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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1519: 59-70</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-10</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58756">http://hdl.handle.net/2433/58756</a></td>
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<td>Type</td>
<td>Departmental Bulletin Paper</td>
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On a problem about the Shilov boundary of a Riemann surface

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1 Notations and a problem

Let $R$ be a Riemann surface and let $H^\infty(R)$ be the algebra of all bounded analytic functions on $R$ with sup-norm $\|f\|_\infty = \|f\|_R = \sup_{p \in R} |f(p)|$.

The maximal ideal space $\mathcal{M}(R)$ of $H^\infty(R)$ is the set of all nonzero continuous homomorphisms of $H^\infty(R)$ to the complex field $\mathbb{C}$. The Gelfand transform $\hat{f}$ of $f \in H^\infty(R)$ is a function on $\mathcal{M}(R)$ defined by $\hat{f}(\phi) = \phi(f)$ for $\phi \in \mathcal{M}(R)$. The maximal ideal space $\mathcal{M}(R)$ is a compact Hausdorff space with respect to the Gelfand topology, the weakest topology among topologies such that every Gelfand transform $\hat{f}$ is to be continuous on $\mathcal{M}(R)$.

A closed subset $E$ of $\mathcal{M}(R)$ is called a boundary for $H^\infty(R)$ if it satisfies $\|\hat{f}\|_E = \max_{p \in E} |\hat{f}(p)| = \|f\|_R$ for all $f \in H^\infty(R)$. The smallest boundary, denoted by $\mathbb{B}(R)$, for

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1The present work is partially supported by Grant-in-Aid for Scientific Research (No.16540132).
$H^\infty(R)$ exists and is called the Shilov boundary of $H^\infty(R)$.

**Theorem A (Gamelin[1, 2])** If $D$ is a domain in the complex plane, then the Shilov boundary $\mathrm{III}(D)$ of $H^\infty(D)$ is extremely disconnected.

It is natural to ask whether the same conclusion remains true for arbitrary Riemann surfaces(cf. [6]). Namely,

**Problem** For any Riemann surface $R$, is the Shilov boundary $\mathrm{III}(R)$ extremely disconnected?

In order to avoid a triviality, one may only consider the case that Riemann surface $R$ admits a nonconstant bounded analytic function; for, otherwise, the Shilov boundary is singleton.

At present we have no counter examples. In this note, we shall give a partial result.

A point evaluation homomorphism $\phi_p$ at $p \in R$, defined by $\phi_p(f) = f(p)$ for $f \in H^\infty(R)$, is an element of $\mathcal{M}(R)$. This induces a natural continuous map from $R$ into $\mathcal{M}(R)$. While this natural map may not be injective in general, we often identify $R$ with its image in $\mathcal{M}(R)$ and regard $R$ as a subset of $\mathcal{M}(R)$. With this convention, Gelfand transform $\hat{f}$ can be regarded as a continuous extension of $f$.

The proof of Theorem A is based on the following simple fact; function $1/(z-p)$ of $z$ has simple pole at $p$ and bounded off any neighborhood of the point $p$. From this fact it follows that $D$ is homeomorphically imbedded as an open subset in $\mathcal{M}(D)$. 
Let $\mathscr{P}_s(R)$ be the set of points $p \in R$ such that there exist a meromorphic function $g$ on $R$ with the following properties: (i) $g$ has a simple pole at $p$, and (ii) $g$ is bounded on $R \setminus U_p$ for any neighborhood $U_p$ of $p$.

**Theorem B ([5])** Let $R$ be a Riemann surface such that $H^\infty(R)$ contains a nonconstant function. Then, a point $p \in R$ belongs to the set $\mathscr{P}_s(R)$ if and only if $p$ has a neighborhood which is homeomorphically imbedded as an open subset in $\mathcal{M}(R)$.

The 'only if' part is easy to see. From this easy part of the theorem one can extend Theorem A to those Riemann surfaces $R$ under the condition $\mathscr{P}_s(R) = R$, whose proof goes in a similar way as Gamelin’s method (cf. [4]).

In this note we consider the case that $\mathscr{P}_s(R)$ is a proper subset of $R$.

## 2 A preliminary observation

In this section we introduce an example of a Riemann surface. First we recall one of the examples constructed in [5]; Let $\Delta = \{z : |z| < 1\}$ be the open unit disc, and set

$$\Delta_k = \Delta \quad (k = 0, 1, 2, \ldots)$$

$$J_k = [a_k, b_k], \quad 0 < a_1 < b_1 < a_2 < b_2 < \cdots, \quad a_k \uparrow 1$$

$$I_k = \bigcup_{j=1}^{n_k} [a_{kj}, b_{kj}], \quad a_1 = a_{k1} < b_{k1} < \cdots < a_{kn_k} < b_{kn_k} = b_k$$

($n_k$ are sufficiently large)

$$D_0 = \Delta_0, \quad D_k = \Delta_k \setminus \bigcup_{\ell=1}^{k-1} J_\ell \quad (k \geq 1)$$
Let $W$ be the Riemann surface obtained by connecting two sides of intervals $I_k$ in the sheet $D_k \setminus I_k$ ($k \geq 1$) with the corresponding two sides in the bottom sheet $D_0 \setminus I_k$ crosswisely. If we choose integers $n_k$ sufficiently large, then the sheets $D_k$ converges to the bottom sheet $D_0$ in the maximal ideal space $\mathcal{M}(W)$ as $k \to \infty$, and we have

$$\mathcal{P}_s(W) = \bigcup_{k=1}^{\infty} (D_k \setminus I_k)$$

Let us consider the following subdomain $W'$ of $W$:

$$\Delta_k' = \Delta' = \{ z : \left| z + \frac{1}{2} \right| \leq \frac{1}{4} \} \quad (k \geq 0)$$

$$D_k' = D_k \setminus \Delta_k' \quad (k \geq 0)$$

$$W' = W \setminus \bigcup_{k=1}^{\infty} \Delta_k'$$

Increasing the number $n_k$ of subintervals forming $I_k$, if necessary, we may further assume that the sheets $D_k'$ converges to $D_0 \setminus \Delta_0'$ in the maximal ideal space $\mathcal{M}(W')$ as $k \to \infty$, and we have

$$\mathcal{P}_s(W') = \Delta_0' \cup \left( \bigcup_{k=1}^{\infty} (D_k' \setminus I_k) \right)$$

The restriction $\tau(f) = f|W'$ is an algebra homomorphism of $H^\infty(W)$ to $H^\infty(W')$, which induces a natural continuous map $\hat{\tau} : \mathcal{M}(W') \to H^\infty(W)$. For $k \geq 1$ set

$$\Gamma_k = \hat{\tau}^{-1}(\partial \Delta_k'),$$

which is homeomorphic to $\mathcal{M}(\Delta) \setminus \Delta$.

Since the sheets $D_k'$ converges to the subdomain $D_0'$ of the bottom sheet $D_0$, one might expect that $\Gamma_k$ converges to a
compact subset, $\partial \Delta'_0$, of the bottom sheet. If this would be true, then the circle $\partial \Delta'_0$ should be a part of the Shilov boundary $\Pi(W')$ and we would have a counterexample to the Problem.

This expectation is false. Namely,

2.1 Theorem The closure of $\bigcup_{k \geq 1} \Gamma_k$ in $\mathcal{M}(W')$ is disjoint from the bottom sheet $D_0$.

Proof: By [3, Theorem 4.1], we have a Cauchy differential

$$\omega(\zeta, z) d\zeta = \left\{ \frac{1}{\zeta - z} + \eta(\zeta, z) \right\} d\zeta$$

on $\mathcal{P}_s(W) \times W$ such that the analytic part $\eta(\zeta, z)$ is bounded on $U \times W$ whenever $U$ is a relatively compact coordinate disc in $\mathcal{P}_s(W)$. Let $0 < \delta < \frac{1}{4}$. Set $f_k(z) = \left( \frac{1}{4z + 2} \right)^{m_k}$ on the sheet $D_k$ for a positive integer $M_k$. On the annulus $\{ z \in D_k : \frac{1}{4} - \delta < |z + \frac{1}{2}| < \frac{1}{4} + \delta \}$, we have

$$f_k(z) = \frac{1}{2\pi i} \left( \int_{|\zeta + \frac{1}{2}| = \frac{1}{4} + \delta} \int_{|\zeta + \frac{1}{2}| = \frac{1}{4}} f_k(\zeta) \omega(\zeta, z) d\zeta \right) = h_k(z) - g_k(z).$$

Choosing $m_k$ large enough, we have $|g_k| \leq \varepsilon_k$ on $W \setminus \{ z \in D'_k : |z + \frac{1}{2}| \leq \frac{1}{4} + \delta \}$ and $|h_k| < 2^{-k-1}$ on $\Delta'_k$. Set $G = \sum_{k \geq 1} g_k$. Since $g_k = h_k - f_k$, it follows that $\frac{1}{2} = 1 - \sum_{k \geq 1} 2^{-k-1} \leq |G| \leq 1 + \sum_{k \geq 1} 2^{-k-1} = \frac{3}{2}$ on each $\partial \Delta'_k$. Hence, $G \in H^\infty(W')$, and $|G| < \sum_{k \geq 1} 2^{-k-1} = \frac{1}{2}$ on the bottom sheet $D_0$. This proves the theorem. $\square$

In the remaining part of the note, we shall prove, moreover,
that the Shilov boundaries $\mathcal{W}(W)$ and $\mathcal{W}(W')$ are both extremely disconnected.
3 Main theorem

3.1 Theorem Let $R$ be a Riemann surface and let $\{Q_k\}$ be the connected components of $\mathcal{D}_s(R)$. Suppose that

$$\mathcal{D}_s(R) \text{ is a dense subset of } R \text{ in } \mathcal{M}(R)$$

and that

- each $Q_k$ contains a point $q_k$ such that $\sup_k |f(q_k)| < \|f\|_R$ for every nonconstant $f \in H^\infty(R)$.

Then, $\mathbb{W}(R)$ is extremely disconnected.

The algebra $H^\infty(R)$ is said to be weakly separating (the points of $R$) if for each pair distinct points $p, q$ of $R$ there is a pair of nonzero functions $f, g$ of $H^\infty(R)$ such that $\hat{f}_g(p) \neq \hat{f}_g(q)$.

For the proof we may assume that $R$ is weakly separating. In fact, if $\tilde{R}$ is the Royden’s resolution of a Riemann surface $R$ with respect to the algebra $H^\infty(R)$, then

(a) $H^\infty(\tilde{R})$ is weakly separating;

(b) $H^\infty(\tilde{R})$ is algebraically isomorphic with $H^\infty(R)$, more precisely, there exists an analytic map $\rho$ of $R$ to $\tilde{R}$ such that $H^\infty(R) = \{ \tilde{f} \circ \rho; \tilde{f} \in \tilde{H}^\infty(\tilde{R}) \}$;

(c) $\tilde{R}$ is $H^\infty(\tilde{R})$-maximal, namely, if $W$ is a Riemann surface containing a proper subdomain being conformally equivalent to $\tilde{R}$, then some elements in $H^\infty(\tilde{R})$ can not be analytically extended to whole $W$;
(d) the Royden's resolution of $(R, H^\infty(R))$ is uniquely determined up to conformal equivalence by properties (a), (b) and (c).

By (b), two Banach algebras $H^\infty(R)$ and $H^\infty(\tilde{R})$ are isometrically isomorphic, that is, $\|\tilde{f} \circ \rho\|_\infty = \|\tilde{f}\|_\infty$. Trivially, $\rho(\mathcal{P}_s(R)) \subset \mathcal{P}_s(\tilde{R})$. Moreover, we have $\mathcal{P}_s(\tilde{R}) = \mathcal{P}(\tilde{R})$ ([5]), where $\mathcal{P}(R)$ denote the pole set which consists of the points $p \in R$ at which a meromorphic function $g$ on $R$, bounded off a compact subset of $R$, has a pole. Therefore, it suffices to show the theorem for $\tilde{R}$ in place of $R$.

To prove the theorem, we can use the same idea due to Gamelin ([1, 2]), where we need some modifications. One is needed because the pole set $\mathcal{P}(R)$ is not be connected and consists of infinitely many connected component. Another difficulty is that we only have local coordinate for a Riemann surface instead of a global coordinate $z$ for the complex plane.

4 Outline of the proof

We assume that $H^\infty(R)$ is weakly separating. For $p \in \mathcal{P}(R)$, we denote by $M_p^\infty$ the set of meromorphic functions with a simple pole at $p$ and bounded off any neighborhood of $p$. For a closed subset $E$ of $M(R)$, we set $\hat{E} = \{\phi \in M(R) : |\hat{f}(\phi)| \leq \|\hat{f}\|_E, f \in H^\infty(R)\}$, called the $H^\infty$-convex hull of $E$, and denote by $H^\infty_E$ the closure of $H^\infty(R)$ with respect to the uniform norm for $E$. Let $M^\infty(R)$ be the set of meromorphic functions on $R$ which are bounded off a compact subset of $R$. It is known that each $g \in M^\infty(R)$ has uniquely defines
a continuous map $\hat{g}$ of $\mathcal{M}(R)$ to the Riemann sphere such that $\hat{g}$ agrees with $g$ on $R$ (regarded as a subset of $\mathcal{M}(R)$) and such that $\hat{f}\hat{g} = \overline{fg}$ on $\mathcal{M}(R) \setminus \{\text{poles of } g\}$ whenever $fg$ belongs to $H^\infty(R)$. For the simplicity of notations, we may identify function $g$ on $R$ with function $\hat{g}$ on $\mathcal{M}(R)$.

The following two lemmas can be prove if one use meromorphic functions in $M^\infty_p$ in place of $1/(z - p)$.

4.1 Lemma Let $E$ be a closed subset of $\mathcal{M}(R)$ and $p \in \mathcal{P}(R) \setminus E$. Then, $p \notin \hat{E}$ if and only if $g \in H^\infty_E$ for some (and hence all) $g \in M^\infty_p$.

4.2 Lemma If $E$ is a closed subset of $\mathcal{M}(R)$, then every connected component $V$ of $\mathcal{P}(R) \setminus E$ satisfies either $V \subset \hat{E}$ or $V \cap \hat{E} = \emptyset$.

A subset $U$ of $R$ is called dominating for $H^\infty(R)$ if $\|f\|_U = \|f\|_R$ for all $f \in H^\infty(R)$. The next lemma is a key.

4.3 Lemma Suppose that $E$ is a closed subset of $\mathcal{M}(R)$ such that $\Pi(R) \not\subset E$, and that $Q$ is a subset of $R$ satisfying either of the following properties;

\[
\|f\|_Q < \|f\|_R \text{ for all nonconstant } f \in H^\infty(R) \tag{4.1}
\]

\[
Q \text{ is contained in the zero set of some } 
\text{nonconstant } g \in H^\infty(R) \tag{4.2}
\]

Then, $E \cup \overline{Q}$ is not a closed boundary for $H^\infty(R)$, and hence, $\overline{E \cup Q}$ does not include any dominating subset of $R$ for $H^\infty(R)$.

**Proof:** Let $U$ be a dominating subset of $R$. Since $E$ is not
a boundary, there this a function $f$ in $H^\infty(R)$ with $\|f\|_E < \|f\|_R$.

If $Q$ satisfies (4.1), then we also have $\|f\|_Q < \|f\|_R$, and hence, $\|f\|_{E \cup Q} < \|f\|_R = \|f\|_U$. This shows the conclusion.

If $Q$ satisfies (4.2), then we have a nonconstant $g \in H^\infty(R)$ with $g = 0$ on $Q$. Since $\|f\|_R > \|f\|_E$, and since $Q$ is nowhere dense in $R$, there exists a point $a$ in $R \setminus Q$ such that $|f(a)| > \|f\|_E$. Multiplying a constant to $f$, we may assume that $f(a) = 1$. For a sufficiently large positive integer $n$, we have $\|f^n g\|_{E \cup \overline{Q}} = \|f^n g\|_E < |g(a)| = |(f^n g)(a)| < \|f^n g\|_R = \|f^n g\|_U$. This yields the conclusion. ∎

The proof of the next lemma is routine.

4.4 **Lemma** If an open subset $U$ of $R$ is dominating for $H^\infty(R)$, then $U$ contains a dominating sequence $S$ for $H^\infty(R)$ such that $S$ has no accumulating points in $U$ (in the standard topology of $R$).

4.5 **Lemma** Suppose (3.1) and that the points $q_k$'s are as in (3.2). Define a linear functional $\Lambda$ on $H^\infty(R)$ by

$$\Lambda(f) = \sum_k f(q_k)2^{-k}. \quad (4.3)$$

If $\mu$ is a measure on $\mathcal{M}(R) \setminus \mathcal{P}(R)$ representing $\Lambda$, i.e., $\Lambda(f) = \int fd\mu$ for $f \in H^\infty(R)$, then $\text{supp}(\mu) \supset \text{III}(R)$. Moreover, among such representing measures there exists $\mu$ with $\text{supp}(\mu) = \text{III}(R)$.

**Proof:** Suppose that $\text{III}(R) \setminus \text{supp}(\mu)$ is not empty. By hypothesis (3.2), the set $Q = \{q_k|k = 1, 2, 3, \ldots\}$ satisfies
Let $E$ be the closure of the set supp$(\mu) \cup Q$. Since $\mathcal{P}(R)$ is dense in $R$, $\mathcal{P}(R) \setminus Q$ is a dominating subset of $R$. By Lemma 4.3, there is a function $h \in H^\infty(R)$ such that $|h(p_0)| > \|h\|_E$ for some point $p_0$ in $\mathcal{P}(R) \setminus Q$. Let $Q_\ell$ be the connected component of $\mathcal{P}(R)$ containing the point $p_0$. By Lemma 4.2, $Q_\ell \setminus Q$ is disjoint from $\hat{E}$. The Shilov idempotent theorem shows that there is a sequence $h_n \in H^\infty(R)$ such that $h_n(p_\ell) \to 1$ and $h_n \to 0$ uniformly on $\hat{E} \setminus \{p_\ell\}$ as $n \to \infty$. For arbitrary $f \in H^\infty(R)$,

$$f(p_\ell)2^{-\ell} = \lim_n \sum_k f(q_k)h_n(q_k)2^{-k} = \lim_n \Lambda(fh_n)$$

$$= \int_{\text{supp}(\mu)} fh_n d\mu = 0,$$

a contradiction. Thus, supp$(\mu) \supset \text{III}(R)$. The last assertion follows from the Hahn-Banach extension theorem and the Riesz representation theorem. $\square$

Now the proof of Theorem 3.1 follows in a similar line due to Gamelin’s. The details will be appear somewhere.

Finally, we note here that the hypothesis (3.2) can be relaxed to the following weaker one in the above argument:

The union of a subfamily $\{Q_k\}$ of the connected components of $\mathcal{P}(R)$ satisfying (3.2) forms a boundary for $H^\infty(R)$

\begin{equation}
\text{(4.4)}
\end{equation}

Instead of (4.1), we may consider the set $Q = \{q_k\}$ satisfying (4.2).
参考文献


