let $\varphi_1, \cdots, \varphi_n$ be the distinct analytic maps of $S(\overline{D})$ and let $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. If $\lambda_1 C_{\varphi_1} + \cdots + \lambda_n C_{\varphi_n}$ is the zero operator on $A$, then $\lambda_1 = \cdots = \lambda_n = 0$.

In [3], Gorkin and Mortini investigated the norms and essential norms of finite linear combinations of composition operators. They also proved that $\langle C(A) \rangle$ is not closed and the multiplication operator $M_z$ is not contained in $\mathcal{L}(A)$. Here we will construct an example of elements of $\mathcal{L}(A) \setminus \langle C(A) \rangle$. For a continuous curve $\{C_{\varphi_t}\}_{t \in [0,1]}$ in $C(A)$, we define that

$$T_n = \sum_{k=1}^{n} \frac{1}{n} C_{\varphi_{\frac{k}{n}}}.$$

Then $\|T_n\| = 1$. For $f \in A$ and $p \in D$, we have that

$$T_n f(p) = \sum_{k=1}^{n} \frac{1}{n} f(\varphi_{\frac{k}{n}}(p)) \to \int_{0}^{1} f(\varphi_t(p)) dt$$

as $n \to \infty$. Since $\{T_n f\}$ is Cauchy sequence in $A$, we have that

$$\int_{0}^{1} f(\varphi_t(z)) dt \in H^\infty.$$
Here we denote by $I_{\varphi_{t}}$ the following integral operator:

$$I_{\varphi_{t}}f(z) = \int_{0}^{1} f(\varphi_{t}(z))dt.$$  \hspace{1cm} (3)

Then the Banach-Steinhaus Theorem implies the following lemma.

**Lemma 2.2** If $\{C_{\varphi_{t}}\}_{t \in [0,1]}$ is a continuous curve in $C(A)$, then the corresponding integral operator $I_{\varphi_{t}}$ is in $U_{\mathcal{L}(A)}$.

**Example 2.3**

(i) Suppose that $C_{\varphi} \sim_{A} C_{\psi}$. Put $\varphi_{t} = (1-t)\varphi + t\psi$. Then $\{C_{\varphi_{t}}\}_{t \in [0,1]}$ is a continuous curve in $C(H^\infty)$ (see [6]) and

$$I_{\varphi_{t}}f(z) = \frac{F(\psi(z)) - F(\varphi(z))}{\psi(z) - \varphi(z)}$$

where $F(z)$ is the primitive function of $f(z)$.

(ii) Suppose that $||\varphi||_{\infty} < 1$. Choose a positive number $r$ such that $r < 1 - ||\varphi||_{\infty}$. We define that $\varphi_{t}(z) = \varphi(z) + re^{2\pi it}z$. Then $||\varphi_{t}||_{\infty} < 1$ for all $t$. Since every $\varphi_{t}(D)$ is a compact subset of $D$, $d_{p}(\varphi_{s}, \varphi_{t}) \to 0$ as $s \to t$. Thus $\{C_{\varphi_{t}}\}_{t \in [0,1]}$ is a closed continuous curve in $C(H^\infty)$. By the Cauchy's Formula, we have that $I_{\varphi_{t}} = C_{\varphi}$.

We remark that the condition $||\varphi||_{\infty} < 1$ induces that $C_{\varphi}$ is not an extreme point of $U_{\mathcal{L}(A)}$. From (ii) of Example 2.3, we have that, for $f \in A$ and $p \in D$,

$$C_{\varphi}f(p) = \int_{0}^{\frac{1}{2}} f(\varphi(p) + rp e^{2\pi it}) dt + \int_{\frac{1}{2}}^{1} f(\varphi(p) + rp e^{2\pi it}) dt$$

Let $\sigma_{t}(z) = \varphi(z) + re^{\pi it}z$ and $\tau_{t}(z) = \varphi(z) - re^{\pi it}z$. By changing variables,

$$C_{\varphi} = \frac{1}{2} I_{\sigma_{t}} + \frac{1}{2} I_{\tau_{t}}.$$  \hspace{1cm} (4)

Since $I_{\sigma_{t}} \neq I_{\tau_{t}}$, we can conclude that $C_{\varphi}$ is not an extreme point. Then we have the following.

**Proposition 2.4** If $C_{\varphi}$ is compact on $A$, then $C_{\varphi}$ is not an extreme point of $U_{\mathcal{L}(A)}$.

Here we state our main result.

**Theorem 2.5** $C_{\varphi}$ is an extreme point of $U_{\mathcal{L}(A)}$ if and only if $C_{\varphi}$ is an isolated point of $C(A)$.

We remark that the same proof of the "only if" part can be applied to $\mathcal{L}(H^{\infty})$. We here present two problems.

**Problem**

(i) Can Theorem 2.5 be applied to $\mathcal{L}(H^{\infty})$?

(ii) Is there other extreme point of the closed unit ball of $\mathcal{L}(A)$?
References


