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Kyoto University
Toeplitz $\Psi^*$-algebras via unitary group representations

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Abstract

As it was pointed out in [12] there are construction methods for spectral invariant Fréchet operator algebras such as $\Psi^*$- and $\Psi_0$-algebras in the bounded operators on a Hilbert space having prescribed properties. For the Segal-Bargmann space $H$ and using systems of unbounded closable Toeplitz operators $T_f$ where $f$ is in a certain class $\operatorname{SP}_{\operatorname{Lip}}(\mathbb{C}^n)$ of symbols we show that these algebras contain all Toeplitz operators $T_h$ with $h \in L^\infty(\mathbb{C}^n)$. Let $\rho$ be the Segal-Bargmann representation of the Heisenberg group $\mathbb{H}_n$ in the bounded operators on $H$. As an application of our results above we characterize a class of smooth Toeplitz operators in the $\Psi^*$-algebra of smooth elements with respect to $\rho$.

1 Introduction

Subsequent to the results in [12] it frequently has been remarked that the abstract concept of (locally) spectral invariant Fréchet algebras such as $\Psi_0$- and $\Psi^*$-algebras successfully can be applied to the structural analysis of certain algebras of pseudo-differential operators. Applications arise in complex analysis, analytic perturbation theory of Fredholm operators and non-abelian cohomology for analyzing isomorphisms of abelian groups in $K$-theory. By generalizing a characterization of the Hörmander classes $\Psi_{\rho,\delta}^0$ \footnote{The author was supported by a JSPS postdoctoral fellowship (PE 05570) for North American and European Researchers.} by commutator conditions (see Theorem 2.1) a construction method for algebras of the above mentioned type with prescribed properties have been given in [12].
Let $H := H^2(\mathbb{C}^n, \mu)$ be the Segal-Bargmann space of Gaussian square integrable entire functions on $\mathbb{C}^n$. We denote by $P$ the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto $H$ and we write $M_f$ for the multiplication with a measurable symbol $f$. In the initial stage of this paper we consider iterated commutators of closable Toeplitz operators $T_f := PM_f$ on $H$ having symbols in a certain class $\text{SP}_\text{Lip}(\mathbb{C}^n)$ of measurable and in general unbounded functions on $\mathbb{C}^n$. For a system $S_m := \{T_{f_1}, \cdots, T_{f_m}\}$ of operators with $f_j \in \text{SP}_\text{Lip}(\mathbb{C}^n)$ and in the sense of [12] the $\Psi_0$-algebra $\Psi^\infty_\infty$ in the bounded operators $\mathcal{L}(H)$ on $H$ can be defined by commutator methods with respect to $S_m$. We show that $\Psi^\infty_\infty$ contains all Toeplitz operators with bounded measurable symbols. More precisely:

**Theorem A** The symbols map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi^\infty_\infty$ is well-defined and continuous.

Let $\mathbb{H}_n$ be the Heisenberg group and $\alpha$ be the Segal-Bargmann representation of $\mathbb{H}_n$ in $\mathcal{L}(H)$, c.f. [10]. The map $\alpha$ is well-known to be unitary, irreducible and strongly continuous. In particular, the $\Psi^*$-algebra $\Psi^\infty(\mathbb{H}_n) \subset \mathcal{L}(H)$ of smooth elements with respect to $\alpha$ arise in a natural way and it can be characterized by commutator methods. We describe a symmetric subspace $S_s \subset L^\infty(\mathbb{C}^n)$ with the induced topology such that:

**Theorem B** The symbols map $S_s \ni h \mapsto T_h \in \Psi^\infty(\mathbb{C}^n)$ is well-defined and continuous.

This result can be stated in terms of the algebra construction. Let $\mathcal{A}$ be the algebra of multiplication operators on $V := L^2(\mathbb{C}^n, \mu)$ with bounded measurable symbols. In a natural way $\alpha$ extends to a representation of $\mathbb{H}_n$ into $\mathcal{L}(V)$ and the corresponding operator algebras $\Psi^k(\mathcal{A}, \mathbb{H}_n)$ of $C^k$-elements in $\mathcal{A}$ form a decreasing scale. Note that $M_f \in \Psi^k(\mathcal{A}, \mathbb{H}_n)$ is related to the smoothness of the symbols $f \in L^\infty(\mathbb{C}^n)$. Clearly, $\mathcal{A}$ projects under $P$ onto the space $\mathcal{A}_P := PA \mathcal{A} P$ of Toeplitz operators with bounded symbols. Theorem B states:

$$P \Psi^k(\mathcal{A}, \mathbb{H}_n) P = P \Psi^{k+1}(\mathcal{A}, \mathbb{H}_n) P \subset \mathcal{L}(H) \quad \text{for all } k \in \mathbb{N}.$$ 

Heuristically, the smoothness of $f$ cannot be recovered by commutator methods from the Toeplitz operator $T_f$. We want to remark here that these results are related to an observation in [14], [3]. Let $\beta : L^2(\mathbb{R}^n) \to H$ be the Bargmann isometry and $f$ a bounded measurable function on $\mathbb{C}^n$. The assignment $\beta^{-1}T_f\beta$ can be shown to be a pseudo-differential operator $W_{\sigma(f)}$ on $L^2(\mathbb{R}^n)$ in its Weyl quantization. By identifying $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ the Weyl symbol $\sigma(f)$ and $f$ are related via the heat equation on $\mathbb{R}^{2n}$. There is $t_0 > 0$ such that:

$$\sigma(f) = e^{-t_0 \Delta} f := \text{solution of the heat equation with initial data } f \text{ at the time } t_0.$$ 

Moreover, $\sigma$ maps the space of continuous functions with compact support into the symbol class $S^{\infty}_{0, \delta}$, $0 \leq \delta < \rho \leq 1$ and $\delta < 1$. Corresponding to Theorem A and B it can be checked that $f \mapsto \sigma(f)$ is continuous with respect to the $L^\infty(\mathbb{C}^n)$ topology and the usual Fréchet topology on $S^{\infty}_{\rho, \delta}$.

In our first section we remind of some basic definitions and results related to the construction of $\Psi_0$- and $\Psi^*$-algebras. For Toeplitz operators having symbols of polynomial growth at infinity an invariant subspace $H_{\exp}(\mathbb{C}^n)$ of $H$ is defined in section 3. Moreover,
the existence of bounded extensions for a class of iterated commutators of Toeplitz operators on $\mathrm{H}_{\exp}(\mathbb{C}^n)$ and Theorem A are proved. Section 4 contains the proof of Theorem B and finally we have added some examples and applications in section 5.

# 2 Fréchet operator algebras with prescribed properties

The following definition due to B. Gramsch have been given in [11]:

**Definition 2.1** Let $\mathcal{B}$ be a Banach-algebra with unit $e$ and let $\mathcal{F}$ be a continuously embedded Fréchet algebra in $\mathcal{B}$ with $e \in \mathcal{F}$. Then $\mathcal{F}$ is called $\Psi_0$-algebra if it is locally spectral invariant in $\mathcal{B}$, i.e. there is $\varepsilon > 0$ with

$$\{ a \in \mathcal{F} : \| e - a \|_{\mathcal{B}} < \varepsilon \} \subset \mathcal{F}^{-1}.$$  

Moreover, one defines:

- If $\mathcal{B}$ is a $C^*$-algebra and $\mathcal{F}$ is a symmetric $\Psi_0$-algebra in $\mathcal{B}$, then $\mathcal{F}$ is called $\Psi^*$-algebra. ($\mathcal{F}$ automatically is spectral invariant, i.e. $\mathcal{F} \cap \mathcal{B}^{-1} = \mathcal{F}^{-1}$).

- If the topology of $\mathcal{F}$ is generated by a system $\{ q_j : j \in \mathbb{N} \}$ of sub-multiplicative semi-norms with $q_j(e) = 1$ for $j \in \mathbb{N}$, then $\mathcal{F}$ is called sub-multiplicative or locally $m$-convex (E. Michael, 1952) $\Psi_0$- resp. $\Psi^*$-algebra.

The concept of $\Psi^*$- and $\Psi_0$-algebras allows to treat phenomenas of local structure. As it was observed for algebras of Pseudo-differential operators, $C^\infty$-properties such as pseudo- or micro-locality are preserved by taking closures in the Fréchet topology. Important examples of $\Psi^*$-algebras are given by the Hörmander classes $\Psi^0_{\rho,\delta}$ of zero order where $\mathcal{B} := \mathcal{L}(L^2(\mathbb{R}^n))$. It is known that $\Psi^0_{\rho,\delta}$ can be described in terms of commutator conditions.

**Theorem 2.1** (R. Beals, '77, [6])

An operator $B : S(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is of class $\Psi^0_{\rho,\delta}$ iff for $\alpha, \beta \in \mathbb{N}_0^n$ all iterated commutators:

$$\text{ad}[-ix]^\alpha \text{ad}[i\partial_x]^\beta(B) : H^{s-\rho|\alpha|+\delta|\beta|} \rightarrow H^s$$  \hspace{1cm} (2.1)

admit bounded extensions between suitable Sobolev spaces to $L^2(\mathbb{R}^n)$.

On the one hand the spectral invariance of $\Psi^0_{\rho,\delta}$ follows from the commutator characterizations in Theorem 2.1, see [19], [20]. On the other hand, by replacing $ix$ and $i\partial_x$ above with a system of closable and densely defined operators, conditions of the type (2.1) have been used to define (submultiplicative) $\Psi_0$-algebras in a fairly general situation, see [12]. Below we give the definitions and remind of some basic results.

\[20 \leq \delta \leq \rho \leq 1 \text{ and } \delta < 1\]
2.1 Commutator Methods

Given a topological vector space $X$ we write $L(X)$ (resp. $\mathcal{L}(X)$) for the linear (resp. bounded linear) operators on $X$.

**Definition 2.2** (Iterated commutators)

For a system $S_m := [A_1, \cdots, A_m]$ where $A_j, B \in L(X)$ we call $m$ the length of $S_m$. We inductively define the iterated commutators $\operatorname{ad}[ \emptyset ](B) := B$ and:

- $\operatorname{ad}[S_j](B) := [A_j, B] = A_j B - BA_j$,
- $\operatorname{ad}[S_j+1](B) := \operatorname{ad}[A_{j+1}](\operatorname{ad}[S_j](B))$ for $j = 1, \cdots, m-1$.

In the case of $A = A_j$ where $j = 1, \cdots, m$ we also write:

- $\operatorname{ad}^0[A](B) := B$ and $\operatorname{ad}^m[A](B) := \operatorname{ad}[S_m](B)$.

With these notations it follows for finite systems $S_j$ and $S_k$ in $L(X)$:

$$\operatorname{ad}[S_j](\operatorname{ad}[S_k](B)) = \operatorname{ad}[S_k, S_j](B).$$

Let $H$ be a Hilbert space and $\mathcal{F} \subset \mathcal{L}(H)$ be a sub-multiplicative $\Psi^*$-algebra. Assume that the topology of $\mathcal{F}$ is generated by a sequence $(\| \cdot \|_{\mathcal{L}(H)})_{j \in \mathbb{N}}$ of semi-norms and without lost of generality let $q_0 := \| \cdot \|_{\mathcal{L}(H)}$. Given a finite system $\mathcal{V}$ of closed and densely defined operators $A : H \supset D(A) \to H$ and following [12] we define:

- $\mathcal{I}(A) := \{ a \in \mathcal{F} : a(D(A)) \subset D(A) \}$,
- $\mathcal{B}(A) := \{ a \in \mathcal{I}(A) : [A, a] \text{ extends to an element } \delta_A(a) \in \mathcal{F} \}$.

Inductively, one obtains:

- $\Psi_0^\mathcal{V} := \mathcal{F}$, with semi-norms $q_{0,j} := q_j$ for $j \in \mathbb{N}$,
- $\Psi_1^\mathcal{V} := \bigcap_{A \in \mathcal{V}} \mathcal{B}(A)$,
- $\Psi_k^\mathcal{V} := \{ a \in \Psi_{k-1}^\mathcal{V} : \delta_A a \in \Psi_{k-1}^\mathcal{V} \text{ for all } A \in \mathcal{V} \}$ where $k \geq 2$,
- $\Psi_\infty^\mathcal{V} := \bigcap_{k \in \mathbb{N}} \Psi_k^\mathcal{V}$.

This process leads to a decreasing scale of algebras in $\mathcal{F}$:

$$\mathcal{F} = \Psi_0^\mathcal{V} \supset \cdots \supset \Psi_n^\mathcal{V} \supset \Psi_{n+1}^\mathcal{V} \supset \cdots \supset \Psi_\infty^\mathcal{V} := \bigcap_{k \in \mathbb{N}} \Psi_k^\mathcal{V}. \quad (2.2)$$

For $n \geq 1$, we inductively define a system $(q_{n,j})_{j \in \mathbb{N}}$ (resp. $(q_{n,j})_{j,n \in \mathbb{N}}$) of norms on $\Psi_n^\mathcal{V}$ (resp. on $\Psi_\infty^\mathcal{V}$) by:

$$q_{n,j}(a) := q_{n-1,j}(a) + \sum_{A \in \mathcal{V}} q_{n-1,j} (\delta_A a). \quad (2.3)$$
According to [12], $\Psi^V_\infty$ is a sub-multiplicative $\Psi_0$-algebra in $\mathcal{F}$. In the case where each $A \in \mathcal{V}$ is symmetric we replace $B(A)$ by:

$$B^\ast(A) := \{ a \in B(A) : a^\ast \in B(A) \}.$$ 

Then the algebras $\Psi^V_n$ are symmetric and $\Psi^V_\infty$ is a $\Psi^\ast$-algebra in $\mathcal{L}(H)$. Let $D \subset H$ be a core for $\mathcal{V}$, i.e. the inclusion $D \hookrightarrow \mathcal{D}(A)$ is dense with respect to the graph norm for all $A \in \mathcal{V}$. Then it was shown in [2], [3]:

**Proposition 2.1** Assume that $a \in \mathcal{F}$ and property $(E_k)$ holds for $k \in \mathbb{N} \cup \{\infty\}$:

$(E_k)$: $D$ is invariant under all $A \in \mathcal{V}$ and $a \in \mathcal{F}$. Moreover, assume that for any system

$$A \subset S_k(\mathcal{V}) := \{ [A_1, \cdots, A_j] : \text{where } A_l \in \mathcal{V} \text{ and } 1 \leq l \leq j \leq k \}.$$ 

$\text{ad}[A](a) : H \supset D \rightarrow H$ has a continuous extensions to $C(A,a) \in \mathcal{F}.$

Then $a \in \Psi^V_k$ and $C(A,a)$ is a bounded extension of ad $[A](a) : H \subset \mathcal{D}(A) \rightarrow H \text{ to } H$ for any operator $A \in \mathcal{V}$.

The (locally) spectral invariance of $A \subset B$ is preserved under projections $p = p^2 \in A$. It is readily verified that $A_p := pA p$ is (locally) spectral invariant in $B_p := B p$. If in addition $B$ is a $C^\ast$-algebra, $A$ is symmetric in $B$ and $p = p^\ast$, then $A_p$ is symmetric and spectral invariant in $B_p$.

With (2.2) and an orthogonal projection $p \in \Psi^V_n$, $n \in \mathbb{N} \cup \{\infty\}$ from $H$ onto a closed subspace $H_0 \subset H$ there is a scale of projected algebras in $\mathcal{L}(H_0)$:

$$\mathcal{L}(H_0) \supset \mathcal{F}_p = \Psi^V_0 \supset \cdots \Psi^V_{n-1} \supset \Psi^V_n.$$  \hspace{1cm} (2.4)

It can be shown that (2.4) arises by commutator methods with a system $\mathcal{V}_p$ of closed operators on $H_0$ where $\mathcal{D}(A_p) := p[\mathcal{D}(A)]$ and

$$\mathcal{V}_p := \{ A_p := pAp : H_0 \supset \mathcal{D}(A_p) \rightarrow H_0 : A \in \mathcal{V} \}.$$

Defining (2.4) by commutator conditions with respect to $\mathcal{V}_p$ only requires that $p \in \Psi^V_1$. Thus this method gives a natural extension of (2.4) to an infinite scale for $n \in \mathbb{N}$.

There is a corresponding scale of $\mathcal{V}$-Sobolev spaces in $H$:

- $\mathcal{H}_{0}^\mathcal{V} := H$ with the norm $p_0 := \| \cdot \|_H$.
- $\mathcal{H}_{1}^\mathcal{V} := \bigcap_{A \in \mathcal{V}} \mathcal{D}(A)$.
- $\mathcal{H}_{k}^\mathcal{V} := \{ x \in \mathcal{H}_{k-1}^\mathcal{V} : Ax \in \mathcal{H}_{k-1}^\mathcal{V} \text{ for all } A \in \mathcal{V} \}, \ k \geq 2.$
- $\mathcal{H}_{\infty}^\mathcal{V} := \bigcap_{k \in \mathbb{N}} \mathcal{H}_{k}^\mathcal{V}.$
We endow $\mathcal{H}_{\mathcal{V}}^{k}$ with the norm

$$p_{k}(x) := p_{k-1}(x) + \sum_{A \in \mathcal{V}} p_{k-1}(Ax), \quad x \in \mathcal{H}_{\mathcal{V}}^{k}.$$ 

Let the topology of $\mathcal{H}_{\mathcal{V}}^{\infty}$ be defined by the system of norms $(p_{k})_{k \in \mathbb{N}_{0}}$. It can be shown that $(\mathcal{H}_{\mathcal{V}}^{k}, p_{k})$ is a Banach space and $(\mathcal{H}_{\mathcal{V}}^{\infty}, (p_{k})_{k \in \mathbb{N}})$ turns into a Fréchet space. Moreover, each $A \in \mathcal{V}$ induces a bounded operator $A_{k} : \mathcal{H}_{\mathcal{V}}^{k} \rightarrow \mathcal{H}_{\mathcal{V}}^{k-1}$.

For $n \in \mathbb{N} \cup \{\infty\}$ it was shown in [12] that all maps

$$\Psi_{k}^{\mathcal{V}} : \mathcal{H}_{\mathcal{V}}^{k} \rightarrow \mathcal{H}_{\mathcal{V}}^{k}$$

are bilinear and continuous. The following result on regularity was proved in [13]:

**Theorem 2.2** Let $A \in \Psi_{\infty}^{\mathcal{V}}$ be a Fredholm operator and $u \in H$ with $Au \in \mathcal{H}_{\mathcal{V}}^{k}$ for some $k \in \mathbb{N} \cup \{\infty\}$. Then it follows that $u \in \mathcal{H}_{\mathcal{V}}^{k}$.

### 3 On the Segal-Bargmann Projection

Throughout this paper we write $(x, y) := x_{1}\overline{y}_{1} + \cdots + x_{n}\overline{y}_{n}$ for the Hermitian inner product on $\mathbb{C}^{n}$ and $|x| := \sqrt{(x, x)}$. For $c > 0$ and the Lebesgue measure $\nu$ let us denote by $\mu_{c}$ the Gaussian measure on $\mathbb{C}^{n}$ given by:

$$d\mu_{c} = c^{n}\pi^{-n} \exp(-c|\cdot|^{2}) \, dv.$$ 

With $\mu := \mu_{1}$ let $H^{2}(\mathbb{C}^{n}, \mu)$ be the Segal-Bargmann space of $\mu$-square integrable entire functions on $\mathbb{C}^{n}$. We denote by $P$ the orthogonal projection from $L^{2}(\mathbb{C}^{n}, \mu)$ onto $H^{2}(\mathbb{C}^{n}, \mu)$. The reproducing kernel $K$ (resp. the normalized kernel $k$) corresponding to $H^{2}(\mathbb{C}^{n}, \mu)$ are known to be:

(a) $K(y, x) := \exp((y, x))$,

(b) $k_{x}(y) := K(y, x) \left\| K(\cdot, x) \right\|^{-1} = \exp((y, x) - \frac{1}{2}|x|^{2})$

where $\left\| \cdot \right\|$ denotes the $L^{2}(\mathbb{C}^{n}, \mu)$-norm. For $z, w \in \mathbb{C}^{n}$ we write $\tau_{w}(z) := z + w$ for the shift by $w$. Consider the space of measurable symbols on $\mathbb{C}^{n}$ given by:

$$T(\mathbb{C}^{n}) := \{ g : g \circ \tau_{x} \in L^{2}(\mathbb{C}^{n}, \mu) \text{ for all } x \in \mathbb{C}^{n} \}.$$ 

For $g \in T(\mathbb{C}^{n})$ and with the natural domain of definition

$$D(T_{g}) := \{ f \in H^{2}(\mathbb{C}^{n}, \mu) : gf \in L^{2}(\mathbb{C}^{n}, \mu) \}$$

the Toeplitz operator $T_{g}$ on $H^{2}(\mathbb{C}^{n}, \mu)$ is densely defined by:

$$T_{g} : D(T_{g}) \ni f \mapsto P(fg).$$

If $g$ has polynomial growth at infinity we can determine an invariant subspace for $T_{g}$.

We inductively define a sequence $(a_{n})_{n \in \mathbb{N}}$ with $a_{1} := \frac{1}{4}$ and $a_{n+1} := [4 \cdot (1 - a_{n})]^{-1}$ for all $n \geq 2$. It can be checked that:
(a) \( a_n < \frac{1}{2}, \quad \forall n \in \mathbb{N}, \)
(b) \((a_n)_{n \in \mathbb{N}}\) is strictly increasing,
(c) \( \lim_{n \to \infty} a_n = \frac{1}{2} \).

Let \( \mathbb{P}[\mathbb{C}^n] \) be the space of all polynomials on \( \mathbb{C}^n \) in the variables \( z := (z_1, \cdots, z_n) \) and \( \bar{z} := (\bar{z}_1, \cdots, \bar{z}_n) \). We write \( \mathbb{P}_a[\mathbb{C}^n] \) for all analytic polynomials and set:

\[
L_{\exp}(\mathbb{C}^n) := \{ f \in L^2(\mathbb{C}^n, \mu) : \exists c < \frac{1}{2}, 0 < D \text{ s.t. } |f(z)| \leq D \exp \left( c |z|^2 \right) \text{ a.e.} \}.
\]

Because of \( \mathbb{P}[\mathbb{C}^n] \subset L_{\exp}(\mathbb{C}^n) \) it follows that \( L_{\exp}(\mathbb{C}^n) \) is dense in \( L^2(\mathbb{C}^n, \mu) \).

With the space \( \mathcal{H}(\mathbb{C}^n) \) of entire functions on \( \mathbb{C}^n \) we define a subspace of \( H^2(\mathbb{C}^n, \mu) \) by:

\[
H_{\exp}(\mathbb{C}^n) := \mathcal{H}(\mathbb{C}^n) \cap L_{\exp}(\mathbb{C}^n).
\]

Consider the symbols having polynomial growth at \( \infty \):

\[
\text{Po1}(\mathbb{C}^n) := \{ f : \exists j \in \mathbb{N} \text{ s.t. } |f(z)| (1 + |z|^2)^{-\frac{j}{2}} \in L^\infty(\mathbb{C}^n) \}.
\]

**Proposition 3.1** It holds \( P[L_{\exp}(\mathbb{C}^n)] \subset H_{\exp}(\mathbb{C}^n) \) and for \( f \) in \( \text{Po1}(\mathbb{C}^n) \):

\[
T_f[H_{\exp}(\mathbb{C}^n)] \subset H_{\exp}(\mathbb{C}^n) \subset D(T_f) \quad (3.2)
\]

**Proof:** It is obvious that \( H_{\exp}(\mathbb{C}^n) \subset D(T_f) \). Because the multiplication by \( f \) clearly maps \( H_{\exp}(\mathbb{C}^n) \) into \( L_{\exp}(\mathbb{C}^n) \) it is sufficient to prove the first assertion of Proposition 3.1. For \( g \in L_{\exp}(\mathbb{C}^n) \) there are \( c < \frac{1}{2} \) and \( D > 0 \) such that a.e.:

\[
|g(z)| \leq D \exp \left( c |z|^2 \right).
\]

By (a), (b) and (c) and with \((a_n)_{n \in \mathbb{N}}\) above we can choose \( n_0 \in \mathbb{N} \) with \( c < a_{n_0} < \frac{1}{2} \).

Using the transformation formula and the reproducing property of \( K \) we obtain:

\[
| \left[ P g \right](z) | \leq \int_{\mathbb{C}^n} |g \exp \{\langle z, \cdot \rangle \}| \, d\mu \\
\leq D \pi^{-n} \int_{\mathbb{C}^n} \exp \left\{ \text{Re} \langle z, \cdot \rangle - \left[ 1 - a_{n_0} \right] |\cdot|^2 \right\} \, dv \\
= D \left( 1 - a_{n_0} \right)^{-n} \int_{\mathbb{C}^n} \exp \left\{ 2 \text{Re} \left( 2^{-1} \left( 1 - a_{n_0} \right)^{-\frac{1}{2}} z, \cdot \right) \right\} \, d\mu \\
= D \left( 1 - a_{n_0} \right)^{-n} \exp \left\{ \left[ 4 \left( 1 - a_{n_0} \right) \right]^{-1} \left| z \right|^2 \right\}.
\]

From (a) above we conclude that \( P g \in H_{\exp}(\mathbb{C}^n) \). \( \square \)

Hence all finite products of Toeplitz operators with symbols in \( \text{Pol}(\mathbb{C}^n) \) are well-defined on the dense subspace \( H_{\exp}(\mathbb{C}^n) \) of \( H^2(\mathbb{C}^n, \mu) \). In particular, all iterated commutators of \( P \) and multiplication operators \( M_f \) with \( f \in \text{Pol}(\mathbb{C}^n) \) can been considered as elements in \( L(L_{\exp}(\mathbb{C}^n)) \). In fact, they can be written as integral operators and a standard application of the *Schur test* leads to a criterion for the boundedness.
Lemma 3.1 Let $L: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ be a measurable function such that:

$$|L(x, y)| \leq |F(x - y)| \exp \left\{ Re\langle x, y \rangle \right\}$$

where $F \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$. Then the integral operator $A$ on $L^2(\mathbb{C}^n, \mu)$ defined by

$$[Af](z) := \int_{\mathbb{C}^n} L(z, \cdot) f \, d\mu$$

is bounded on $L^2(\mathbb{C}^n, \mu)$ with $\|A\| \leq 2^n \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$.

Proof: With $p := q = \exp(\frac{1}{2} |\cdot|^2)$ on $\mathbb{C}^n$ it follows that:

$$\int_{\mathbb{C}^n} |L(\cdot, y)| \, p \, d\mu \leq \frac{1}{\pi^n} \int_{\mathbb{C}^n} |F(\cdot - y)| \exp \left\{ Re\langle \cdot, y \rangle - \frac{1}{2} |\cdot|^2 \right\} \, dv$$

$$= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |F(\cdot + y, y) - \frac{1}{2} |\cdot + y|^2 \right\} \, dv$$

$$= 2^n p(y) \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}.$$

Similarly, we get $\int |L(x, \cdot)| \, p \, d\mu \leq 2^n p(x) \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$. Applying the Schur test we obtain the desired result.

Consider the subspace $\text{SP}_{Lip}(\mathbb{C}^n)$ of $\text{Pol}(\mathbb{C}^n)$ defined by:

$$\text{SP}_{Lip}(\mathbb{C}^n) := \{ f \in \text{Pol}(\mathbb{C}^n) : \exists c, D > 0 \text{ s.t. } |f(z) - f(w)| \leq D \exp(c |z - w|) \}.$$

As an application of Lemma (3.1) we can prove:

Proposition 3.2 Let $m \in \mathbb{N}$ and $S_m := \{M_{f_1}, \cdots, M_{f_m}\}$ with $f_j \in \text{SP}_{Lip}(\mathbb{C}^n)$. Then the commutator $\text{ad}[S_m](P) \in L(L_{\exp}(\mathbb{C}^n))$ has a continuous extension to $L^2(\mathbb{C}^n, \mu)$.

Proof: It is easy to check that the commutator $\text{ad}[S_m](P)$ can be written as an integral operator on $L^2(\mathbb{C}^n, \mu)$ with kernel:

$$K_m(z, u) = \exp\left( \langle z, u \rangle \right) \prod_{j=1}^{m} \{ f_j(z) - f_j(u) \}. \quad (3.3)$$

By (3.3) and our assumptions on $f_j \in S_m$ we can choose $c, D > 0$ such that

$$|K_m(z, u)| \leq D \exp \left( c |z - u| + Re\langle z, u \rangle \right).$$

Because of $F := D \exp(c |\cdot|) \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$ Lemma 3.1 implies the assertion.

We remark that by (3.3) the maps $\text{ad}[S_m](P)$ are invariant under permutations of the system $S_m$. Now, we can prove the boundedness of a class of iterated commutators.
Corollary 3.1 Let \( g \in L^\infty(\mathbb{C}^n) \) and \( S_m := \{ M_{f_1}, \ldots, M_{f_m} \} \) with \( f_j \in SP_{Lip}(\mathbb{C}^n) \). Then the commutator
\[
\text{ad}[S_m]\left( [P, M_g] \right) \in L( L_{\exp}(\mathbb{C}^n) )
\]
has a bounded extensions \( A(S_m, g) \) to \( L^2(\mathbb{C}^n, \mu) \) and (3.4) below is continuous between Banach spaces:
\[
L^\infty(\mathbb{C}^n) \ni g \mapsto A(S_m, g) \in L( L^2(\mathbb{C}^n, \mu) ). \tag{3.4}
\]
Proof: It can be checked by induction or our remark following Proposition 3.2 that:
\[
\text{ad}[S_m]\left( [P, M_g] \right) = \left[ \text{ad}[S_m](P), M_g \right] \in L( L_{\exp}(\mathbb{C}^n) ).
\]
Because \( M_g \) is bounded and \( \text{ad}[S_m](P) \) has a bounded extension to \( L^2(\mathbb{C}^n, \mu) \) by Proposition 3.2 we conclude the desired result. \( \square \)

Given a finite set \( \mathbf{X} := \{ X_1, \ldots, X_n \} \subset L(L^2(\mathbb{C}^n, \mu)) \) we denote by \( A(\mathbf{X}) \) the algebra generated by \( \mathbf{X} \). Moreover, we write:
\[
A_P(\mathbf{X}) := P A(\mathbf{X}) P := \{ PAP : A \in A(\mathbf{X}) \},
\]
for the corresponding projected algebra in \( L(L^2(\mathbb{C}^n, \mu)) \). By Proposition 3.1 and for all \( m \geq 1 \) it follows that the commutator:
\[
\text{ad}[S_{m-1}]( [P, M_{f_m}] ) = -\text{ad}[S_m](P)
\]
can be regarded as bounded operators on \( L^2(\mathbb{C}^n, \mu) \).

Proposition 3.3 Let \( g \in L^\infty(\mathbb{C}^n) \) and \( T_m := \{ T_{f_1}, \ldots, T_{f_m} \} \) with \( f_j \in SP_{Lip}(\mathbb{C}^n) \). Then
\[
\text{ad}[T_m](T_g) \in L(H_{\exp}(\mathbb{C}^n))
\]
is well-defined. More precisely, with \( S_m := \{ M_{f_1}, \ldots, M_{f_m} \} \) it holds:
\[
\text{ad}[T_m](T_g) \in A_P\left\{ \text{ad}[N](P), M_g : N \subset S_m \right\} \tag{3.5}
\]
and \( \text{ad}[T_m](T_g) \) has a bounded extension \( C(T_m, g) \) to \( H^2(\mathbb{C}^n, \mu) \). Moreover, the symbols map
\[
L^\infty(\mathbb{C}^n) \ni g \mapsto C(T_m, g) \in L( H^2(\mathbb{C}^n, \mu) ) \tag{3.6}
\]
is continuous between Banach spaces.

Proof: By Proposition 3.1 the iterated commutators \( \text{ad}[T_m](T_g) \) are well-defined. It is a straightforward computation that:
\[
\text{ad}[T_1](T_g) = P \left[ [P, M_{f_1}], [P, M_g] \right] P
\]
which proves (3.5) in the case $m = 1$. By induction assume \( \text{ad} [T_j] (T_g) \) has the form:

\[
\text{ad} [T_j] (T_g) = \sum_{l \in \mathcal{I}} P A_l M_g B_l P \tag{3.7}
\]

where \( \mathcal{I} \) is a finite index set, \( I \) the identity operator and

\[
A_l, B_l \in \mathcal{A}(S_j) := \mathcal{A} \{ \text{ad} [\mathcal{N}] (P), I : \text{ with } \mathcal{N} \subset S_j \}.
\tag{3.8}
\]

Then it follows that:

\[
\text{ad} [T_{j+1}] (T_g) = \sum_{l \in \mathcal{I}} [T_{f_{j+1}}, P A_l M_g B_l P].
\]

To prove (3.7) in the case \( j + 1 \) it is sufficient to show for all \( l \in \mathcal{I} \) the existence of a finite set \( \tilde{\mathcal{I}} \subset \mathbb{N} \) and operators \( C_k, D_k \in \mathcal{A}(S_{j+1}) \) such that

\[
[T_{f_{j+1}}, P A_l M_g B_l P] = \sum_{k \in \tilde{\mathcal{I}}} P C_k M_g D_k P. \tag{3.9}
\]

Note that (3.9) follows from \( T_{f_{j+1}} P A_l M_g B_l P = P M_{f_{j+1}} P A_l M_g B_l P \) and

\[
[M_{f_{j+1}}, Q] \in \mathcal{A}(S_{j+1})
\]

for \( Q \in \{ P, A_l, B_l \} \). The continuity of (3.6) is a direct consequence of (3.7).

As an immediate consequence of Proposition 3.2 we remark:

**Lemma 3.2** Let \( f \in SP_{Lip}(\mathbb{C}^n) \) and \( \mathcal{D}(T_f) \) as in (3.1). Then the Toeplitz operator \( T_f \) is densely defined and closed on \( \mathcal{D}(T_f) \).

**Proof:** Because of \( f \in T(\mathbb{C}^n) \) it follows that \( T_f \) is densely defined. Moreover,

\[
M_f = T_f + [M_f, P] : \mathcal{D}(T_f) \subset H^2(\mathbb{C}^n, \mu) \longrightarrow L^2(\mathbb{C}^n, \mu).
\tag{3.10}
\]

Proposition 3.2 with \( j = 1 \) shows that the commutator \( [M_f, P] \) has a continuous extension to \( H^2(\mathbb{C}^n, \mu) \). Choose a sequence \( (h_n)_{n \in \mathbb{N}} \subset \mathcal{D}(T_f) \) such that:

(i) \( \lim_{n \to \infty} h_n = h \in H^2(\mathbb{C}^n, \mu) \),

(ii) \( \lim_{n \to \infty} T_fh_n = g \in H^2(\mathbb{C}^n, \mu) \).

Then we conclude from the continuity of \( [M_f, P] \) and (3.10) that

\[
fh = \lim_{n \to \infty} fh_n \in L^2(\mathbb{C}^n, \mu)
\]

Hence \( h \in \mathcal{D}(T_f) \) and \( g = \lim_{n \to \infty} P(fh_n) = T_fh \).
Let $\mathcal{T}_m := \{ T_{f_1}, \ldots, T_{f_m} \}$ be a system of Toeplitz operators where $f_j \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ for $j = 1, \ldots, n$. From Lemma 3.2 it follows that the domains $\mathcal{D}(T_{f_j})$ are closed with respect to the graph norm $\| \cdot \|_{\text{gr}} := \| \cdot \| + \| T_{f_j} \cdot \|$. Consider $D_j \subset H^2(\mathbb{C}^n, \mu)$ defined by:

$$D_j := \| \cdot \|_{\text{gr}} - \text{closure of } H_{\exp}(\mathbb{C}^n) \text{ in } \mathcal{D}(T_{f_j}).$$

If we consider $T_{f_j}$ as a closed operator on $D_j$ we can define a scale of algebras (2.2) by commutator methods with the system $S_m$. By Lemma 2.1 with $D := H_{\exp}(\mathbb{C}^n)$ our result in Proposition 3.3 can be formulated as follows:

**Theorem 3.1** The symbol map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi^S_{\infty}$ is well-defined and continuous.

Note that an application of Theorem 2.2 in the case of $\mathcal{V} := S_m$ gives a regularity result for Fredholm Toeplitz operators with bounded symbols.

4 **Toeplitz $\Psi^*$-algebras via the Segal-Bargmann representation**

There is a unitary representation of the Heisenberg group $\mathbb{H}_n$ in $\mathcal{L}(L^2(\mathbb{C}, \mu))$. By identifying $\mathbb{H}_n$ with $\mathbb{C}^n \times \mathbb{R}$ the group law is given by, [10]:

$$(z, t) \ast (w, s) := (z + w, t + s + 2^{-1} \text{Im} \langle w, z \rangle).$$

For $z \in \mathbb{C}^n$ and $f \in L^2(\mathbb{C}^n, \mu)$ we define the operator $W_z f := k_z \cdot f \circ \tau_z$. It follows by an easy calculation:

**Lemma 4.1** $H^2(\mathbb{C}^n, \mu)$ is an invariant subspace for all $W_z$ where $z \in \mathbb{C}^n$. Moreover,

1. $W_z$ is unitary with $W_z^* = W_{-z} = W_z^{-1},$
2. The commutator $\text{ad}[P]W_z$ vanishes,
3. For $z, w \in \mathbb{C}^n : W_z W_w = \exp(i \text{Im} \langle w, z \rangle) W_{z+w}.$

By Lemma 4.1 a unitary representation $\tilde{\rho} : \mathbb{H}_n \rightarrow \mathcal{L}(L^2(\mathbb{C}, \mu))$ of $\mathbb{H}_n$ is given by:

$$\tilde{\rho}(z, t) := e^{it} W_\frac{z}{\sqrt{2}}.$$

Moreover, the restriction of $\tilde{\rho}(z, t)$ to $H^2(\mathbb{C}^n, \mu)$ gives rise to a unitary representation $\rho$ of $\mathbb{H}_n$ in $\mathcal{L}(H^2(\mathbb{C}, \mu))$. It is well-known that $\rho$ is irreducible and strongly continuous and it is referred to as **Segal-Bargmann representation**, c.f. [10].

For any $A \in B := \mathcal{L}(H^2(\mathbb{C}, \mu))$ we define the map:

$$\Phi_A : \mathbb{H}_n \rightarrow B$$

$$(z, t) \mapsto \rho(z, t) A \rho(z, t)^{-1} = W_\frac{z}{\sqrt{2}} A W_\frac{-z}{\sqrt{2}}.$$ (4.1)
In particular, note that for $f \in L^\infty(\mathbb{C}^n)$
\[ \Phi_{T_f}(z, t) = T_{f \circ T_{-z}}. \]

For $k \in \mathbb{N} \cup \{\infty\}$ we consider the $C^k$-elements
\[ \Psi^k := \{ A \in B : \Phi_A \in C^k(\mathbb{H}_n, B) \} \]
defined via $\rho$. To any $z \in \mathbb{C}^n$ we associate $\varphi^z_A : \mathbb{R} \to B$ by $\varphi^z_A(s) := W_{sz}A W_{-sz}$. According to (4.1) it follows that:
\[ \Psi^k = \bigcap_{z \in \mathbb{C}^n} \Psi^{k, z} \]

where
\[ \Psi^{k, z} := \{ A \in B : \varphi^z_A \in C^k(\mathbb{R}, B) \}. \]

Here we characterize the $C^k$-Toeplitz operators (i.e. the Toeplitz operators $T_f \in \Psi^k$) in terms of their symbols. We use a characterization of $\Psi^\infty$ by commutator conditions and apply our results of the previous section.

For all $z \in \mathbb{C}^n$ the map $(W_{sz})_{s \in \mathbb{R}} \subset B$ defines a strongly continuous unitary group. By $V^z$ we denote its infinitesimal generator with domain of definition:
\[ D(V^z) := \{ h \in H^2(\mathbb{C}^n, \mu) : V^zh := \lim_{s \to 0} s^{-1}(W_{sz} - 1)h \text{ exists} \}. \]

By Stone's Theorem $iV^z$ is selfadjoint and associated to $V^z := [iV^z]$ there is a scale:
\[ B := \Psi^0 \supset \cdots \supset \Psi^n \supset \cdots \supset \Psi^\infty := \bigcap_{k \in \mathbb{N}} \Psi^k \]

of algebras in $B$ defined by commutator methods with $V^z$ as it was described in (2.2) of section 2.1. In particular, $\Psi^\infty$ is a $\Psi^*$-algebra and it is well-known that (4.2) and (4.3) are related as follows, see [16]:

**Proposition 4.1** For $z \in \mathbb{C}^n$ let $V^z := [iV^z]$ then:

(i) $\Psi^{k, z} \subset \Psi^k$ for $k \in \mathbb{N}$,

(ii) $\Psi^{k+1} \subset \Psi^{k, z}$ for $k \in \mathbb{N}_0$ and $\Psi^\infty = \Psi^\infty$.

Using the fact that convergence in $H^2(\mathbb{C}^n, \mu)$ implies uniformly compact convergence on $\mathbb{C}^n$ we can calculate $V^z$ explicitly. Let $h \in D(V^z)$ and $w \in \mathbb{C}^n$:
\[ [V^zh](w) = \frac{d}{ds} [k_{sz}(w) h(w - sz)]_{s=0} = \{ \langle w, z \rangle - \sum_{j=1}^n z_j \frac{\partial}{\partial w_j} \} h(w). \]

It easily can be seen that all the monomials $m_\alpha(z) := z^\alpha$ for $\alpha \in \mathbb{N}_0^n$ are contained in the domain $D(V^z)$. Moreover, from the standard identities $M_{w_j} := T_{w_j}$ and $\frac{\partial}{\partial w_j} := T_{\overline{w_j}}$ it follows that the restriction of $V^z$ to $P_a[\mathbb{C}^n]$ coincides with an unbounded Toeplitz operator:
\[ V^zp := (\langle \cdot, x \rangle - \langle z, \cdot \rangle) p = 2i \int_{\mathbb{C}^n} T_{\overline{w(z)}} p, \quad p \in P_a[\mathbb{C}^n]. \]
In the following we write:

$$g_z := 2i \text{Im} \langle \cdot, z \rangle$$

for the symbol of the Toeplitz operator appearing above. Consider the space $\mathcal{D}(T_{g_z})$ with the graph norm $\| \cdot \|_{gr} := \| \cdot \| + \| T_{g_z} \cdot \|$. By Lemma 3.2 it follows that $(\mathcal{D}(T_{g_z}), \| \cdot \|_{gr})$ is a Banach space containing $\mathbb{P}_a[\mathbb{C}^n]$ and $H_{\exp}(\mathbb{C}^n)$.

**Lemma 4.2** For all $z \in \mathbb{C}^n$ the embedding $\mathbb{P}_a[\mathbb{C}^n] \hookrightarrow H_{\exp}(\mathbb{C}^n)$ is dense with respect to the graph norm topology. Moreover,

$$H_{\exp}(\mathbb{C}^n) \subset \mathcal{D}(V^z) \cap \mathcal{D}(T_{g_z}) \quad (4.5)$$

and the restrictions of $V^z$ and $T_{g_z}$ to $H_{\exp}(\mathbb{C}^n)$ coincide.

**Proof:** For $f \in H_{\exp}(\mathbb{C}^n)$ we can choose $c_1 \in (0, \frac{1}{2})$ and $D_1 > 0$ such that:

$$|f(w)| \leq D_1 \exp(c_1|w|^2)$$

for all $z \in \mathbb{C}^n$. Hence, $f \in L^2(\mathbb{C}^n, \mu_r)$ for all $r \in (2c_1, 1)$. Fix $c_2, c_3$ with $2c_1 < c_2 < c_3 < 1$ and choose $D_2 > 0$ with

$$|w|^2 \leq D_2 \exp\left(\left[ c_3 - c_2 \right]|w|^2\right)$$

for all $w \in \mathbb{C}^n$. Then we obtain for all $p \in \mathbb{P}_a[\mathbb{C}^n]$:

$$\| T_{g_z} (f - p) \|^2 \leq \| g_z (f - p) \|^2$$

$$\leq 2 |z|^2 \int_{\mathbb{C}^n} |\cdot|^2 |f - p|^2 d\mu$$

$$\leq 2D_2 |z|^2 r^{-n} \| f - p \|^2_{L^2(\mathbb{C}^n, \mu_r)} < \infty$$

where $r = 1 - c_3 + c_2 \in (2c_1, 1)$. Because $\mathbb{P}_a[\mathbb{C}^n]$ is dense in $L^2(\mathbb{C}^n, \mu_r) \cap \mathcal{H}(\mathbb{C}^n)$ for all $r > 0$ the first assertion follows.

Now, (4.5) immediately can be derived from $T_{g_z}p = V^z p$ for $p \in \mathbb{P}_a[\mathbb{C}^n]$ and the density result above which implies that:

$$H_{\exp}(\mathbb{C}^n) \subset \text{closure}(\mathbb{P}_a[\mathbb{C}^n], \| \cdot \|_{gr}) \subset \mathcal{D}(V^z) \cap \mathcal{D}(T_{g_z})$$

Finally, we apply the continuity of $V^z, T_{g_z} : (\mathbb{P}_a[\mathbb{C}^n], \| \cdot \|_{gr}) \to H^2(\mathbb{C}^n, \mu)$. □

For $z \in \mathbb{C}^n$ we denote by $\tilde{V}^z$ the infinitesimal generator of $(W_{sz})_{s \in \mathbb{R}}$ considered as strongly continuous group of unitary operators on $L^2(\mathbb{C}^n, \mu)$. Let $\mathcal{D}(\tilde{V}^z)$ be its domain of definition, then $V^z$ can be obtained by restricting $\tilde{V}^z$ to $\mathcal{D}(V^z)$. For $f \in \text{SP}_{\text{lip}}(\mathbb{C}^n)$ and $r \in \mathbb{N}$ we write

$$\mathcal{A}_r(f) := \mathcal{A}([M_f, \cdots, M_f]) \subset \mathcal{L}(L^2(\mathbb{C}^n, \mu))$$

where the algebra on the right hand side was defined in (3.8) of Proposition 3.3.
Lemma 4.3 The domain $D(\tilde{V}^z)$ is invariant under $A \in \mathcal{A}_r(f)$ where $f$ is a linear function on $\mathbb{C}^n$. Moreover, the commutator $[A, \tilde{V}^z]$ vanishes as an operator on $D(\tilde{V}^z)$.

Proof: It is sufficient to show that for all $j \in \mathbb{N}$ the space $D(\tilde{V}^z)$ is invariant under the operators

$$a_j(f) := \text{ad}^j \left[ M_f \right] (P).$$

Note that $L_{\exp}(\mathbb{C}^n)$ is an invariant under $W_z$ and it holds $W_z M_f W_z = M_{f \circ \tau_z}$. Because $W_z$ commutes with $P$ it follows that:

$$W_z a_j(f) W_z = \text{ad}^j \left[ M_{f \circ \tau_z} \right] (P) = a_j(f).$$

We have used the linearity of $f$ for the second equality. Hence, the commutator $[A, W_z]$ vanishes for all $A \in \mathcal{A}_r(f)$. Fix $h \in D(\tilde{V}^z)$ and $A \in \mathcal{A}_r(f)$, then:

$$\frac{1}{s} \left\{ W_z s - I \right\} A h = A \frac{1}{s} \left\{ W_z s - I \right\} h \rightarrow A \tilde{V}^z h$$

as $s$ tends to 0. It follows that $Ah \in D(\tilde{V}^z)$ with $\tilde{V}^z Ah = A \tilde{V}^z h$.

Remark 4.1 Let $W$ be any subspace of $H := H^2(\mathbb{C}^n, \mu)$ such that $H_{\exp}(\mathbb{C}^n) \subset W$. Consider the operators:

$$\mathcal{O}_W := \{ A \in \text{L}(W, H) : H_{\exp}(\mathbb{C}^n) \text{ is an invariant space for } A \}.$$

Let $A \in \mathcal{O}_W$ and assume there is $A^* \in \mathcal{O}_W$ with $\langle Af, g \rangle = \langle f, A^* g \rangle$ for all $f, g \in W$. Because of $K(\cdot, \lambda) \in H_{\exp}(\mathbb{C}^n)$ for all $\lambda \in \mathbb{C}^n$ it follows that $A$ can be written as an integral operator with kernel:

$$K_A(\cdot, z) = \overline{A^* K(\cdot, z)}(w). \quad (4.6)$$

In particular, $A$ completely is determined by the restriction of $A^*$ to $H_{\exp}(\mathbb{C}^n)$. Assume that $A$ has a continuous extensions $\tilde{A}$ from $H_{\exp}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. Fix $g \in H^2(\mathbb{C}^n, \mu)$ and a sequence $(g_n)_n \subset H_{\exp}(\mathbb{C}^n)$ with $g = \lim_{n \rightarrow \infty} g_n$. Then it follows for $z \in \mathbb{C}^n$:

$$\left[ \tilde{A} g \right](z) = \lim_{n \rightarrow \infty} \left\langle A g_n, K(\cdot, z) \right\rangle = \lim_{n \rightarrow \infty} \left\langle g_n, A^* K(\cdot, z) \right\rangle = \left\langle g, A^* K(\cdot, z) \right\rangle$$

and $\tilde{A}$ is given by the same integral formula. In particular, $A$ has a (unique) extension from $W$ to $H^2(\mathbb{C}^n, \mu)$.

Let $h \in L^\infty(\mathbb{C}^n)$ and $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a linear function. We write $C_j(f, h)$ for the continuous extensions of the commutators

$$\text{ad}^j \left[ T_f \right] (T_h) \in \text{L}(H_{\exp}(\mathbb{C}^n))$$

to $H^2(\mathbb{C}^n, \mu)$, (note that $f \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ and Proposition 3.3).
Corollary 4.1 Let $h \in L^{\infty}(\mathbb{C}^{n})$. Assume that $\mathcal{D}(\tilde{V}^{z})$ is invariant under the multiplication operator $M_{h}$. Then $\mathcal{D}(V^{z})$ is invariant under $C_{j}(f, h)$ for all $j \in \mathbb{N}$.

Proof: According to (3.7) there is a finite index set $I$ and $A_{i}, B_{i} \in A_{j}(f)$ such that

$$\text{ad}^{j}[T_{f}](T_{h}) = \sum_{i \in I} P A_{i} M_{h} B_{i} P.$$ 

Due to our assumption on $h$ and by Lemma 4.3 the assertion follows. \hfill $\square$

Now, we can proof our main result on the smoothness of Toeplitz operators with respect to the Segal-Bargmann representation $\rho$ of the Heisenberg group:

Theorem 4.1 Let $h \in \mathcal{S}_{s} := \mathcal{S} \cap \overline{\mathcal{S}}$ where $\overline{\mathcal{S}} = \{ \tilde{h} : h \in \mathcal{S} \}$ and

$$\mathcal{S} := \{ h \in L^{\infty}(\mathbb{C}^{n}) : \text{s. t.} \mathcal{D}(\tilde{V}^{z}) \text{ is invariant under } M_{h} \text{ for all } z \in \mathbb{C}^{n} \}.$$ 

Then the symbol map into the $\Psi^{*}$-algebra $\Psi^{\infty}$ given by:

$$\mathcal{S}_{s} \ni h \mapsto T_{h} \in \Psi^{\infty}$$

is well-defined and continuous if $\mathcal{S}_{s}$ carries the $L^{\infty}(\mathbb{C}^{n})$-topology.

Proof: Using our notation in (4.2) and (4.3) we must show that $T_{h} \in \Psi^{\infty}_{z} = \Psi_{\tilde{V}^{z}}$ for all complex directions $z \in \mathbb{C}^{n}$ and $V^{z} := [iV^{z}]$:

$\mathcal{D}(V^{z})$ is invariant under $T_{q}$ for $q \in \{ h, \tilde{h} \} \subset \mathcal{S}_{s}$ and by Lemma 4.2 it follows that the commutators $A_{1} := [iV^{z}, T_{q}]$ and $[T_{ig_{z}}, T_{q}]$ coincide on $\mathcal{H}_{\exp}(\mathbb{C}^{n})$. Because $iV^{z}$ is self-adjoint we can define $A_{1}^{*} := [T_{q}, iV^{z}]$ and $W := \mathcal{D}(V^{z})$ in Remark 4.1. The operator $[T_{ig_{z}}, T_{q}]$ has a bounded extension $C_{1}(ig_{z}, q)$ from $\mathcal{H}_{\exp}(\mathbb{C}^{n})$ to $H^{2}(\mathbb{C}^{n}, \mu)$. We conclude from Remark 4.1 that $C_{1}(ig_{z}, q)$ is an extension of $A_{1}$ from $W$ to $H^{2}(\mathbb{C}^{n}, \mu)$ and $T_{q} \in \Psi_{1}^{\infty}$. By induction we must prove for $j \in \mathbb{N}$:

1. The domain of definition $\mathcal{D}(V^{z})$ is invariant under $C_{j}(ig_{z}, q)$,

2. The commutators $A_{j+1} := [iV^{z}, C_{j}(ig_{z}, q)]$ have the bounded extension $C_{j+1}(ig_{z}, q)$ from $\mathcal{D}(V^{z})$ to $H^{2}(\mathbb{C}^{n}, \mu)$.

Assertion (1) is a direct consequence of Corollary 4.1 and (2) can be derived from Remark 4.1 with $A_{j+1}^{*} := [C_{j}(ig_{z}, q)^{*}, iV^{z}]$ on $W := \mathcal{D}(V^{z})$ and the fact that $A_{j+1}$ has the continuous extension $C_{j+1}(ig_{z}, q)$ from $\mathcal{H}_{\exp}(\mathbb{C}^{n})$ to $H^{2}(\mathbb{C}^{n}, \mu)$. The continuity of the symbols map follows from (2.3) together with the continuity of (3.6) in Proposition 3.3. \hfill $\square$

\textsuperscript{3}Note that by Corollary 4.1 and the identity $C_{j}(ig_{z}, q)^{*} = (-1)^{j}C_{j}(ig_{z}, \bar{q})$ the commutator $A_{j+1}^{*}$ is well-defined on $\mathcal{D}(V^{z})$. 

W. Bauer
5 Examples and Applications

Let $A$ denote the subalgebra of $L( L^2(\mathbb{C}^n, \mu) )$ of all multiplication operators with bounded symbols $h \in L^\infty(\mathbb{C}^n)$. For $z \in \mathbb{C}^n$ and with $\tilde{V}^z := [ i\tilde{V}^z ]$ there is a scale of algebras arising by commutator methods:

$$A \supset \Psi^*_1 \supset \ldots \supset \Psi^*_n \supset \ldots \supset \Psi^*_\infty = \bigcap_{n \in \mathbb{N}} \Psi^*_n. \quad (5.1)$$

In general, the inclusions above will be proper. As an immediate consequence of Theorem 4.1 it follows for the projected scale of vector spaces:

$$A_P \supset \Psi^*_1 P = \cdots \supset \Psi^*_n P = \Psi^*_n P = \cdots = \Psi^*_\infty P. \quad (5.2)$$

Here $A_P \subset L( H^2(\mathbb{C}^n, \mu) )$ is the space of Toeplitz operators with bounded measurable symbols. By passing from (5.1) to the scale (5.2) the underlying $C^k$-structure is lost.

We give an example of a class of bounded functions $g$ such that $D(\tilde{V}^z)$ is an invariant subspace for $M_g$ and $M_h$ for all $z \in \mathbb{C}^n$.

**Example 5.1** Denote by $C_\infty^c(\mathbb{C}^n)$ the space of compactly supported smooth functions. For $z = ( z_1, \ldots, z_n ) \in \mathbb{C}^n$ we write $z_j := x_j + iy_j$ and with $\alpha, \beta \in \mathbb{N}_0^n$:

$$z^{\alpha,\beta} := x^\alpha y^\beta, \quad \partial^{\alpha,\beta} := \frac{\partial^{\left|\alpha\right|}}{\partial x^\alpha} \frac{\partial^{\left|\beta\right|}}{\partial y^\beta}.$$

Fix $h \in D(\tilde{V}^z)$ and $z \in \mathbb{C}^n$. For $g \in C_\infty^c(\mathbb{C}^n)$ (real valued) and $s \neq 0$ we write:

$$\frac{1}{s} \left[ W_{sz} - I \right] M_g h = \frac{1}{s} \left[ M_{g \circ \tau_{sz}} - M_g \right] W_{sz} h + M_g \frac{1}{s} \left[ W_{sz} - I \right] h. \quad (5.3)$$

The second term converges in $L^2(\mathbb{C}^n, \mu)$ as $s \to 0$. Consider the smooth and compactly supported function $d_g (z, \cdot) := - (\text{grad } g(\cdot), z)_{\mathbb{R}^{2n}}$. Then:

$$C_{s,z} := \left\| \frac{1}{s} \left[ M_{g \circ \tau_{sz}} - M_g \right] - M_{d_g(z,\cdot)} \right\| \leq \sum_{\left|\alpha\right| + \left|\beta\right| = 2} \frac{s}{(\alpha + \beta)!} \left\| \partial^{\alpha,\beta} g \right\|_\infty \left| z^{\alpha,\beta} \right|.$$

Hence $\lim_{s \to 0} C_{s,z} = 0$ and the right hand side of

$$\left\| \frac{1}{s} \left[ M_{g \circ \tau_{sz}} - M_g \right] W_{sz} h - M_{d_g(z,\cdot)} W_{sz} h \right\| \leq C_{s,z} \left\| h \right\| + \left\| d_g (z, \cdot) \right\|_\infty \left\| (W_{sz} - I) h \right\|$$

tends to 0 as $s \to 0$. It follows $gh \in D(\tilde{V}^z)$. With our notation of Theorem 4.1 we conclude that $C_\infty^c(\mathbb{C}^n) \subset S_s$. By the continuity of

$$L^\infty(\mathbb{C}^n) \subset S_s \ni h \mapsto T_h \in \Psi^\infty$$

and the fact that $C_\infty^c(\mathbb{C}^n)$ is uniformly dense in the space $C_0(\mathbb{C}^n)$ of all continuous functions vanishing at infinity it follows that $\{ T_h : h \in C_0(\mathbb{C}^n) \} \subset \Psi^\infty$. 
In our second example we construct a compact operator $A \in \mathcal{B} := \mathcal{L}(H^2(\mathbb{C}, \mu))$ which
is not contained in $\Psi^{1,z}$ for any $z \in \mathbb{C}$ (with our notation in (4.2)). As a consequence and
using Example 5.1 $A$ is not limit point of finite sums of finite products of Toeplitz operators
with symbols in $C_0(\mathbb{C})$ and with respect to the Fréchet topology of $\Psi^{\infty,z}$. However, since
$A$ is compact it can be approximated by Toeplitz operators with smooth and compactly
supported symbols in the topology of $\mathcal{B}$, c.f. [8].

**Example 5.2** For $j \in \mathbb{N}_0$ let $P_j \in \mathcal{B}$ be the rank one projection onto
span$\{ m_j := z^j \}$. With a sequence $a := (a_n)_{n \in \mathbb{N}}$ tending to zero consider the compact diagonal operator:

$$A := \sum_{j \in \mathbb{N}} a_j P_j \in \mathcal{B}. $$

With $z \in \mathbb{C}$, $|z| = 1$ and $g_z := 2i \text{Im} \langle \cdot, z \rangle$ we compute $[T_{g_z}, A] m_j = [V^z, A] m_j$ explicitly
for all $j \in \mathbb{N}$. By (4.4) one obtains that:

$$[T_{g_z}, A] m_j = a_j T_{g_z} m_j - A[\bar{z} m_{j+1} - j z m_{j-1}]$$

$$= a_j (\bar{z} m_{j+1} - j z m_{j-1}) - (a_{j+1} \bar{z} m_{j+1} - j a_{j-1} z m_{j-1})$$

$$= (a_j - a_{j+1}) \bar{z} m_{j+1} - j z (a_j - a_{j-1}) m_{j-1}. $$

With $e_j := (j!)^{-\frac{1}{2}} z^j$ we have $\langle e_j, e_l \rangle_2 = \delta_{i,j}$ for all $j, l \in \mathbb{N}$. Hence it follows that

$$\| [T_{g_z}, A] e_j \|_2^2 = (j + 1) | a_j - a_{j+1} |^2 + j | a_j - a_{j-1} |^2. $$

(5.4)

We choose $a$ such that the right hand side of (5.4) tends to infinity for $j \to \infty$. This
can be done by the choice of an oscillating sequence $a_j := (-1)^j j^{-\frac{1}{2}}$. Then it follows

$$(j + 1) | a_j - a_{j+1} |^2 = (j + 1) | j^{-\frac{1}{2}} + (j + 1)^{-\frac{1}{2}} |^2 \geq \sqrt{j + 1}$$

and so the right hand side of (5.4) is unbounded for $j \to \infty$. Hence $[T_{g_z}, A]$ has no bounded
extension to $H^2(\mathbb{C}, \mu)$ and $A \notin \Psi^{1,z}$ by Proposition 4.1.

Let $\beta : L^2(\mathbb{R}^n) \to H^2(\mathbb{C}^n, \mu)$ denote the Bargmann isometrie, c.f. [10]. Our results on
Toeplitz operators on $H^2(\mathbb{C}^n, \mu)$ can be used in the analysis of a class of *Gabor-Daubechies
windowed localization operators* $L_h := \beta^{-1} T_h \beta $ on $L^2(\mathbb{R}^n)$ where $h \in L^\infty(\mathbb{C}^n)$, c.f. [9]. It
was remarked in [14] the operator $L_h$ can be considered as a pseudodifferential operator
$W_{\sigma(h)}$ in Weyl quantization with *Weyl symbol* $\sigma(h)$ on $\mathbb{R}^{2n}$. Via the identification of $\mathbb{R}^{2n}$
and $\mathbb{C}^n$ the correspondence between $h$ and $\sigma(h)$ can be expressed in terms of the heat
equation on $\mathbb{R}^{2n}$. More precisely, $\sigma(h)$ is a solution with initial data $h$ at a fixed time
$t_0 > 0$. In the next example we describe how the operators introduced in the previous
sections transform under $\beta$, c.f. [10].

---

\[ \text{Here} \text{ the window is a Hermite function on } \mathbb{R}^n \]
**Example 5.3** For \( u \in L^2(\mathbb{R}^n) \) it is well-known that \( \beta u \) can be expressed by the integral:

\[
[\beta u](z) = (2\pi)^{-\frac{n}{4}} \int_{\mathbb{R}^n} u(x) \exp \{ \langle x, z \rangle - \frac{1}{4} |x|^2 - \frac{1}{2} \langle z, \bar{z} \rangle \} dx.
\]

Fix \( a = p + iq \in \mathbb{C}^n \), then it can be checked that \( W_a \in L(H^2(\mathbb{C}^n, \mu)) \) transform as:

\[
B_a u := [\beta^{-1} W_a \beta ] (u) = u(\cdot - 2p) \exp \{ iq(p \cdot \cdot) \}.
\]

In particular, in the case \( q = 0 \) the unitary operator \( B_a \) is a usual shift in direction \( 2p \).

For \( j = 1, \ldots, n \) it is readily verified that \( T_{z_j} \) and \( T_{\overline{z}_j} \) transform in the following way:

(i) \( \beta^{-1} T_{z_j} \beta = \frac{1}{2} x_j - \partial_{x_j} \),

(ii) \( \beta^{-1} T_{\overline{z}_j} \beta = \frac{1}{2} x_j + \partial_{x_j} \)

From (i), (ii) and for \( \alpha \in \mathbb{N}_0^n \) one obtains the identity:

\[
\beta \partial_x^\alpha = (-1)^{|\alpha|} T_{i\Im z_1}^{\alpha_1} \cdots T_{i\Im z_n}^{\alpha_n} \beta =: (-1)^{|\alpha|} T_{i\Im z}^\alpha \beta.
\]

Let \( g \in D(\mathbb{R}^n) \) be a test function and fix \( f \in H_{\exp}(\mathbb{C}^n) \). It follows that:

\[
\langle \beta^{-1} f, \partial_x^\alpha g \rangle_{L^2(\mathbb{R}^n)} = \langle f, \beta \partial_x^\alpha g \rangle = \langle \beta^{-1} T_{i\Im z_1}^{\alpha_1} \cdots T_{i\Im z_n}^{\alpha_n} f, g \rangle_{L^2(\mathbb{R}^n)}.
\]

Here we have used the fact that \( H_{\exp}(\mathbb{C}^n) \) is invariant under all unbounded Toeplitz operators \( T_{i\Im z} \) which was proved in Proposition 3.1. It follows that:

\[
\mathcal{D} := \beta^{-1}[H_{\exp}(\mathbb{C}^n)] \subset H^\infty(\mathbb{R}^n) = \bigcap_{k \in \mathbb{N}} H^k(\mathbb{R}^n)
\]

where \( H^s(\mathbb{R}^n) \) denotes the \( k \)-th Sobolev space. Hence, for \( \alpha, \beta \in \mathbb{N}_0^n \) the restriction of (2.1) in Theorem 2.1 to \( \mathcal{D} \):

\[
\text{ad}[-iz]^\alpha \text{ad}[i\partial_x]^\beta (B) : \mathcal{D} \rightarrow \mathcal{D}
\]

is well-defined for any \( B \in L(\mathcal{D}) \). With the choice \( h \in L^\infty(\mathbb{C}^n) \) and \( L_h := \beta^{-1} T_h \beta \in L(\mathcal{D}) \) we obtain by conjugating (5.5) with \( \beta \) and using (i), (ii) above:

\[
\text{ad}[iT_{2\Re z}]^\alpha \text{ad}[T_{i\Im z}]^\beta (T_h) : H_{\exp}(\mathbb{C}^n) \rightarrow H_{\exp}(\mathbb{C}^n).
\]

It follows by Proposition 3.3 that the operators in (5.6) have bounded extensions to \( H^2(\mathbb{C}^n, \mu) \) and so (5.5) can be extended continuously to \( L^2(\mathbb{R}^n) \). Hence we have proved a weaker version of the defining property (2.1) for \( \Psi^0_{\rho, \delta} \) in Theorem 2.1.

Since the Gaussian measure \( \mu \) is invariant under unitary transformations of \( \mathbb{C}^n \), there is a natural group representation of \( U_n \) in \( L(H^2(\mathbb{C}^n, \mu)) \) generating \( \Psi^\ast \)-algebras of smooth elements. As a final example we want to remark:
Example 5.4 Let $A \in \mathbb{R}^{n \times n}$ be self-adjoint and consider the unitary group:
\[ \mathbb{R} \ni t \mapsto e^{itA} \in U_n. \]

The group of unitary composition operators $C_t f := f \circ e^{itA}$ on $H^2(\mathbb{C}^n, \mu)$ can be shown to be strongly continuous, cf. [3]. The restriction of the infinitesimal generator $L_A$ of $(C_t)_{t \in \mathbb{R}}$ to $\mathbb{P}_a[\mathbb{C}^n]$ coincides with an (unbounded) Toeplitz operator. More precisely, it was shown in [3] that:
\[ L_{Ap} = \left[ T_{\langle Az, z \rangle} - n \cdot \text{trace}(A) \right] p, \quad p \in \mathbb{P}_a[\mathbb{C}^n]. \]

Hence, in general the symbol of $L_A$ regarded as a Toeplitz operator is a polynomial of degree 2, which is not globally lipschitz continuous on $\mathbb{C}^n$. Proposition 3.3 cannot be applied in this situation and the smoothness of a Toeplitz operator $T_f$ with bounded symbols $f$ with respect to $(C_t)_t$ requires further assumption on the symbol $f$. For a more detailed calculation we refer to [3].

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References


TOEPLITZ $\Psi^*$-ALGEBRAS


