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On a problem about the Shilov boundary of a Riemann surface

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1 Notations and a problem

[1]

Let R be a Riemann surface and let $H^{\infty}(R)$ be the algebra of all bounded analytic functions on R with sup-norm $\|f\|_{\infty}=\|f\|_{R}=\sup_{p\in R}|f(p)|$.

The **maximal ideal space** $\mathscr{M}(R)$ of $H^{\infty}(R)$ is the set of all nonzero continuous homomrphisms of $H^{\infty}(R)$ to the complex field $\mathbf{C}$. The **Gelfand transform** $\hat{f}$ of $f\in H^{\infty}(R)$ is a function on $\mathscr{M}(R)$ defined by $\hat{f}(\phi)=\phi(f)$ for $\phi\in\mathscr{M}(R)$. The maximal ideal space $\mathscr{M}(R)$ is a compact Hausdorff space with respect to the **Gelfand toplogy**, the weakest topology among toplogies such that every Gelfand transform $\hat{f}$ is to be continuous on $\mathscr{M}(R)$.

A closed subset E of $\mathscr{M}(R)$ is called a **boundary** for $H^{\infty}(R)$ if it statisfies $\|\hat{f}\|_{E}=\max_{p\in E}|\hat{f}(p)|=\|f\|_{R}$ for all $f\in H^{\infty}(R)$. The smallest boundary, denoted by $Ш(R)$, for

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$H^\infty(R)$ exists and is called the **Shilov boundary** of $H^\infty(R)$.

Theorem A (Gamelin[1, 2]) *If D is a domain in the complex plane, then the Shilov boundary $\text{Ш}(D)$ of $H^\infty(D)$ is extremely disconnected.*

It is natural to ask whether the same conclusion remains true for arbtrary Riemann surfaces(cf. [6]). Namely,

Problem *For any Riemann surface R, is the Shilov boundary $\text{Ш}(R)$ extremely disconnected?*

In order to avoid a triviality, one may only consider the case that Riemann surface R admits a nonconstant bounded analytic function; for, otherwise, the Shilov boundary is singleton.

At present we have no counter examples. In this note, we shall give a partial result.

A point evaluation homomorphism ϕ_p at $p \in R$, defined by $\phi_p(f) = f(p)$ for $f \in H^\infty(R)$, is an element of $\mathscr{M}(R)$. This induces a natural continuous map from R into $\mathscr{M}(R)$. While this natural map may not be injective in general, we often identify R with its image in $\mathscr{M}(R)$ and regard R as a subset of $\mathscr{M}(R)$. With this convention, Gelfand transform $\hat{f}$ can be regarded as a continuous extension of f.

The proof of Theorem A is based on the following simple fact; function $1/(z-p)$ of z has simple pole at p and bounded off any neighborhood of the point p. From this fact it follows that D is homeomorphically imbedded as an open subset in $\mathscr{M}(D)$.

Let $\mathscr{P}_s(R)$ be the set of points $p \in R$ such that there exist a meromorphic function g on R with the following properties: (i) g has a simple pole at p, and (ii) g is bounded on $R \setminus U_p$ for any neighborhood U_p of p.

Theorem B ([5]) *Let R be a Riemann surface such that $H^\infty(R)$ contains a nonconstant function. Then, a point $p \in R$ belongs to the set $\mathscr{P}_s(R)$ if and only if p has a neighborhood which is homeomorphically imbedded as an open subset in $\mathscr{M}(R)$.*

The 'only if' part is easy to see. From this easy part of the theorem one can extend Theorem A to those Riemann surfaces R under the condition $\mathscr{P}_s(R) = R$, whose proof goes in a similar way as Gamelin's method(cf. [4]).

In this note we consider the case that $\mathscr{P}_s(R)$ is a proper subset of R.

2 A preliminary observatrion

In this section we introduce an example of a Riemann surface. First we recall one of the examples constructed in [5]; Let $\Delta = \{z : |z| < 1\}$ be the open unit disc, and set

$$\begin{aligned}
\Delta_k &= \Delta \quad (k = 0, 1, 2, \dots) \\
J_k &= [a_k, b_k], \quad 0 < a_1 < b_1 < a_2 < b_2 < \cdots, \quad a_k \uparrow 1 \\
I_k &= \cup_{j=1}^{n_k} [a_{kj}, b_{kj}], \quad a_1 = a_{k1} < b_{k1} < \cdots < a_{kn_k} < b_{kn_k} = b_k \\
&\quad (n_k \text{ are sufficiently large}) \\
D_0 &= \Delta_0, \quad D_k = \Delta_k \setminus \cup_{\ell=1}^{k-1} J_\ell \quad (k \geqq 1)
\end{aligned}$$

Let W be the Reimann surface obtained by connecting two sides of intervals I_k in the sheet $D_k \setminus I_k$ $(k \geqq 1)$ with the corresponding two sides in the bottom sheet $D_0 \setminus I_k$ crosswisely. If we choose integers n_k sufficiently large, then the sheets D_k converges to the bottom sheet D_0 in the maximal ideal space $\mathscr{M}(W)$ as $k \to \infty$, and we have

$$\mathscr{P}_s(W) = \cup_{k=1}^{\infty}(D_k \setminus I_k)$$

Let us consider the following subdomain W' of W:

$$\begin{aligned} \Delta'_k =& \Delta' = \{z : |z + \frac{1}{2}| \leqq \frac{1}{4}\} \quad (k \geqq 0) \\ D'_k =& D_k \setminus \Delta'_k \quad (k \geqq 0) \\ W' =& W \setminus \cup_{k=1}^{\infty} \Delta'_k \end{aligned}$$

Incresing the number n_k of subintervals forming I_k, if necessary, we may further assume that the sheets D'_k converges to $D_0 \setminus \Delta'_0$ in the maximal ideal space $\mathscr{M}(W')$ as $k \to \infty$, and we have

$$\mathscr{P}_s(W') = \Delta'_0 \cup (\cup_{k=1}^{\infty}(D'_k \setminus I_k))$$

The restrinction $\tau(f) = f|W'$ is an algebra homomorphism of $H^\infty(W)$ to $H^\infty(W')$, which induces a natural continuous map $\hat{\tau} : \mathscr{M}(W') \to H^\infty(W)$. For $k \geqq 1$ set

$$\Gamma_k = \hat{\tau}^{-1}(\partial \Delta'_k),$$

which is homeomorphic to $\mathscr{M}(\Delta) \setminus \Delta$.

Since the sheets D'_k converges to the subdomain D'_0 of the bottom sheet D_0, one might expect that Γ_k converges to a compact subset, $\partial\Delta'_0$, of the bottom sheet. If this would be true, then the circle $\partial\Delta'_0$ should be a part of the Shilov boundary $Ш(W')$ and we would have a counter example to the Problem.

This expectation is false. Namely,

2.1 Theorem *The closure of* $\cup_{k\geqq 1}\Gamma_k$ *in* $\mathscr{M}(W')$ *is disjoint from the bottom sheet* D_0.

Proof: By [3, Theorem 4.1], we have a Cauchy differential

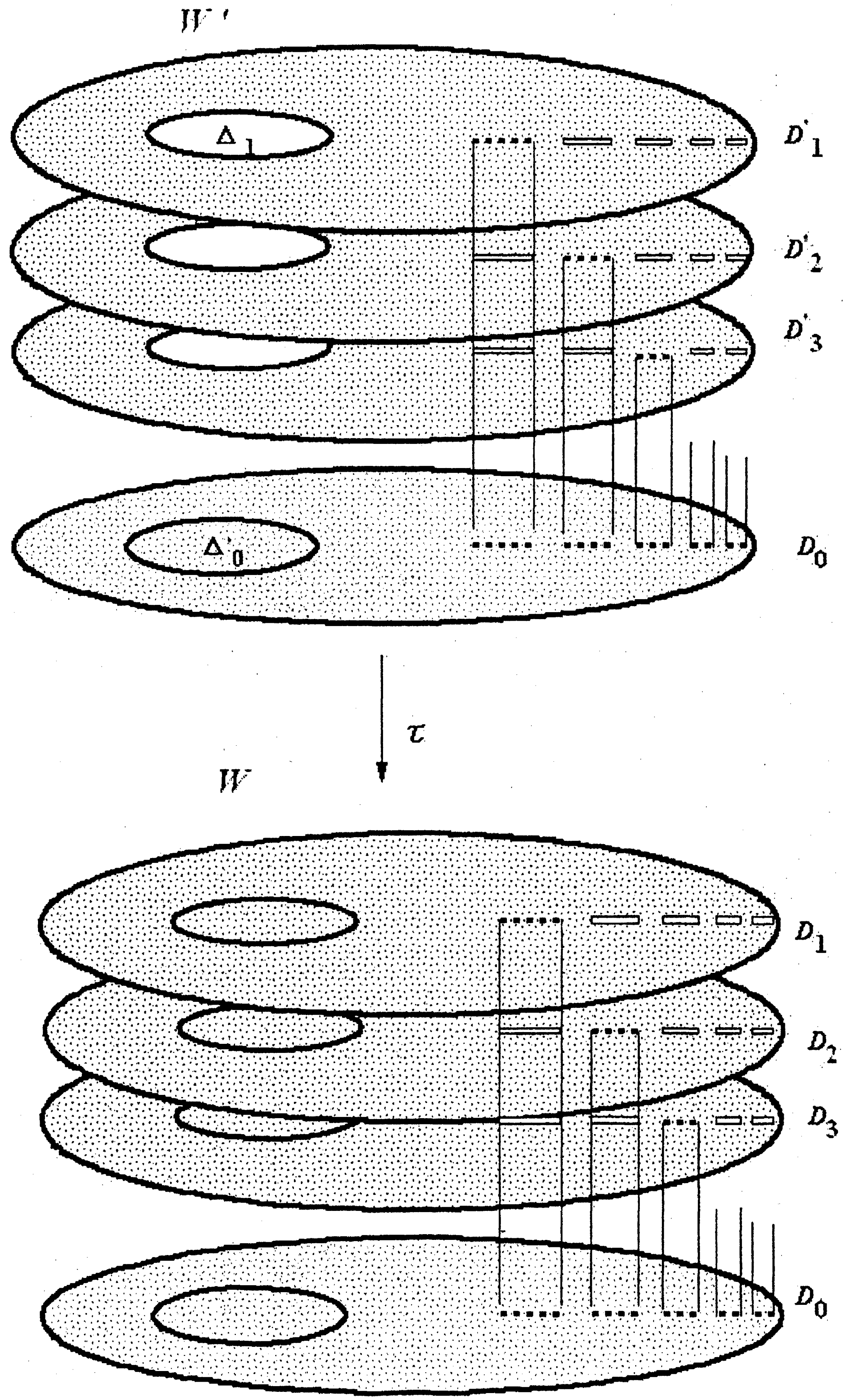
$$\omega(\zeta, z)d\zeta = \left\{\frac{1}{\zeta - z} + \eta(\zeta, z)\right\} d\zeta$$

on $\mathscr{P}_s(W) \times W$ such that the analyitc part $\eta(\zeta.z)$ is bounded on $U \times W$ whenever U is a relatively compact coordinate disc in $\mathscr{P}_s(W)$. Let $0 < \delta < \frac{1}{4}$. Set $f_k(z) = \left(\frac{1}{4z+2}\right)^{m_k}$ on the sheet D_k for a positive interger M_K. On the annulus $\{z \in D_k : \frac{1}{4} - \delta < |z + \frac{1}{2}| < \frac{1}{4} + \delta\}$, we have

$$\begin{aligned} f_k(z) &= \frac{1}{2\pi i}\left(\int_{|\zeta+\frac{1}{2}|=\frac{1}{4}+\delta} - \int_{|\zeta+\frac{1}{2}|=\frac{1}{4}}\right) f_k(\zeta)\omega(\zeta, z)d\zeta \\ &= h_k(z) - g_k(z). \end{aligned}$$

Choosing m_k large enough, we have $|g_k| \leqq \varepsilon_k$ on $W\backslash\{z \in D'_k : |z + \frac{1}{2}| \leqq \frac{1}{4} + \delta\}$ and $|h_k| < 2^{-k-1}$ on Δ'_k. Set $G = \sum_{k\geqq 1} g_k$. Since $g_k = h_k - f_k$, it follows that $\frac{1}{2} = 1 - \sum_{k\geqq 1} 2^{-k-1} \leqq |G| \leqq 1 + \sum_{k\geqq 1} 2^{-k-1} = \frac{3}{2}$ on each $\partial\Delta'_\ell$. Hence, $G \in H^\infty(W')$, and $|G| < \sum_{k\geqq 1} 2^{-k-1} = \frac{1}{2}$ on the bottom sheet D_0. This proves the theorem. □

In the remaing part of the note, we shall prove, moreover, that the Shilov boundaries Ш(W) and Ш(W') are both extremely disconnected.



3 Main theorem

3.1 Theorem *Let R be a Riemann surface and let $\{Q_k\}$ be the connected componets of $\mathscr{P}_s(R)$. Suppose that*

$$\mathscr{P}_s(R) \text{ is a dense subset of } R \text{ in } \mathscr{M}(R) \tag{3.1}$$

and that

$$\text{each } Q_k \text{ contains a point } q_k \text{ such that } \sup_k |f(q_k)| < \|f\|_R \text{ for every nonconstant } f \in H^\infty(R). \tag{3.2}$$

Then, Ш(R) is extremely disconnected.

The algebra $H^\infty(R)$ is said to be **weakly separating** (the points of R) if for each pair distinct points p, q of R there is a pair of nonzero functions f, g of $H^\infty(R)$ such that $\frac{f}{g}(p) \neq \frac{f}{g}(q)$.

For the proof we may assume that R is weakly separating. In fact, if $\tilde{R}$ is the Royden's resulution of a Riemann surface R with respect to the algebra $H^\infty(R)$, then

(a) $H^\infty(\tilde{R})$ is weakly separating;

(b) $H^\infty(\tilde{R})$ is algebraically isomorphic with $H^\infty(R)$, more precisely, there exists an analyitc map ρ of R to $\tilde{R}$ such that $H^\infty(R) = \{\tilde{f} \circ \rho; \tilde{f} \in \tilde{H}^\infty(\tilde{R})\}$;

(c) $\tilde{R}$ is $H^\infty(\tilde{R})$-maximal, namely, if W is a Riemann surface containing a proper subdomain being conformally equivalent to $\tilde{R}$, then some elements in $H^\infty(\tilde{R})$ can not be analytically extended to whole W;

(d) the Royden's resulution of $(R, H^\infty(R))$ is uniquely determined up to confomal equivalence by propeties (a), (b) and (c).

By (b), two Banach algebras $H^\infty(R)$ and $H^\infty(\tilde{R})$ are isomtrically isomorphic, that is, $\|\tilde{f}\circ\rho\|_\infty = \|\tilde{f}\|_\infty$. Trivially, $\rho(\mathscr{P}_s(R)) \subset \mathscr{P}_s(\tilde{R})$. Moreover, we have $\mathscr{P}_s(\tilde{R}) = \mathscr{P}(\tilde{R})$ ([5]), where $\mathscr{P}(R)$ denote the **pole set** which consists of the points $p \in R$ at which a meromorphic function g on R, bounded off a compact subset of R, has a ploe. Therefore, it suffices to show the theorem for $\tilde{R}$ in place of R.

To prove the theorem, we can use the same idea due to Gamelin ([1, 2]), where we need some modifications. One is needed because the pole set $\mathscr{P}(R)$ is not be connected and consists of infinitely many connected component. Another difficulty is that we only have local coordinate for a Riemann surface instead of a global coordinate z for the complex plane.

4 Outline of the proof

We assume that $H^\infty(R)$ is weakly separating. For $p \in \mathscr{P}(R)$, we denote by M_p^∞ the set of meromorphic functions with a simple pole at p and bounded off any neighborhood of p. For a closed subset E of $\mathscr{M}(R)$, we set $\hat{E} = \{\phi \in \mathscr{M}(R) : |\hat{f}(\phi)| \leqq \|\hat{f}\|_E, f \in H^\infty(R)\}$, called the H^∞-convex hull of E, and denote by H_E^∞ the closure of $H^\infty(R)$ with respect to the uniform norm for E. Let $M^\infty(R)$ be the set of meromorphic functions on R which are bounded off a compact subset of R. It is known that each $g \in M^\infty(R)$ has uniquely defines a continuous map $\hat{g}$ of $\mathscr{M}(R)$ to the Riemann sphere such that $\hat{g}$ agrees with g on R (regarded as a subset of $\mathscr{M}(R)$) and such that $\hat{f}\hat{g} = \widehat{fg}$ on $\mathscr{M}(R) \setminus \{\text{poles of } g\}$ whenever fg belongs to $H^\infty(R)$. For the simplicity of notations, we may identify function g on R with function $\hat{g}$ on $\mathscr{M}(R)$.

The following two lemmas can be prove if one use meromorphic functions in M_p^∞ in place of $1/(z-p)$.

4.1 Lemma *Let E be a closed subset of $\mathscr{M}(R)$ and $p \in \mathscr{P}(R) \setminus E$. Then, $p \notin \hat{E}$ if and only if $g \in H_E^\infty$ for some (and hence all) $g \in M_p^\infty$.*

4.2 Lemma *If E is a closed subset of $\mathscr{M}(R)$, then every connected component V of $\mathscr{P}(R) \setminus E$ satisfies either $V \subset \hat{E}$ or $V \cap \hat{E} = \emptyset$.*

A subset U of R is called **dominating** for $H^\infty(R)$ if $\|f\|_U = \|f\|_R$ for all $f \in H^\infty(R)$. The next lemma is a key.

4.3 Lemma *Suppose that E is a closed subset of $\mathscr{M}(R)$ such that $\text{III}(R) \not\subset E$, and that Q is a subset of R satisfying either of the following properties;*

$$\|f\|_Q < \|f\|_R \text{ for all nonconstant } f \in H^\infty(R) \tag{4.1}$$

$$Q \text{ is contained in the zero set of some nonconstant } g \in H^\infty(R) \tag{4.2}$$

Then, $E \cup \overline{Q}$ is not a closed boundary for $H^\infty(R)$, and hence, $\widehat{E \cup \overline{Q}}$ does not include any dominating subset of R for $H^\infty(R)$.

Proof: Let U be a dominating subset of R. Since E is not a boundary, there this a function f in $H^\infty(R)$ with $\|f\|_E < \|f\|_R$.

If Q satisfies (4.1), then we also have $\|f\|_Q < \|f\|_R$, and hence, $\|f\|_{E\cup\overline{Q}} < \|f\|_R = \|f\|_U$. This shows the conclusion.

If Q satisfies (4.2), then we have a nonconstant $g \in H^\infty(R)$ with $g = 0$ on Q. Since $\|f\|_R > \|f\|_E$, and since Q is nowhere dense in R, there exists a point a in $R \setminus Q$ such that $|f(a)| > \|f\|_E$. Multiplying a constant to f, we may assume that $f(a) = 1$. For a sufficently large positive integer n, we have $\|f^n g\|_{E\cup\overline{Q}} = \|f^n g\|_E < |g(a)| = |(f^n g)(a)| < \|f^n g\|_R = \|f^n g\|_U$. This yields the conclusion. □

The proof of the next lemma is routine.

4.4 Lemma *If an open subset U of R is dominating for $H^\infty(R)$, then U contains a dominating sequence S for $H^\infty(R)$ such that S has no accumulating points in U (in the standard topology of R).*

4.5 Lemma *Suppose (3.1) and that the points q_k's are as in (3.2). Define a linear functional Λ on $H^\infty(R)$ by*

$$\Lambda(f) = \sum_k f(q_k)2^{-k}. \tag{4.3}$$

If μ is a measure on $\mathscr{M}(R) \setminus \mathscr{P}(R)$ representing Λ, i.e., $\Lambda(f) = \int f d\mu$ for $f \in H^\infty(R)$, then $\operatorname{supp}(\mu) \supset \text{Ш}(R)$. *Moreover, among such representing meausures there exists μ with* $\operatorname{supp}(\mu) = \text{Ш}(R)$.

Proof: Suppose that $\text{Ш}(R) \setminus \operatorname{supp}(\mu)$ is not empty. By hypothesis (3.2), the set $Q = \{q_k | k = 1, 2, 3, \ldots\}$ satisfies

(4.1). Let E be the clusure of the set supp$(\mu) \cup Q$. Since $\mathscr{P}(R)$ is dense in R, $\mathscr{P}(R) \setminus Q$ is a dominating subset of R. By Lemma4.3, there is a function $h \in H^\infty(R)$ such that $|h(p_0)| > \|h\|_E$ for some point p_0 in $\mathscr{P}(R) \setminus Q$. Let Q_ℓ be the connected component of $\mathscr{P}(R)$ containing the point p_0. By Lemma 4.2, $Q_\ell \setminus Q$ is disjoint from $\hat{E}$. The Shilov idenpotent theorem shows that there is a sequence $h_n \in H^\infty(R)$ such that $h_n(p_\ell) \to 1$ and $h_n \to 0$ uniformly on $\hat{E} \setminus \{p_\ell\}$ as $n \to \infty$. For arbitrary $f \in H^\infty(R)$,

$$
\begin{aligned}
f(p_\ell)2^{-\ell} &= \lim_n \sum_k f(q_k)h_n(q_k)2^{-k} = \lim_n \Lambda(fh_n) \\
&= \int_{\text{supp}(\mu)} fh_n d\mu = 0,
\end{aligned}
$$

a contradiction. Thus, supp$(\mu) \supset \text{Ш}(R)$. The last assurtion follows form the Hahn-Banach extension theorem and the Riesz representation theorem. □

Now the proof of Theorem3.1 follows in a similar line due to Gamelin's. The details will be appear somewhere.

Finally, we note here that the hypothesis (3.2) can be relaxed to the following weaker one in the above argument:

> The union of a subfamily $\{Q_k\}$ of the connected components of $\mathscr{P}(R)$ satisfying (3.2) forms a boundary for $H^\infty(R)$

(4.4)

Instead of (4.1), we may consider the set $Q = \{q_k\}$ satisfying (4.2) .

参考文献

[1] T. W. Gamelin, Lectures on $H^\infty(D)$, La Plata Notas de Mathematica, 1972.

[2] T. W. Gamelin, The Shilov boundary of $H^\infty(U)$, Amer. J. Math. 154(1974), 79–103.

[3] T. W. Gamelin and M. Hayashi, The algebra of bounded analytic functions on a Riemann surface, J. Reine Angew. Math. 382(1978), 49–73.

[4] M. Hayashi, Hardy classes on Riemann surfaces, thesis, University of California at Los Angeles, 1979.

[5] M. Hayashi, The maximal ideal space of the bounded analytic functions on a Riemann surface, J. Math. Soc. Japan 39 (1987), 337–344.

[6] N. K. Nikol'skiĭ, V. P. Havin and S. V. Hruščev (Compiled and edited), Исследования по линейным операторам и теории функций. (Russian) [Investigations on linear operators and the theory of functions] 99 нерешенных задач линейного и комплексного анализа. [99 unsolved problems in linear and complex analysis] Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 81 (1978). "Nauka" Leningrad. Otdel., Leningrad, 1978. pp. 1–296.