Toeplitz $\Psi^*$-algebras via unitary group representations

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Abstract

As it was pointed out in [12] there are construction methods for spectral invariant Fréchet operator algebras such as $\Psi^*$- and $\Psi_0$-algebras in the bounded operators on a Hilbert space having prescribed properties. For the Segal-Bargmann space $H$ and using systems of unbounded closable Toeplitz operators $T_f$ where $f$ is in a certain class $\text{SP}_{\text{Lip}}(\mathbb{C}^n)$ of symbols we show that these algebras contain all Toeplitz operators $T_h$ with $h \in L^\infty(\mathbb{C}^n)$. Let $\rho$ be the Segal-Bargmann representation of the Heisenberg group $\mathbb{H}_n$ in the bounded operators on $H$. As an application of our results above we characterize a class of smooth Toeplitz operators in the $\Psi^*$-algebra of smooth elements with respect to $\rho$.

1 Introduction

Subsequent to the results in [12] it frequently has been remarked that the abstract concept of (locally) spectral invariant Fréchet algebras such as $\Psi_0$- and $\Psi^*$-algebras successfully can be applied to the structural analysis of certain algebras of pseudo-differential operators. Applications arise in complex analysis, analytic perturbation theory of Fredholm operators and non-abelian cohomology for analyzing isomorphisms of abelian groups in $K$-theory. By generalizing a characterization of the Hörmander classes $\Psi_{\rho,\delta}^0$ by commutator conditions (see Theorem 2.1) a construction method for algebras of the above mentioned type with prescribed properties have been given in [12].

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$0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$
Let \( H := H^2(\mathbb{C}^n, \mu) \) be the Segal-Bargmann space of Gaussian square integrable entire functions on \( \mathbb{C}^n \). We denote by \( P \) the orthogonal projection from \( L^2(\mathbb{C}^n, \mu) \) onto \( H \) and we write \( M_f \) for the multiplication with a measurable symbol \( f \). In the initial stage of this paper we consider iterated commutators of closable Toeplitz operators \( T_f := PM_f \) on \( H \) having symbols in a certain class \( \text{SP}_{\text{lip}}(\mathbb{C}^n) \) of measurable and in general unbounded functions on \( \mathbb{C}^n \). For a system \( \mathcal{S}_m := \{T_{f_1}, \ldots, T_{f_m}\} \) of operators with \( f_j \in \text{SP}_{\text{lip}}(\mathbb{C}^n) \) and in the sense of [12] the \( \Psi_0 \)-algebra \( \Psi^{S_m}_\infty \) in the bounded operators \( \mathcal{L}(H) \) on \( H \) can be defined by commutator methods with respect to \( \mathcal{S}_m \). We show that \( \Psi^{S_m}_\infty \) contains all Toeplitz operators with bounded measurable symbols. More precisely:

**Theorem A** The symbols map \( L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi^{S_m}_\infty \) is well-defined and continuous.

Let \( \mathbb{H}_n \) be the Heisenberg group and \( \alpha \) be the Segal-Bargmann representation of \( \mathbb{H}_n \) in \( \mathcal{L}(H) \), c.f. [10]. The map \( \alpha \) is well-known to be unitary, irreducible and strongly continuous. In particular, the \( \Psi^* \)-algebra \( \Psi^\infty(\mathbb{H}_n) \subset \mathcal{L}(H) \) of smooth elements with respect to \( \alpha \) arise in a natural way and it can be characterized by commutator methods. We describe a symmetric subspace \( \mathcal{S}_s \subset L^\infty(\mathbb{C}^n) \) with the induced topology such that:

**Theorem B** The symbols map \( \mathcal{S}_s \ni h \mapsto T_h \in \Psi^\infty(\mathbb{H}_n) \) is well-defined and continuous.

This result can be stated in terms of the algebra construction. Let \( A \) be the algebra of multiplication operators on \( V := L^2(\mathbb{C}^n, \mu) \) with bounded measurable symbols. In a natural way \( \alpha \) extends to a representation of \( \mathbb{H}_n \) into \( \mathcal{L}(V) \) and the corresponding operator algebras \( \Psi^k(A, \mathbb{H}_n) \) of \( C^k \)-elements in \( A \) form a decreasing scale. Note that \( M_f \in \Psi^k(A, \mathbb{H}_n) \) is related to the smoothness of the symbols \( f \in L^\infty(\mathbb{C}^n) \). Clearly, \( A \) projects under \( P \) onto the space \( A_P := PAP \) of Toeplitz operators with bounded symbols. Theorem B states:

\[
P \Psi^k(A, \mathbb{H}_n) P = P \Psi^{k+1}(A, \mathbb{H}_n) P \subset \mathcal{L}(H) \quad \text{for all } k \in \mathbb{N}.
\]

Heuristically, the smoothness of \( f \) cannot be recovered by commutator methods from the Toeplitz operator \( T_f \). We want to remark here that these results are related to an observation in [14], [3]. Let \( \beta : L^2(\mathbb{R}^n) \rightarrow H \) be the Bargmann isometry and \( f \) a bounded measurable function on \( \mathbb{C}^n \). The assignment \( \beta^{-1}T_f\beta \) can be shown to be a pseudo-differential operator \( W_{\sigma(f)} \) on \( L^2(\mathbb{R}^n) \) in its Weyl quantization. By identifying \( \mathbb{R}^{2n} \) and \( \mathbb{C}^n \) the Weyl symbol \( \sigma(f) \) and \( f \) are related via the heat equation on \( \mathbb{R}^{2n} \). There is \( t_0 > 0 \) such that:

\[
\sigma(f) = e^{-t_0 \Delta} f := \text{solution of the heat equation with initial data } f \text{ at the time } t_0.
\]

Moreover, \( \sigma \) maps the space of continuous functions with compact support into the symbol class \( S_{\rho, \delta}^\infty \), \( 0 \leq \delta \leq \rho \leq 1 \) and \( \delta < 1 \). Corresponding to Theorem A and B it can be checked that \( f \mapsto \sigma(f) \) is continuous with respect to the \( L^\infty(\mathbb{C}^n) \) topology and the usual Fréchet topology on \( S_{\rho, \delta}^\infty \).

In our first section we remind of some basic definitions and results related to the construction of \( \Psi_0 \)- and \( \Psi^* \)-algebras. For Toeplitz operators having symbols of polynomial growth at infinity an invariant subspace \( H_{\exp}(\mathbb{C}^n) \) of \( H \) is defined in section 3. Moreover,
the existence of bounded extensions for a class of iterated commutators of Toeplitz operators on $H_{\exp}(\mathbb{C}^{n})$ and Theorem A are proved. Section 4 contains the proof of Theorem B and finally we have added some examples and applications in section 5.

2 Fréchet operator algebras with prescribed properties

The following definition due to B. Gramsch have been given in [11]:

**Definition 2.1** Let $B$ be a Banach-algebra with unit $e$ and let $\mathcal{F}$ be a continuously embedded Fréchet algebra in $B$ with $e \in \mathcal{F}$. Then $\mathcal{F}$ is called $\Psi_{0}$-algebra if it is locally spectral invariant in $B$, i.e. there is $\varepsilon > 0$ with

$$\{ a \in \mathcal{F} : \| e - a \|_{B} < \varepsilon \} \subset \mathcal{F}^{-1}.$$

Moreover, one defines:

- If $B$ is a $C^{*}$-algebra and $\mathcal{F}$ is a symmetric $\Psi_{0}$-algebra in $B$, then $\mathcal{F}$ is called $\Psi^{*}$-algebra. ($\mathcal{F}$ automatically is spectral invariant, i.e. $\mathcal{F} \cap B^{-1} = \mathcal{F}^{-1}$).

- If the topology of $\mathcal{F}$ is generated by a system $[ q_{j} : j \in \mathbb{N} ]$ of sub-multiplicative semi-norms with $q_{j}(e) = 1$ for $j \in \mathbb{N}$, then $\mathcal{F}$ is called sub-multiplicative or locally $m$-convex (E. Michael, 1952) $\Psi_{0}$- resp. $\Psi^{*}$-algebra.

The concept of $\Psi^{*}$- and $\Psi_{0}$-algebras allows to treat phenomenas of local structure. As it was observed for algebras of Pseudo-differential operators, $C^{\infty}$-properties such as pseudo- or micro-locality are preserved by taking closures in the Fréchet topology. Important examples of $\Psi^{*}$-algebras are given by the Hörmander classes $\Psi_{\rho,\delta}^{0}$ 2 of zero order where $B := \mathcal{L}(L^{2}(\mathbb{R}^{n}))$. It is known that $\Psi_{\rho,\delta}^{0}$ can be described in terms of commutator conditions.

**Theorem 2.1** (R. Beals, '77, [6])

An operator $B : S(\mathbb{R}^{n}) \to S'(\mathbb{R}^{n})$ is of class $\Psi_{\rho,\delta}^{0}$ iff for $\alpha, \beta \in \mathbb{N}_{0}^{n}$ all iterated commutators:

$$ad[-ix]^{\alpha} ad[i\partial_{x}]^{\beta}(B) : H^{s-\rho|\alpha|+\delta|\beta|} \to H^{s}$$

admit bounded extensions between suitable Sobolev spaces to $L^{2}(\mathbb{R}^{n})$.

On the one hand the spectral invariance of $\Psi_{\rho,\delta}^{0}$ follows from the commutator characterizations in Theorem 2.1, see [19], [20]. On the other hand, by replacing $ix$ and $i\partial_{x}$ above with a system of closable and densely defined operators, conditions of the type (2.1) have been used to define (submultiplicative) $\Psi_{0}$-algebras in a fairly general situation, see [12]. Below we give the definitions and remind of some basic results.

\[ 0 \leq \delta \leq \rho \leq 1 \text{ and } \delta < 1 \]
2.1 Commutator Methods

Given a topological vector space $X$ we write $L(X)$ (resp. $\mathcal{L}(X)$) for the linear (resp. bounded linear) operators on $X$.

**Definition 2.2** (Iterated commutators)

For a system $S_m := [A_1, \cdots, A_m]$ where $A_j, B \in L(X)$ we call $m$ the length of $S_m$. We inductively define the iterated commutators $\text{ad}[\emptyset](B) := B$ and:

- $\text{ad}[S_j](B) := [A_j, B] = A_jB - BA_j$,
- $\text{ad}[S_{j+1}](B) := \text{ad}[A_{j+1}](\text{ad}[S_j](B))$ for $j = 1, \cdots, m - 1$.

In the case of $A = A_j$ where $j = 1, \cdots, m$ we also write:

- $\text{ad}^0[A](B) := B$ and $\text{ad}^m[A](B) := \text{ad}[S_m](B)$.

With these notations it follows for finite systems $S_j$ and $S_k$ in $L(X)$:

$$\text{ad}[S_j](\text{ad}[S_k](B)) = \text{ad}[S_k, S_j](B).$$

Let $H$ be a Hilbert space and $\mathcal{F} \subset \mathcal{L}(H)$ be a sub-multiplicative $\Psi^*$-algebra. Assume that the topology of $\mathcal{F}$ is generated by a sequence $(q_j)_{j \in \mathbb{N}}$ of semi-norms and without loss of generality let $q_0 := \| \cdot \|_{\mathcal{L}(H)}$. Given a finite system $\mathcal{V}$ of closed and densely defined operators $A : H \supset D(A) \rightarrow H$ and following [12] we define:

- $\mathcal{I}(A) := \{ a \in \mathcal{F} : a(D(A)) \subset D(A) \}$,
- $B(A) := \{ a \in \mathcal{I}(A) : [A, a] \text{ extends to an element } \delta_A(a) \in \mathcal{F} \}$.

Inductively, one obtains:

- $\Psi_0^\mathcal{V} := \mathcal{F}$, with semi-norms $q_{0,j} := q_j$ for $j \in \mathbb{N}$,
- $\Psi_1^\mathcal{V} := \bigcap_{A \in \mathcal{V}} B(A)$,
- $\Psi_k^\mathcal{V} := \{ a \in \Psi_{k-1}^\mathcal{V} : \delta_A a \in \Psi_{k-1}^\mathcal{V} \text{ for all } A \in \mathcal{V} \}$ where $k \geq 2$,
- $\Psi_\infty^\mathcal{V} := \bigcap_{k \in \mathbb{N}} \Psi_k^\mathcal{V}$.

This process leads to a decreasing scale of algebras in $\mathcal{F}$:

$$\mathcal{F} = \Psi_0^\mathcal{V} \supset \cdots \supset \Psi_n^\mathcal{V} \supset \Psi_{n+1}^\mathcal{V} \supset \cdots \supset \Psi_\infty^\mathcal{V} := \bigcap_{k \in \mathbb{N}} \Psi_k^\mathcal{V}. \quad (2.2)$$

For $n \geq 1$, we inductively define a system $(q_{n,j})_{j \in \mathbb{N}}$ (resp. $(q_{n,j})_{j,n \in \mathbb{N}}$) of norms on $\Psi_n^\mathcal{V}$ (resp. on $\Psi_\infty^\mathcal{V}$) by:

$$q_{n,j}(a) := q_{n-1,j}(a) + \sum_{A \in \mathcal{V}} q_{n-1,j}(\delta_A a). \quad (2.3)$$
According to [12], $\Psi^\infty_0$ is a sub-multiplicative $\Psi_0$-algebra in $\mathcal{F}$. In the case where each $A \in \mathcal{V}$ is symmetric we replace $B(A)$ by:

$$B^*(A) := \{ a \in B(A) : a^* \in B(A) \}.$$

Then the algebras $\Psi^n_0$ are symmetric and $\Psi^\infty_0$ is a $\Psi^*$-algebra in $\mathcal{L}(H)$. Let $D \subset H$ be a core for $\mathcal{V}$, i.e. the inclusion $D \hookrightarrow \mathcal{D}(A)$ is dense with respect to the graph norm for all $A \in \mathcal{V}$. Then it was shown in [2], [3]:

**Proposition 2.1** Assume that $a \in \mathcal{F}$ and property $(E_k)$ holds for $k \in \mathbb{N} \cup \{\infty\}$:

$(E_k)$: $D$ is invariant under all $A \in \mathcal{V}$ and $a \in \mathcal{F}$. Moreover, assume that for any system

$$A \subset S_k(\mathcal{V}) := \{ [A_1, \cdots, A_j] : \text{where } A_l \in \mathcal{V} \text{ and } 1 \leq l \leq j \leq k \}.$$

Then $a \in \Psi^n_k$ and $C(A, a)$ is a bounded extension of $\text{ad}[A](a) : H \subset \mathcal{D}(A) \rightarrow H$ to $H$ for any operator $A \in \mathcal{V}$.

The (locally) spectral invariance of $\mathcal{A} \subset \mathcal{B}$ is preserved under projections $p = p^2 \in \mathcal{A}$. It is readily verified that $\mathcal{A}_p := p \mathcal{A} p$ is (locally) spectral invariant in $\mathcal{B}_p := p \mathcal{B} p$. If in addition $\mathcal{B}$ is a $C^*$-algebra, $\mathcal{A}$ is symmetric in $\mathcal{B}$ and $p = p^*$, then $\mathcal{A}_p$ is symmetric and spectral invariant in $\mathcal{B}_p$.

With (2.2) and an orthogonal projection $p \in \Psi^n_k$, $n \in \mathbb{N} \cup \{\infty\}$ from $H$ onto a closed subspace $H_0 \subset H$ there is a scale of projected algebras in $\mathcal{L}(H_0)$:

$$\mathcal{L}(H_0) \supset \mathcal{F}_p = \Psi^1_{p} \supset \cdots \supset \Psi^1_{-1} \supset \Psi^1_{p}.$$

It can be shown that (2.4) arises by commutator methods with a system $\mathcal{V}_p$ of closed operators on $H_0$ where $\mathcal{D}(A_p) := p[\mathcal{D}(A)]$ and

$$\mathcal{V}_p := \{ A_p := p A p : H_0 \supset \mathcal{D}(A_p) \rightarrow H_0 : A \in \mathcal{V} \}.$$

Defining (2.4) by commutator conditions with respect to $\mathcal{V}_p$ only requires that $p \in \Psi_1^\infty$. Thus this method gives a natural extension of (2.4) to an infinite scale for $n \in \mathbb{N}$.

There is a corresponding scale of $\mathcal{V}$-Sobolev spaces in $H$:

- $\mathcal{H}^0_\mathcal{V} := H$ with the norm $p_0 := \| \cdot \|_H$.
- $\mathcal{H}^1_\mathcal{V} := \bigcap_{A \in \mathcal{V}} \mathcal{D}(A)$.
- $\mathcal{H}^k_\mathcal{V} := \{ x \in \mathcal{H}^{k-1}_\mathcal{V} : Ax \in \mathcal{H}^{k-1}_\mathcal{V} \text{ for all } A \in \mathcal{V} \}, \ k \geq 2$.
- $\mathcal{H}^\infty_\mathcal{V} := \bigcap_{k \in \mathbb{N}} \mathcal{H}^k_\mathcal{V}$. 

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We endow $\mathcal{H}_V^k$ with the norm
\[ p_k(x) := p_{k-1}(x) + \sum_{A \in \mathcal{V}} p_{k-1}(Ax), \quad x \in \mathcal{H}_V^k. \]

Let the topology of $\mathcal{H}_V^\infty$ be defined by the system of norms $(p_k)_{k \in \mathbb{N}_0}$. It can be shown that $(\mathcal{H}_V^k, p_k)$ is a Banach spaces and $(\mathcal{H}_V^\infty, (p_k)_{k \in \mathbb{N}})$ turns into a Fréchet space. Moreover, each $A \in \mathcal{V}$ induces a bounded operator $A_k : \mathcal{H}_V^k \to \mathcal{H}_V^{k-1}$. For $n \in \mathbb{N} \cup \{\infty\}$ it was shown in [12] that all maps
\[ \Psi^*_k \times \mathcal{H}_V^k \to \mathcal{H}_V^k : (a, x) \mapsto a(x) \]
are bilinear and continuous. The following result on regularity was proved in [13]:

\begin{theorem}
Let $A \in \Psi^*_\infty$ be a Fredholm operator and $u \in H$ with $Au = f \in \mathcal{H}_V^k$ for some $k \in \mathbb{N} \cup \{\infty\}$. Then it follows that $u \in \mathcal{H}_V^k$.
\end{theorem}

3 On the Segal-Bargmann Projection

Throughout this paper we write $(x, y) := x_1 \bar{y}_1 + \cdots x_n \bar{y}_n$ for the Hermitian inner product on $\mathbb{C}^n$ and $|x| := \sqrt{(x, x)}$. For $c > 0$ and the Lebesgue measure $v$ let us denote by $\mu_c$ the Gaussian measure on $\mathbb{C}^n$ given by:
\[ d\mu_c = c^n \pi^{-n} \exp(-c |\cdot|^2) \, dv. \]

With $\mu := \mu_1$ let $H^2(\mathbb{C}^n, \mu)$ be the Segal-Bargmann space of $\mu$-square integrable entire functions on $\mathbb{C}^n$. We denote by $P$ the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto $H^2(\mathbb{C}^n, \mu)$. The reproducing kernel $K$ (resp. the normalized kernel $k$) corresponding to $H^2(\mathbb{C}^n, \mu)$ are known to be:

(a) $K(y, x) := \exp((y, x))$,

(b) $k_x(y) := K(y, x) \| K(\cdot, x) \|^{-1} = \exp((y, x) - \frac{1}{2} |x|^2)$

where $\| \cdot \|$ denotes the $L^2(\mathbb{C}^n, \mu)$-norm. For $z, w \in \mathbb{C}^n$ we write $\tau_w(z) := z + w$ for the shift by $w$. Consider the space of measurable symbols on $\mathbb{C}^n$ given by:
\[ T(\mathbb{C}^n) := \{ g : g \circ \tau_x \in L^2(\mathbb{C}^n, \mu) \text{ for all } x \in \mathbb{C}^n \}. \]

For $g \in T(\mathbb{C}^n)$ and with the natural domain of definition
\[ D(T_g) := \{ f \in H^2(\mathbb{C}^n, \mu) : gf \in L^2(\mathbb{C}^n, \mu) \} \quad (3.1) \]
the Toeplitz operator $T_g$ on $H^2(\mathbb{C}^n, \mu)$ is densely defined by:
\[ T_g : D(T_g) \ni f \mapsto P(fg). \]

If $g$ has polynomial growth at infinity we can determine an invariant subspace for $T_g$.

We inductively define a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_1 := \frac{1}{4}$ and $a_{n+1} := [4 \cdot (1 - a_n)]^{-1}$ for all $n \geq 2$. It can be checked that:
(a) $a_n < \frac{1}{2}, \quad \forall n \in \mathbb{N},$
(b) $(a_n)_{n \in \mathbb{N}}$ is strictly increasing,
(c) $\lim_{n \to \infty} a_n = \frac{1}{2}.$

Let $\mathbb{P}[\mathbb{C}^n]$ be the space of all polynomials on $\mathbb{C}^n$ in the variables $z := (z_1, \cdots, z_n)$ and $\overline{z} := (\overline{z}_1, \cdots, \overline{z}_n).$ We write $\mathbb{P}_a[\mathbb{C}^n]$ for all analytic polynomials and set:

$$L_{\exp}(\mathbb{C}^n) := \{ f \in L^2(\mathbb{C}^n, \mu) : \exists c < \frac{1}{2}, 0 < D \text{ s.t. } |f(z)| \leq D \exp\left( c |z|^2 \right) \text{ a.e.} \}.$$

Because of $\mathbb{P}[\mathbb{C}^n] \subset L_{\exp}(\mathbb{C}^n)$ it follows that $L_{\exp}(\mathbb{C}^n)$ is dense in $L^2(\mathbb{C}^n, \mu).$

With the space $\mathcal{H}(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$ we define a subspace of $H^2(\mathbb{C}^n, \mu)$ by:

$$H_{\exp}(\mathbb{C}^n) := \mathcal{H}(\mathbb{C}^n) \cap L_{\exp}(\mathbb{C}^n).$$

Consider the symbols having polynomial growth at $\infty$:

$$\mathrm{Pol}(\mathbb{C}^n) := \{ f : \exists j \in \mathbb{N} \text{ s.t. } |f(z)| (1 + |z|^2)^{-\frac{1}{2}} \in L^\infty(\mathbb{C}^n) \}.$$

**Proposition 3.1** It holds $P[L_{\exp}(\mathbb{C}^n)] \subset H_{\exp}(\mathbb{C}^n)$ and for $f$ in $\mathrm{Pol}(\mathbb{C}^n):$

$$T_f[H_{\exp}(\mathbb{C}^n)] \subset H_{\exp}(\mathbb{C}^n) \subset D(T_f) \quad (3.2)$$

**Proof:** It is obvious that $H_{\exp}(\mathbb{C}^n) \subset D(T_f).$ Because the multiplication by $f$ clearly maps $H_{\exp}(\mathbb{C}^n)$ into $L_{\exp}(\mathbb{C}^n)$ it is sufficient to prove the first assertion of Proposition 3.1. For $g \in L_{\exp}(\mathbb{C}^n)$ there are $c < \frac{1}{2}$ and $D > 0$ such that a.e.:

$$|g(z)| \leq D \exp\left( c |z|^2 \right).$$

By (a), (b) and (c) with $(a_n)_{n \in \mathbb{N}}$ above we can choose $n_0 \in \mathbb{N}$ with $c < a_{n_0} < \frac{1}{2}.$

Using the transformation formula and the reproducing property of $K$ we obtain:

$$|\mathbb{P}g(z)| \leq \int_{\mathbb{C}^n} |g \exp\{ \langle z, \cdot \rangle \}| d\mu$$

$$\leq D \pi^{-n} \int_{\mathbb{C}^n} \exp\left\{ \mathrm{Re}\langle z, \cdot \rangle - \left( 1 - a_{n_0} \right) |\cdot|^2 \right\} d\nu$$

$$= D \left( 1 - a_{n_0} \right)^{-n} \int_{\mathbb{C}^n} \exp\left\{ \frac{\mathrm{Re}\langle 2^{-1} (1 - a_{n_0})^{-\frac{1}{2}} z, \cdot \rangle}{\sqrt{4(1 - a_{n_0})}} \right\} d\mu$$

$$= D \left( 1 - a_{n_0} \right)^{-n} \exp\left\{ \left[ \frac{4(1 - a_{n_0})}{2^{-1}} \right]^{-\frac{1}{2}} |z|^2 \right\}.$$

From (a) above we conclude that $Pg \in H_{\exp}(\mathbb{C}^n).$ 

Hence all finite products of Toeplitz operators with symbols in $\mathrm{Pol}(\mathbb{C}^n)$ are well-defined on the dense subspace $H_{\exp}(\mathbb{C}^n)$ of $H^2(\mathbb{C}^n, \mu).$ In particular, all iterated commutators of $P$ and multiplication operators $M_f$ with $f \in \mathrm{Pol}(\mathbb{C}^n)$ can be considered as elements in $L(L_{\exp}(\mathbb{C}^n)).$ In fact, they can be written as integral operators and a standard application of the Schur test leads to a criterion for the boundedness.
Lemma 3.1 Let $L : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a measurable function such that:

$$|L(x, y)| \leq |F(x - y)| \exp \left\{ Re \langle x, y \rangle \right\}$$

where $F \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$. Then the integral operator $A$ on $L^2(\mathbb{C}^n, \mu)$ defined by

$$[A f](z) := \int_{\mathbb{C}^n} L(z, \cdot) f d\mu$$

is bounded on $L^2(\mathbb{C}^n, \mu)$ with $\| A \| \leq 2^n \| F \|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$.

Proof: With $p := q = \exp(\frac{1}{2} |\cdot|^2)$ on $\mathbb{C}^n$ it follows that:

$$\int_{\mathbb{C}^n} |L(\cdot, y)| p d\mu \leq \frac{1}{\pi^n} \int_{\mathbb{C}^n} |F(\cdot - y)| \exp \left\{ Re \langle \cdot, y \rangle - \frac{1}{2} |\cdot|^2 \right\} dv$$

$$= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |F| \exp \left\{ Re \langle \cdot + y, y \rangle - \frac{1}{2} |\cdot + y|^2 \right\} dv$$

$$= 2^n p(y) \| F \|_{L^q(\mathbb{C}^n, \mu)}.$$  

Similarly, we get $\int |L(x, \cdot)| p d\mu \leq 2^n p(x) \| F \|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$. Applying the Schur test we obtain the desired result.

Consider the subspace $\text{SP}_{\text{Lip}}(\mathbb{C}^n)$ of $\text{Pol}(\mathbb{C}^n)$ defined by:

$$\text{SP}_{\text{Lip}}(\mathbb{C}^n) := \{ f \in \text{Pol}(\mathbb{C}^n) : \exists c, D > 0 \text{ s.t. } |f(z) - f(w)| \leq D \exp(c |z - w|) \}.$$

As an application of Lemma (3.1) we can prove:

Proposition 3.2 Let $m \in \mathbb{N}$ and $S_m := \{ M_{f_1}, \ldots, M_{f_m} \}$ with $f_j \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$. Then the commutator $\text{ad}[S_m](P) \in L(L^2(\mathbb{C}^n, \mu))$ has a continuous extension to $L^2(\mathbb{C}^n, \mu)$.

Proof: It is easy to check that the commutator $\text{ad}[S_m](P)$ can be written as an integral operator on $L^2(\mathbb{C}^n, \mu)$ with kernel:

$$K_m(z, u) = \exp(\langle z, u \rangle) \prod_{j=1}^m \{ f_j(z) - f_j(u) \}.$$  \hspace{1cm} (3.3)

By (3.3) and our assumptions on $f_j \in S_m$ we can choose $c, D > 0$ such that

$$|K_m(z, u)| \leq D \exp(\langle c |z - u| + \text{Re} \langle z, u \rangle \rangle).$$

Because of $F := D \exp(\langle c |\cdot| \rangle) \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$ Lemma 3.1 implies the assertion.

We remark that by (3.3) the maps $\text{ad}[S_m](P)$ are invariant under permutations of the system $S_m$. Now, we can prove the boundedness of a class of iterated commutators.
Corollary 3.1 Let $g \in L^\infty(\mathbb{C}^n)$ and $S_m := \{M_{f_1}, \ldots, M_{f_m}\}$ with $f_j \in SP_{Lip}(\mathbb{C}^n)$. Then the commutator
\[ \text{ad} [S_m] \left( [P, M_g] \right) \in L(L^\infty(\mathbb{C}^n)) \]
has a bounded extension $A(S_m, g)$ to $L^2(\mathbb{C}^n, \mu)$ and (3.4) below is continuous between Banach spaces:
\[ L^\infty(\mathbb{C}^n) \ni g \mapsto A(S_m, g) \in \mathcal{L}(L^2(\mathbb{C}^n, \mu)). \] (3.4)

Proof: It can be checked by induction or our remark following Proposition 3.2 that:
\[ \text{ad} [S_m] \left( [P, M_g] \right) = \left[ \text{ad} [S_m] (P), M_g \right] \in L(L^\infty(\mathbb{C}^n)). \]

Because $M_g$ is bounded and $\text{ad} [S_m] (P)$ has a bounded extension to $L^2(\mathbb{C}^n, \mu)$ by Proposition 3.2 we conclude the desired result. □

Given a finite set $X := \{X_1, \ldots, X_n\} \subset \mathcal{L}(L^2(\mathbb{C}^n, \mu))$ we denote by $\mathcal{A}(X)$ the algebra generated by $X$. Moreover, we write:
\[ A_P(X) := P \mathcal{A}(X) P := \{PAP : A \in \mathcal{A}(X)\}. \]
for the corresponding projected algebra in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$. By Proposition 3.1 and for all $m \geq 1$ it follows that the commutator:
\[ \text{ad} [S_{m-1}] \left( [P, M_{f_m}] \right) = -\text{ad} [S_m] (P) \]
can be regarded as bounded operators on $L^2(\mathbb{C}^n, \mu)$.

Proposition 3.3 Let $g \in L^\infty(\mathbb{C}^n)$ and $T_m := \{T_{f_1}, \ldots, T_{f_m}\}$ with $f_j \in SP_{Lip}(\mathbb{C}^n)$. Then
\[ \text{ad} [T_m] \left( T_g \right) \in L(H^\infty(\mathbb{C}^n)) \]
is well-defined. More precisely, with $S_m := \{M_{f_1}, \ldots, M_{f_m}\}$ it holds:
\[ \text{ad} [T_m] \left( T_g \right) \in A_P \left\{ \text{ad} [N] (P), M_g : \text{with } N \subset S_m \right\} \] (3.5)
and $\text{ad} [T_m] \left( T_g \right)$ has a bounded extension $C(T_m, g)$ to $H^2(\mathbb{C}^n, \mu)$. Moreover, the symbols map
\[ L^\infty(\mathbb{C}^n) \ni g \mapsto C(T_m, g) \in \mathcal{L}(H^2(\mathbb{C}^n, \mu)) \] (3.6)
is continuous between Banach spaces.

Proof: By Proposition 3.1 the iterated commutators $\text{ad} [T_m] \left( T_g \right)$ are well-defined. It is a straightforward computation that:
\[ \text{ad} [T_1] \left( T_g \right) = P \left[ \left[ P, M_{f_1} \right], \left[ P, M_g \right] \right] P \]
which proves (3.5) in the case $m = 1$. By induction assume $\text{ad}[T_j](T_g)$ has the form:

$$\text{ad}[T_j](T_g) = \sum_{l \in \mathcal{I}} P A_l M_g B_l P$$

(3.7)

where $\mathcal{I}$ is a finite index set, $I$ the identity operator and

$$A_l, B_l \in \mathcal{A}(S_j) := \mathcal{A}\{\text{ad}[\mathcal{N}](P), I : \text{ with } \mathcal{N} \subset S_j\}.$$  

(3.8)

Then it follows that:

$$\text{ad}[T_{j+1}](T_g) = \sum_{l \in \mathcal{I}} [T_{f_{j+1}}, P A_l M_g B_l P].$$

To prove (3.7) in the case $j + 1$ it is sufficient to show for all $l \in \mathcal{I}$ the existence of a finite set $\tilde{\mathcal{I}} \subset \mathbb{N}$ and operators $C_k, D_k \in \mathcal{A}(S_{j+1})$ such that

$$[T_{f_{j+1}}, P A_l M_g B_l P] = \sum_{k \in \tilde{\mathcal{I}}} P C_k M_g D_k P.$$  

(3.9)

Note that (3.9) follows from $T_{f_{j+1}} P A_l M_g B_l P = P M_{f_{j+1}} P A_l M_g B_l P$ and

$$[M_{f_{j+1}}, Q] \in \mathcal{A}(S_{j+1})$$

for $Q \in \{P, A_l, B_l\}$. The continuity of (3.6) is a direct consequence of (3.7).

As an immediate consequence of Proposition 3.2 we remark:

**Lemma 3.2** Let $f \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ and $\mathcal{D}(T_f)$ as in (3.1). Then the Toeplitz operator $T_f$ is densely defined and closed on $\mathcal{D}(T_f)$.

**Proof:** Because of $f \in T(\mathbb{C}^n)$ it follows that $T_f$ is densely defined. Moreover,

$$M_f = T_f + [M_f, P] : \mathcal{D}(T_f) \subset H^2(\mathbb{C}^n, \mu) \rightarrow L^2(\mathbb{C}^n, \mu).$$

(3.10)

Proposition 3.2 with $j = 1$ shows that the commutator $[M_f, P]$ has a continuous extension to $H^2(\mathbb{C}^n, \mu)$. Choose a sequence $(h_n)_{n \in \mathbb{N}} \subset \mathcal{D}(T_f)$ such that:

(i) $\lim_{n \to \infty} h_n = h \in H^2(\mathbb{C}^n, \mu),$

(ii) $\lim_{n \to \infty} T_f h_n = g \in H^2(\mathbb{C}^n, \mu).$

Then we conclude from the continuity of $[M_f, P]$ and (3.10) that

$$fh = \lim_{n \to \infty} fh_n \in L^2(\mathbb{C}^n, \mu)$$

Hence $h \in \mathcal{D}(T_f)$ and $g = \lim_{n \to \infty} P(fh_n) = T_f h.$

$\square$
Let $\mathcal{T}_m := \{T_{f_1}, \cdots, T_{f_m}\}$ be a system of Toeplitz operators where $f_j \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ for $j = 1, \cdots, n$. From Lemma 3.2 it follows that the domains $\mathcal{D}(T_{f_j})$ are closed with respect to the graph norm $\| \cdot \|_{\text{gr}} := \| \cdot \| + \| T_{f_j} \cdot \|$. Consider $D_j \subset H^2(\mathbb{C}^n, \mu)$ defined by:

$$D_j := \| \cdot \|_{\text{gr}} - \text{closure of } \text{H}_{\exp}(\mathbb{C}^n) \text{ in } \mathcal{D}(T_{f_j}).$$

If we consider $T_{f_j}$ as a closed operator on $D_j$ we can define a scale of algebras (2.2) by commutator methods with the system $\mathcal{S}_m$. By Lemma 2.1 with $D := \text{H}_{\exp}(\mathbb{C}^n)$ our result in Proposition 3.3 can be formulated as follows:

**Theorem 3.1** The symbol map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi_{\infty}^{S_m}$ is well-defined and continuous.

Note that an application of Theorem 2.2 in the case of $\mathcal{V} := \mathcal{S}_m$ gives a regularity result for Fredholm Toeplitz operators with bounded symbols.

## 4 Toeplitz $\Psi^*$-algebras via the Segal-Bargmann representation

There is a unitary representation of the Heisenberg group $\mathbb{H}_n$ in $\mathcal{L}(L^2(\mathbb{C}, \mu))$. By identifying $\mathbb{H}_n$ with $\mathbb{C}^n \times \mathbb{R}$ the group law is given by, [10]:

$$(z, t) \ast (w, s) := (z + w, t + s + 2^{-1} \text{Im}(w, z)).$$

For $z \in \mathbb{C}^n$ and $f \in L^2(\mathbb{C}^n, \mu)$ we define the operator $W_z f := k_z \cdot f \circ \tau_z$. It follows by an easy calculation:

**Lemma 4.1** $H^2(\mathbb{C}^n, \mu)$ is an invariant subspace for all $W_z$ where $z \in \mathbb{C}^n$. Moreover,

1. $W_z$ is unitary with $W_z^* = W_{-z} = W_z^{-1}$,
2. The commutator $\text{ad}[P] W_z$ vanishes,
3. For $z, w \in \mathbb{C}^n : W_z W_w = \exp(i \text{Im}(w, z)) W_{z+w}$.

By Lemma 4.1 a unitary representation $\tilde{\rho} : \mathbb{H}_n \to \mathcal{L}(L^2(\mathbb{C}^n, \mu))$ of $\mathbb{H}_n$ is given by:

$$\tilde{\rho}(z, t) := e^{it} W_{\frac{z}{\sqrt{2}}}.$$

Moreover, the restriction of $\tilde{\rho}(z, t)$ to $H^2(\mathbb{C}^n, \mu)$ gives rise to a unitary representation $\rho$ of $\mathbb{H}_n$ in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$. It is well-known that $\rho$ is irreducible and strongly continuous and it is referred to as Segal-Bargmann representation, c.f. [10].

For any $A \in B := \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ we define the map:

$$\Phi_A : \mathbb{H}_n \to B$$

$$\begin{array}{c}
(z, t) \mapsto \rho(z, t) A \rho(z, t)^{-1} = W_{\frac{z}{\sqrt{2}}} A W_{\frac{z}{\sqrt{2}}}^{-1}.
\end{array}$$

(4.1)
In particular, note that for \( f \in L^\infty(\mathbb{C}^n) \)
\[
\Phi_{T_f}(z, t) = T_{f \circ \tau_{-\frac{t}{2}}}.
\]

For \( k \in \mathbb{N} \cup \{ \infty \} \) we consider the \( C^k \)-elements
\[
\Psi^k := \{ A \in B : \Phi_A \in C^k(\mathbb{H}_n, B) \}
\]
defined via \( \rho \). To any \( z \in \mathbb{C}^n \) we associate \( \varphi_A^z : \mathbb{R} \rightarrow B \) by \( \varphi_A^z(s) := W_{sz}AW_{-sz} \). According to (4.1) it follows that:
\[
\Psi^k = \bigcap_{z \in \mathbb{C}^n} \Psi^{k,z}
\]
where \( \Psi^{k,z} := \{ A \in B : \varphi_A^z \in C^k(\mathbb{R}, B) \} \).

Here we characterize the \( C^k \)-Toeplitz operators (i.e. the Toeplitz operators \( T_f \in \Psi^k \)) in terms of their symbols. We use a characterization of \( \Psi^\infty \) by commutator conditions and apply our results of the previous section.

For all \( z \in \mathbb{C}^n \) the map \( (W_{sz})_{s \in \mathbb{R}} \subset B \) defines a strongly continuous unitary group. By \( V^z \) we denote its infinitesimal generator with domain of definition:
\[
\mathcal{D}(V^z) := \{ h \in H^2(\mathbb{C}^n, \mu) : V^z h := \lim_{s \rightarrow 0} s^{-1}(W_{sz} - I)h \text{ exists} \}.
\]

By Stone's Theorem \( iV^z \) is selfadjoint and associated to \( V^z := [iV^z] \) there is a scale:
\[
B := \Psi^V_0 \supset \cdots \supset \Psi^V_n \supset \cdots \supset \Psi^V_{\infty} := \bigcap_{k \in \mathbb{N}} \Psi^V_k
\]
of algebras in \( B \) defined by commutator methods with \( V^z \) as it was described in (2.2) of section 2.1. In particular, \( \Psi^V_\infty \) is a \( \Psi^* \)-algebra and it is well-known that (4.2) and (4.3) are related as follows, see [16]:

**Proposition 4.1** For \( z \in \mathbb{C}^n \) let \( V^z := [iV^z] \) then:

(i) \( \Psi^{k,z} \subset \Psi^V_k \) for \( k \in \mathbb{N} \),

(ii) \( \Psi^V_{k+1} \subset \Psi^{k,z} \) for \( k \in \mathbb{N}_0 \) and \( \Psi^V_{\infty} = \Psi^V_{\infty,z} \).

Using the fact that convergence in \( H^2(\mathbb{C}^n, \mu) \) implies uniformly compact convergence on \( \mathbb{C}^n \) we can calculate \( V^z \) explicitly. Let \( h \in \mathcal{D}(V^z) \) and \( w \in \mathbb{C}^n \):
\[
[V^z h](w) = \frac{d}{ds} [k_{sz}(w) h(w - sz)]_{s=0} = \{ \langle w, z \rangle - \sum_{j=1}^{n} z_j \frac{\partial}{\partial w_j} \} h(w).
\]

It easily can be seen that all the monomials \( m_\alpha(z) := z^\alpha \) for \( \alpha \in \mathbb{N}_0^n \) are contained in the domain \( \mathcal{D}(V^z) \). Moreover, from the standard identities \( M_{w_j} := T_{w_j} \) and \( \frac{\partial}{\partial w_j} := T_{\overline{w_j}} \) it follows that the restriction of \( V^z \) to \( \mathbb{P}_a[\mathbb{C}^n] \) coincides with an unbounded Toeplitz operator:
\[
V^z p := T_{\langle \cdot, z \rangle - (z, \cdot)} p = 2i T_{\text{Im} \langle \cdot, z \rangle} p, \quad p \in \mathbb{P}_a[\mathbb{C}^n].
\]
In the following we write:

$$g_z := 2i \text{Im} \langle \cdot, z \rangle$$

for the symbol of the Toeplitz operator appearing above. Consider the space $D(T_{g_z})$ with the graph norm $\| \cdot \|_{gr} := \| \cdot \| + \| T_{g_z} \cdot \|$. By Lemma 3.2 it follows that $(D(T_{g_z}), \| \cdot \|_{gr})$ is a Banach space containing $\mathbb{P}_a[\mathbb{C}^n]$ and $H_{\exp}(\mathbb{C}^n)$.

**Lemma 4.2** For all $z \in \mathbb{C}^n$ the embedding $\mathbb{P}_a[\mathbb{C}^n] \hookrightarrow H_{\exp}(\mathbb{C}^n)$ is dense with respect to the graph norm topology. Moreover,

$$H_{\exp}(\mathbb{C}^n) \subset D(V^z) \cap D(T_{g_z}) \quad (4.5)$$

and the restrictions of $V^z$ and $T_{g_z}$ to $H_{\exp}(\mathbb{C}^n)$ coincide.

**Proof:** For $f \in H_{\exp}(\mathbb{C}^n)$ we can choose $c_1 \in (0, \frac{1}{2})$ and $D_1 > 0$ such that:

$$|f(w)| \leq D_1 \exp (c_1 |w|^2)$$

for all $z \in \mathbb{C}^n$. Hence, $f \in L^2(\mathbb{C}^n, \mu_r)$ for all $r \in (2c_1, 1)$. Fix $c_2, c_3$ with $2c_1 < c_2 < c_3 < 1$ and choose $D_2 > 0$ with

$$|w|^2 \leq D_2 \exp (c_3 - c_2 |w|^2)$$

for all $w \in \mathbb{C}^n$. Then we obtain for all $p \in \mathbb{P}_a[\mathbb{C}^n]$:

$$\| T_{g_z} (f - p) \|^2 \leq \| g_z (f - p) \|^2 \leq 2 |z|^2 \int_{\mathbb{C}^n} |:|^2 |f - p|^2 d\mu$$

$$\leq 2D_2 |z|^2 r^{-n} \| f - p \|_{L^2(\mathbb{C}^n, \mu_r)}^2 < \infty$$

where $r = 1 - c_3 + c_2 \in (2c_1, 1)$. Because $\mathbb{P}_a[\mathbb{C}^n]$ is dense in $L^2(\mathbb{C}^n, \mu_r) \cap \mathcal{H}(\mathbb{C}^n)$ for all $r > 0$ the first assertion follows.

Now, (4.5) immediately can be derived from $T_{g_z} p = V^z p$ for $p \in \mathbb{P}_a[\mathbb{C}^n]$ and the density result above which implies that:

$$H_{\exp}(\mathbb{C}^n) \subset \text{closure}(\mathbb{P}_a[\mathbb{C}^n], \| \cdot \|_{gr}) \subset D(V^z) \cap D(T_{g_z}).$$

Finally, we apply the continuity of $V^z, T_{g_z} : (\mathbb{P}_a[\mathbb{C}^n], \| \cdot \|_{gr}) \to H^2(\mathbb{C}^n, \mu)$. □

For $z \in \mathbb{C}^n$ we denote by $\tilde{V}^z$ the infinitesimal generator of $(W_{sz})_{s \in \mathbb{R}}$ considered as strongly continuous group of unitary operators on $L^2(\mathbb{C}^n, \mu)$. Let $\mathcal{D}(\tilde{V}^z)$ be its domain of definition, then $V^z$ can be obtained by restricting $\tilde{V}^z$ to $\mathcal{D}(V^z)$. For $f \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ and $r \in \mathbb{N}$ we write

$$\mathcal{A}_r(f) := \mathcal{A}([M_{\rho}, \cdots, M_{\rho}]) \subset \mathcal{L}(L^2(\mathbb{C}^n, \mu))$$

where the algebra on the right hand side was defined in (3.8) of Proposition 3.3.
Lemma 4.3 The domain $D(\tilde{V}^z)$ is invariant under $A \in \mathcal{A}_r( f )$ where $f$ is a linear function on $\mathbb{C}^n$. Moreover, the commutator $[ A, \tilde{V}^z ]$ vanishes as an operator on $D(\tilde{V}^z)$.

Proof: It is sufficient to show that for all $j \in \mathbb{N}$ the space $D(\tilde{V}^z)$ is invariant under the operators

$$a_j( f ) := \text{ad}^j[ M_f ]( P ).$$

Note that $L_{\exp}(\mathbb{C}^n)$ is an invariant under $W_z$ and it holds $W_z M_f W_z = M_{f \exp_z}$. Because $W_z$ commutes with $P$ it follows that:

$$W_{-z} a_j( f ) W_z = \text{ad}^j[ M_{f \exp_z} ]( P ) = a_j( f ).$$

We have used the linearity of $f$ for the second equality. Hence, the commutator $[ A, W_z ]$ vanishes for all $A \in \mathcal{A}_r( f )$. Fix $h \in D(\tilde{V}^z)$ and $A \in \mathcal{A}_r( f )$, then:

$$\frac{1}{s} \{ W_z - I \} A h = A \frac{1}{s} \{ W_z - I \} h \rightarrow A \tilde{V}^z h$$

as $s$ tends to 0. It follows that $Ah \in D(\tilde{V}^z)$ with $\tilde{V}^z Ah = A \tilde{V}^z h$. \qed

Remark 4.1 Let $W$ be any subspace of $H := H^2(\mathbb{C}^n, \mu)$ such that $H_{\exp}(\mathbb{C}^n) \subset W$. Consider the operators:

$$O_W := \{ A \in \mathcal{L}( W, H ) : H_{\exp}(\mathbb{C}^n) \text{ is an invariant space for } A \}.$$ 

Let $A \in O_W$ and assume there is $A^* \in O_W$ with $\langle Af, g \rangle = \langle f, A^* g \rangle$ for all $f, g \in W$. Because of $K(\cdot, \lambda) \in H_{\exp}(\mathbb{C}^n)$ for all $\lambda \in \mathbb{C}^n$ it follows that $A$ can be written as an integral operator with kernel:

$$K_A( z, w ) = A^* K(\cdot, z)( w ). \quad (4.6)$$

In particular, $A$ completely is determined by the restriction of $A^*$ to $H_{\exp}(\mathbb{C}^n)$. Assume that $A$ has a continuous extensions $\tilde{A}$ from $H_{\exp}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. Fix $g \in H^2(\mathbb{C}^n, \mu)$ and a sequence $(g_n)_n \subset H_{\exp}(\mathbb{C}^n)$ with $g = \lim_{n \rightarrow \infty} g_n$. Then it follows for $z \in \mathbb{C}^n$:

$$[ \tilde{A} g ]( z ) = \lim_{n \rightarrow \infty} \langle Ag_n, K(\cdot, z) \rangle = \lim_{n \rightarrow \infty} \langle g_n, A^* K(\cdot, z) \rangle = \langle g, A^* K(\cdot, z) \rangle$$

and $\tilde{A}$ is given by the same integral formula. In particular, $A$ has a (unique) extension from $W$ to $H^2(\mathbb{C}^n, \mu)$.

Let $h \in L^\infty(\mathbb{C}^n)$ and $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a linear function. We write $C_j( f, h )$ for the continuous extensions of the commutators

$$\text{ad}^j[ T_f ]( T_h ) \in \mathcal{L}( H_{\exp}(\mathbb{C}^n) )$$

to $H^2(\mathbb{C}^n, \mu)$, (note that $f \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ and Proposition 3.3).
Corollary 4.1 Let \( h \in L^{\infty}(\mathbb{C}^{n}) \). Assume that \( D(V^{z}) \) is invariant under the multiplication operator \( M_{h} \). Then \( D(V^{z}) \) is invariant under \( C_{j}(f, h) \) for all \( j \in \mathbb{N} \).

Proof: According to (3.7) there is a finite index set \( I \) and \( A_{i}, B_{i} \in \mathcal{A}_{j}(f) \) such that

\[
\text{ad}^{t}[T_{f}] (T_{h}) = \sum_{i \in I} P A_{i} M_{h} B_{i} P.
\]

Due to our assumption on \( h \) and by Lemma 4.3 the assertion follows. \( \square \)

Now, we can proof our main result on the smoothness of Toeplitz operators with respect to the Segal-Bargmann representation \( \rho \) of the Heisenberg group:

Theorem 4.1 Let \( h \in S_{\rho} := S \cap \overline{S} \) where \( \overline{S} = \{ \overline{h} : h \in S \} \) and

\[
S := \{ h \in L^{\infty}(\mathbb{C}^{n}) : \text{s. t. } D(V^{z}) \text{ is invariant under } M_{h} \text{ for all } z \in \mathbb{C}^{n} \}.
\]

Then the symbol map into the \( \Psi^{*} \)-algebra \( \Psi^{\infty} \) given by:

\[
S_{\rho} \ni h \mapsto T_{h} \in \Psi^{\infty}
\]

is well-defined and continuous if \( S_{\rho} \) carries the \( L^{\infty}(\mathbb{C}^{n}) \)-topology.

Proof: Using our notation in (4.2) and (4.3) we must show that \( T_{h} \in \Psi^{\infty,z} = \Psi^{\infty} \) for all complex directions \( z \in \mathbb{C}^{n} \) and \( V^{z} := [iv^{z}] \):

\[
D(V^{z}) \text{ is invariant under } T_{q} \text{ for } q \in \{ h, \overline{h} \} \subset S_{\rho} \text{ and by Lemma 4.2 it follows that }
\]

the commutators \( A_{1} := [iv^{z}, T_{q}] \) and \( [T_{ig_{z}}, T_{q}] \) coincide on \( H_{\exp}(\mathbb{C}^{n}) \). Because \( iv^{z} \) is self-adjoint we can define \( A_{1}^{*} := [T_{q}, iv^{z}] \) and \( W := D(V^{z}) \) in Remark 4.1. The operator \( [T_{ig_{z}}, T_{q}] \) has a bounded extension \( C_{1}(ig_{z}, q) \) from \( H_{\exp}(\mathbb{C}^{n}) \) to \( H^{2}(\mathbb{C}^{n}, \mu) \). We conclude from Remark 4.1 that \( C_{1}(ig_{z}, q) \) is an extension of \( A_{1} \) from \( W \) to \( H^{2}(\mathbb{C}^{n}, \mu) \) and \( T_{q} \in \Psi_{1}^{\infty} \).

By induction we must prove for \( j \in \mathbb{N} \):

(1) The domain of definition \( D(V^{z}) \) is invariant under \( C_{j}(ig_{z}, q) \),

(2) The commutators \( A_{j+1} := [iv^{z}, C_{j}(ig_{z}, q)] \) have the bounded extension \( C_{j+1}(ig_{z}, q) \) from \( D(V^{z}) \) to \( H^{2}(\mathbb{C}^{n}, \mu) \).

Assertion (1) is a direct consequence of Corollary 4.1 and (2) can be derived from Remark 4.1 with \( A_{j+1}^{*} := [C_{j}(ig_{z}, q)^{*}, iv^{z}] \) on \( W := D(V^{z}) \) \(^{3}\) and the fact that \( A_{j+1} \) has the continuous extension \( C_{j+1}(ig_{z}, q) \) from \( H_{\exp}(\mathbb{C}^{n}) \) to \( H^{2}(\mathbb{C}^{n}, \mu) \). The continuity of the symbols map follows from (2.3) together with the continuity of (3.6) in Proposition 3.3. \( \square \)

\(^{3}\)Note that by Corollary 4.1 and the identity \( C_{j}(ig_{z}, q)^{*} = (-1)^{j}C_{j}(ig_{z}, \overline{q}) \) the commutator \( A_{j+1}^{*} \) is well-defined on \( D(V^{z}) \).
5 Examples and Applications

Let $A$ denote the subalgebra of $\mathcal{L}(L^2(\mathbb{C}^n, \mu))$ of all multiplication operators with bounded symbols $h \in L^\infty(\mathbb{C}^n)$. For $z \in \mathbb{C}^n$ and with $\tilde{V}^z := [i\tilde{V}^z]$ there is a scale of algebras arising by commutator methods:

$$A \supset \Psi_1^{\tilde{V}^z} \supset \cdots \supset \Psi_n^{\tilde{V}^z} \supset \cdots \supset \Psi_{\infty}^{\tilde{V}^z} = \bigcap_{n \in \mathbb{N}} \Psi_n^{\tilde{V}^z}. \quad (5.1)$$

In general, the inclusions above will be proper. As an immediate consequence of Theorem 4.1 it follows for the projected scale of vector spaces:

$$A_P \supset \Psi_1^{-P}^{\tilde{V}^{\sim}} = \cdots = \Psi_{nP}^{-P}^{\tilde{V}^{\sim}} = \Psi_{n+1_P}^{-P} = \cdots = \Psi_{\infty P}^{-P}^{\tilde{V}^{\sim}}z. \quad (5.2)$$

Here $A_P \subset \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ is the space of Toeplitz operators with bounded measurable symbols. By passing from (5.1) to the scale (5.2) the underlying $C^k$-structure is lost.

We give an example of a class of bounded functions $g$ such that $D(V^z)$ is an invariant subspace for $M_g$ and $M_\mathbb{H}$ for all $z \in \mathbb{C}^n$.

**Example 5.1** Denote by $C^\infty_c(\mathbb{C}^n)$ the space of compactly supported smooth functions. For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ we write $z_j := x_j + iy_j$ and with $\alpha, \beta \in \mathbb{N}_0^n$:

$$z^{\alpha,\beta} := x^\alpha y^\beta, \quad \partial^{\alpha,\beta} := \frac{\partial^{\left|\alpha\right|}}{\partial x^\alpha} \frac{\partial^{\left|\beta\right|}}{\partial y^\beta}.$$  

Fix $h \in D(\tilde{V}^z)$ and $z \in \mathbb{C}^n$. For $g \in C^\infty_c(\mathbb{C}^n)$ (real valued) and $s \neq 0$ we write:

$$\frac{1}{s} \left[ W_{sz} - I \right] M_g h = \frac{1}{s} \left[ M_{g \circ \tau_{-sz}} - M_g \right] W_{sz} h + M_g \frac{1}{s} \left[ W_{sz} - I \right] h. \quad (5.3)$$

The second term converges in $L^2(\mathbb{C}^n, \mu)$ as $s \to 0$. Consider the smooth and compactly supported function $dg(z, \cdot) = -\langle \text{grad} g(\cdot), z \rangle_{\mathbb{R}^{2n}}$. Then:

$$C_{s,z} := \left\| \frac{1}{s} \left[ M_{g \circ \tau_{-sz}} - M_g \right] - M_{dg(z, \cdot)} \right\| \leq \sum_{\left|\alpha \right| + \left|\beta\right| = 2} \frac{|s|}{(\alpha + \beta)!} \left\| \partial^{\alpha,\beta} g \right\|_\infty \left| z^{\alpha,\beta} \right|.$$  

Hence $\lim_{s \to 0} C_{s,z} = 0$ and the right hand side of

$$\left\| \frac{1}{s} \left[ M_{g \circ \tau_{-sz}} - M_g \right] W_{sz} h - M_{dg(z, \cdot)} h \right\| \leq C_{s,z} \left\| h \right\| + \left\| dg(z, \cdot) \right\|_\infty \left\| (W_{sz} - I) h \right\|$$

tends to $0$ as $s \to 0$. It follows $gh \in D(V^z)$. With our notation of Theorem 4.1 we conclude that $C^\infty_c(\mathbb{C}^n) \subset S_s$. By the continuity of $L^\infty(\mathbb{C}^n) \subset S_s \ni h \mapsto T_h \in \Psi^\infty$

and the fact that $C^\infty_c(\mathbb{C}^n)$ is uniformly dense in the space $C_0(\mathbb{C}^n)$ of all continuous functions vanishing at infinity it follows that $\{ T_h : h \in C_0(\mathbb{C}^n) \} \subset \Psi^\infty$. 


In our second example we construct a compact operator $A \in \mathcal{B} := \mathcal{L}(H^2(\mathbb{C}, \mu))$ which is not contained in $\Psi^{1,z}$ for any $z \in \mathbb{C}$ (with our notation in (4.2)). As a consequence and using Example 5.1 $A$ is not limit point of finite sums of finite products of Toeplitz operators with symbols in $C_0(\mathbb{C})$ and with respect to the Fréchet topology of $\Psi^{\infty,z}$. However, since $A$ is compact it can be approximated by Toeplitz operators with smooth and compactly supported symbols in the topology of $\mathcal{B}$, c.f. [8].

**Example 5.2** For $j \in \mathbb{N}_0$ let $P_j \in \mathcal{B}$ be the rank one projection onto $\text{span}\{m_j := z^j\}$. With a sequence $a := (a_n)_{n \in \mathbb{N}}$ tending to zero consider the compact diagonal operator:

$$A := \sum_{j \in \mathbb{N}} a_j P_j \in \mathcal{B}.$$ 

With $z \in \mathbb{C}$, $|z| = 1$ and $g_z := 2i \text{Im} \langle \cdot, z \rangle$ we compute $[T_{g_z}, A] m_j = [V^z, A] m_j$ explicitly for all $j \in \mathbb{N}$. By (4.4) one obtains that:

$$[T_{g_z}, A] m_j = a_j T_{g_z} m_j - A[\bar{z} m_{j+1} - j z m_{j-1}]$$

$$= a_j (\bar{z} m_{j+1} - j z m_{j-1}) - (a_{j+1} \bar{z} m_{j+1} - j a_{j-1} z m_{j-1})$$

$$= (a_j - a_{j+1}) \bar{z} m_{j+1} - j z (a_j - a_{j-1}) m_{j-1}.$$ 

With $e_j := (j!)^{-\frac{1}{2}} z^j$ we have $\langle e_j, e_l \rangle_2 = \delta_{i,j}$ for all $j, l \in \mathbb{N}$. Hence it follows that

$$\| [T_{g_z}, A] e_j \|_2^2 = (j + 1) |a_j - a_{j+1}|^2 + j |a_j - a_{j-1}|^2. \quad (5.4)$$

We choose $a$ such that the right hand side of (5.4) tends to infinity for $j \to \infty$. This can be done by the choice of an oscillating sequence $a_j := (-1)^j j^{-\frac{1}{4}}$. Then it follows

$$(j + 1) |a_j - a_{j+1}|^2 = (j + 1) |j^{-\frac{1}{4}} + (j + 1)^{-\frac{1}{4}}|^2 \geq \sqrt{j+1}$$

and so the right hand side of (5.4) is unbounded for $j \to \infty$. Hence $[T_{g_z}, A]$ has no bounded extension to $H^2(\mathbb{C}, \mu)$ and $A \notin \Psi^{1,z}$ by Proposition 4.1.

Let $\beta : L^2(\mathbb{R}^n) \to H^2(\mathbb{C}^n, \mu)$ denote the Bargmann isometrie, c.f. [10]. Our results on Toeplitz operators on $H^2(\mathbb{C}^n, \mu)$ can be used in the analysis of a class of *Gabor-Daubechies windowed localization operators* $L_h := \beta^{-1} T_h \beta$ on $L^2(\mathbb{R}^n)$ where $h \in L^\infty(\mathbb{C}^n)$, c.f. [9]. It was remarked in [14] the operator $L_h$ can be considered as a pseudodifferential operator $W_{\sigma(h)}$ in Weyl quantization with *Weyl symbol* $\sigma(h)$ on $\mathbb{R}^{2n}$. Via the identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ the correspondence between $h$ and $\sigma(h)$ can be expressed in terms of the heat equation on $\mathbb{R}^{2n}$. More precisely, $\sigma(h)$ is a solution with initial data $h$ at a fixed time $t_0 > 0$. In the next example we describe how the operators introduced in the previous sections transform under $\beta$, c.f. [10].

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4Here the window is a Hermite function on $\mathbb{R}^n$
Example 5.3 For \( u \in L^2(\mathbb{R}^n) \) it is well-known that \( \beta u \) can be expressed by the integral:
\[
[\beta u](z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) \exp \left\{ \langle x, z \rangle - \frac{1}{4} |x|^2 - \frac{1}{2} \langle z, \overline{z} \rangle \right\} dx.
\]

Fix \( a = p + iq \in \mathbb{C}^n \), then it can be checked that \( W_a \in L(H^2(\mathbb{C}^n, \mu)) \) transform as:
\[
B_a u := [\beta^{-1}W_a \beta](u) = u(\cdot - 2p) \exp \{ iq(p - \cdot) \}.
\]

In particular, in the case \( q = 0 \) the unitary operator \( B_a \) is a usual shift in direction \( 2p \).

For \( j = 1, \ldots, n \) it is readily verified that \( T_{z_j} \) and \( T_{\overline{z}_j} \) transform in the following way:

(i) \( \beta^{-1}T_{z_j}\beta = \frac{1}{2}x_j - \partial_{x_j} \),
(ii) \( \beta^{-1}T_{\overline{z}_j}\beta = \frac{1}{2}x_j + \partial_{x_j} \).

From (i), (ii) and for \( \alpha \in \mathbb{N}_0^n \) one obtains the identity:
\[
\beta \partial_x^\alpha = (-1)^{|\alpha|}T_{i\text{Im} z_1}^\alpha \cdots T_{i\text{Im} z_n}^\alpha \beta =: (-1)^{|\alpha|}T_{i\text{Im} z}^\alpha \beta.
\]

Let \( g \in D(\mathbb{R}^n) \) be a test function and fix \( f \in H_{\exp}(\mathbb{C}^n) \). It follows that:
\[
\langle \beta^{-1}f, \partial_x^\alpha g \rangle_{L^2(\mathbb{R}^n)} = \langle f, \beta \partial_x^\alpha g \rangle = \langle \beta^{-1}T_{i\text{Im} z_1}^\alpha \cdots T_{i\text{Im} z_n}^\alpha f, g \rangle_{L^2(\mathbb{R}^n)}.
\]

Here we have used the fact that \( H_{\exp}(\mathbb{C}^n) \) is invariant under all unbounded Toeplitz operators \( T_{i\text{Im} z_j} \) which was proved in Proposition 3.1. It follows that:
\[
\mathcal{D} := \beta^{-1}[H_{\exp}(\mathbb{C}^n)] \subset H^\infty(\mathbb{R}^n) = \bigcap_{k \in \mathbb{N}} H^k(\mathbb{R}^n)
\]
where \( H^s(\mathbb{R}^n) \) denotes the \( k \)-th Sobolev space. Hence, for \( \alpha, \beta \in \mathbb{N}_0^n \) the restriction of (2.1) in Theorem 2.1 to \( \mathcal{D} \):
\[
\text{ad}[-ix]^\alpha \text{ad}[i\partial_x]^\beta(B) : \mathcal{D} \to \mathcal{D}
\]
is well-defined for any \( B \in L(\mathcal{D}) \). With the choice \( h \in L^\infty(\mathbb{C}^n) \) and \( L_h := \beta^{-1}T_h\beta \in L(\mathcal{D}) \) we obtain by conjugating (5.5) with \( \beta \) and using (i), (ii) above:
\[
\text{ad}[iT_{2\text{Re} z_j}]^\alpha \text{ad}[T_{i\text{Im} z_j}]^\beta(T_h) : H_{\exp}(\mathbb{C}^n) \to H_{\exp}(\mathbb{C}^n).
\]

It follows by Proposition 3.3 that the operators in (5.6) have bounded extensions to \( H^2(\mathbb{C}^n, \mu) \) and so (5.5) can be extended continuously to \( L^2(\mathbb{R}^n) \). Hence we have proved a weaker version of the defining property (2.1) for \( \Psi_{\rho, \delta}^{0} \) in Theorem 2.1.

Since the Gaussian measure \( \mu \) is invariant under unitary transformations of \( \mathbb{C}^n \), there is a natural group representation of \( U_n \) in \( \mathcal{L}(H^2(\mathbb{C}^n, \mu)) \) generating \( \Psi^*-\text{algebras} \) of smooth elements. As a final example we want to remark:
Example 5.4 Let $A \in \mathbb{R}^{n \times n}$ be self-adjoint and consider the unitary group:

$$\mathbb{R} \ni t \mapsto e^{itA} \in U_n.$$ 

The group of unitary composition operators $C_t f := f \circ e^{itA}$ on $H^2(\mathbb{C}^n, \mu)$ can be shown to be strongly continuous, cf. [3]. The restriction of the infinitesimal generator $L_A$ of $(C_t)_{t \in \mathbb{R}}$ to $\mathbb{P}_a[\mathbb{C}^n]$ coincides with an (unbounded) Toeplitz operator. More precisely, it was shown in [3] that:

$$L_AP = \left[ T_{(A_x,x)} - n \cdot \text{trace}(A) \right] p, \quad p \in \mathbb{P}_a[\mathbb{C}^n].$$ 

Hence, in general the symbol of $L_A$ regarded as a Toeplitz operator is a polynomial of degree 2, which is not globally lipschitz continuous on $\mathbb{C}^n$. Proposition 3.3 cannot be applied in this situation and the smoothness of a Toeplitz operator $T_f$ with bounded symbols $f$ with respect to $(C_t)_t$ requires further assumption on the symbol $f$. For a more detailed calculation we refer to [3].

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References


TOEPLITZ $\Psi^*$-ALGEBRAS


