<table>
<thead>
<tr>
<th>Title</th>
<th>Toeplitz $\Psi^*$-algebras via unitary group representations (Analytic Function Spaces and Their Operators)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>BAUER, W.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1519: 1-20</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58759">http://hdl.handle.net/2433/58759</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Toeplitz $\Psi^*$-algebras via unitary group representations

W. Bauer *

Science University of Tokyo,
Department of Mathematics,
Noda, Chiba (278-8510)
Japan
Email: bauерwolfram@web.de

Abstract

As it was pointed out in [12] there are construction methods for spectral invariant Fréchet operator algebras such as $\Psi^*$- and $\Psi_0$-algebras in the bounded operators on a Hilbert space having prescribed properties. For the Segal-Bargmann space $H$ and using systems of unbounded closable Toeplitz operators $T_f$ where $f$ is in a certain class $\text{SP}_{\text{Lip}}(\mathbb{C}^n)$ of symbols we show that these algebras contain all Toeplitz operators $T_h$ with $h \in L^\infty(\mathbb{C}^n)$. Let $\rho$ be the Segal-Bargmann representation of the Heisenberg group $\mathbb{H}_n$ in the bounded operators on $H$. As an application of our results above we characterize a class of smooth Toeplitz operators in the $\Psi^*$-algebra of smooth elements with respect to $\rho$.

1 Introduction

Subsequent to the results in [12] it frequently has been remarked that the abstract concept of (locally) spectral invariant Fréchet algebras such as $\Psi_0$- and $\Psi^*$-algebras successfully can be applied to the structural analysis of certain algebras of pseudo-differential operators. Applications arise in complex analysis, analytic perturbation theory of Fredholm operators and non-abelian cohomology for analyzing isomorphisms of abelian groups in $K$-theory. By generalizing a characterization of the Hörmander classes $\Psi_{\rho,\delta}^0$ by commutator conditions (see Theorem 2.1) a construction method for algebras of the above mentioned type with prescribed properties have been given in [12].

*The author was supported by a JSPS postdoctoral fellowship (PE 05570) for North American and European Researchers.

$^0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$
TOEPLITZ $\Psi^*$-ALGEBRAS

Let $H := H^2(\mathbb{C}^n, \mu)$ be the Segal-Bargmann space of Gaussian square integrable entire functions on $\mathbb{C}^n$. We denote by $P$ the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto $H$ and we write $M_f$ for the multiplication with a measurable symbol $f$. In the initial stage of this paper we consider iterated commutators of closable Toeplitz operators $T_f := PM_f$ on $H$ having symbols in a certain class $\text{SP}_{\text{Lip}}(\mathbb{C}^n)$ of measurable and in general unbounded functions on $\mathbb{C}^n$. For a system $S_m := \{T_{f_1}, \ldots, T_{f_m}\}$ of operators with $f_j \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ and in the sense of [12] the $\Psi_0$-algebra $\Psi_\infty^S$ in the bounded operators $\mathcal{L}(H)$ on $H$ can be defined by commutator methods with respect to $S_m$. We show that $\Psi_\infty^S$ contains all Toeplitz operators with bounded measurable symbols. More precisely:

**Theorem A** The symbols map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi_\infty^S$ is well-defined and continuous.

Let $\mathbb{H}_n$ be the Heisenberg group and $\alpha$ be the Segal-Bargmann representation of $\mathbb{H}_n$ in $\mathcal{L}(H)$, c.f. [10]. The map $\alpha$ is well-known to be unitary, irreducible and strongly continuous. In particular, the $\Psi^*$-algebra $\Psi^\infty(\mathbb{H}_n) \subset \mathcal{L}(H)$ of smooth elements with respect to $\alpha$ arise in a natural way and it can be characterized by commutator methods. We describe a symmetric subspace $S_s \subset L^\infty(\mathbb{C}^n)$ with the induced topology such that:

**Theorem B** The symbols map $S_s \ni h \mapsto T_h \in \Psi^\infty(\mathbb{H}_n)$ is well-defined and continuous.

This result can be stated in terms of the algebra construction. Let $A$ be the algebra of multiplication operators on $V := L^2(\mathbb{C}^n, \mu)$ with bounded measurable symbols. In a natural way $\alpha$ extends to a representation of $\mathbb{H}_n$ into $\mathcal{L}(V)$ and the corresponding operator algebras $\Psi^k(A, \mathbb{H}_n)$ of $C^k$-elements in $A$ form a decreasing scale. Note that $M_f \in \Psi^k(A, \mathbb{H}_n)$ is related to the smoothness of the symbols $f \in L^\infty(\mathbb{C}^n)$. Clearly, $A$ projects under $P$ onto the space $A_P := PAP$ of Toeplitz operators with bounded symbols. Theorem B states:

$$P \Psi^k(A, \mathbb{H}_n) P = P \Psi^{k+1}(A, \mathbb{H}_n) P \subset \mathcal{L}(H) \quad \text{for all } k \in \mathbb{N}.$$

Heuristically, the smoothness of $f$ cannot be recovered by commutator methods from the Toeplitz operator $T_f$. We want to remark here that these results are related to an observation in [14], [3]. Let $\beta : L^2(\mathbb{R}^n) \rightarrow H$ be the Bargmann isometry and $f$ a bounded measurable function on $\mathbb{C}^n$. The assignment $\beta^{-1} T_f \beta$ can be shown to be a pseudo-differential operator $W_{\sigma(f)}$ on $L^2(\mathbb{R}^n)$ in its Weyl quantization. By identifying $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ the Weyl symbol $\sigma(f)$ and $f$ are related via the heat equation on $\mathbb{R}^{2n}$. There is $t_0 > 0$ such that:

$$\sigma(f) = e^{-t_0 \Delta} f := \text{solution of the heat equation with initial data } f \text{ at the time } t_0.$$

Moreover, $\sigma$ maps the space of continuous functions with compact support into the symbol class $S_0^\infty$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Corresponding to Theorem A and B it can be checked that $f \mapsto \sigma(f)$ is continuous with respect to the $L^\infty(\mathbb{C}^n)$ topology and the usual Fréchet topology on $S_0^{-\infty}$.

In our first section we remind of some basic definitions and results related to the construction of $\Psi_0$- and $\Psi^*$-algebras. For Toeplitz operators having symbols of polynomial growth at infinity an invariant subspace $\mathcal{H}_{\text{exp}}(\mathbb{C}^n)$ of $H$ is defined in section 3. Moreover,
the existence of bounded extensions for a class of iterated commutators of Toeplitz operators on $H_{\exp}(\mathbb{C}^n)$ and Theorem A are proved. Section 4 contains the proof of Theorem B and finally we have added some examples and applications in section 5.

2 Fréchet operator algebras with prescribed properties

The following definition due to B. Gramsch have been given in [11]:

Definition 2.1 Let $B$ be a Banach-algebra with unit $e$ and let $\mathcal{F}$ be a continuously embedded Fréchet algebra in $B$ with $e \in \mathcal{F}$. Then $\mathcal{F}$ is called $\Psi_{0}$-algebra if it is locally spectral invariant in $B$, i.e. there is $\epsilon > 0$ with

$$\{ a \in \mathcal{F} : \| e - a \|_B < \epsilon \} \subset \mathcal{F}^{-1}.$$  

Moreover, one defines:

- If $B$ is a $C^*$-algebra and $\mathcal{F}$ is a symmetric $\Psi_{0}$-algebra in $B$, then $\mathcal{F}$ is called $\Psi^*$-algebra. ($\mathcal{F}$ automatically is spectral invariant, i.e. $\mathcal{F} \cap B^{-1} = \mathcal{F}^{-1}$).

- If the topology of $\mathcal{F}$ is generated by a system $\{ q_j : j \in \mathbb{N} \}$ of sub-multiplicative semi-norms with $q_j(e) = 1$ for $j \in \mathbb{N}$, then $\mathcal{F}$ is called sub-multiplicative or locally $m$-convex (E. Michael, 1952) $\Psi_0$- resp. $\Psi^*$-algebra.

The concept of $\Psi^*$- and $\Psi_0$-algebras allows to treat phenomena of local structure. As it was observed for algebras of Pseudo-differential operators, $C^\infty$-properties such as pseudo- or micro- locality are preserved by taking closures in the Fréchet topology. Important examples of $\Psi^*$-algebras are given by the Hörmander classes $\Psi_{\rho,\delta}^0 \text{ of zero order where } B := L(L^2(\mathbb{R}^n))$. It is known that $\Psi_{\rho,\delta}^0$ can be described in terms of commutator conditions.

Theorem 2.1 (R. Beals, '77, [6])

An operator $B : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ is of class $\Psi_{\rho,\delta}^0$ iff for $\alpha, \beta \in \mathbb{N}_0^n$ all iterated commutators:

$$ad[-ix]^\alpha ad[i\partial_x]^\beta(B) : H^{s-\rho|\alpha|+\delta|\beta|} \rightarrow H^s$$

admit bounded extensions between suitable Sobolev spaces to $L^2(\mathbb{R}^n)$.

On the one hand the spectral invariance of $\Psi_{\rho,\delta}^0$ follows from the commutator characterizations in Theorem 2.1, see [19], [20]. On the other hand, by replacing $ix$ and $i\partial_x$ above with a system of closable and densely defined operators, conditions of the type (2.1) have been used to define (submultiplicative) $\Psi_0$-algebras in a fairly general situation, see [12]. Below we give the definitions and remind of some basic results.

\[20 \leq \delta \leq \rho \leq 1\text{ and } \delta < 1\]
2.1 Commutator Methods

Given a topological vector space $X$ we write $L(X)$ (resp. $\mathcal{L}(X)$) for the linear (resp. bounded linear) operators on $X$.

**Definition 2.2 (Iterated commutators)**

For a system $S_m := [A_1, \cdots, A_m]$ where $A_j, B \in L(X)$ we call $m$ the length of $S_m$. We inductively define the iterated commutators $\text{ad}^m \{B\} := B$ and:

- $\text{ad}^1 \{A_j\}(B) := [A_j, B] = A_jB - BA_j$,

In the case of $A = A_j$ where $j = 1, \cdots, m$ we also write:

- $\text{ad}^0 \{A\}(B) := B$ and $\text{ad}^m \{A\}(B) := \text{ad}^m \{S_m\}(B)$.

With these notations it follows for finite systems $S_j$ and $S_k$ in $L(X)$:

$$\text{ad}^m \{S_j\}(\text{ad}^n \{S_k\}(B)) = \text{ad}^m \{S_k, S_j\}(B).$$

Let $H$ be a Hilbert space and $F \subset \mathcal{L}(H)$ be a sub-multiplicative $\Psi^*$-algebra. Assume that the topology of $F$ is generated by a sequence $(q_j)_{j \in \mathbb{N}}$ of semi-norms and without loss of generality let $q_0 := \|\cdot\|_{\mathcal{L}(H)}$. Given a finite system $V$ of closed and densely defined operators $A : H \supset D(A) \to H$ and following [12] we define:

- $I(A) := \{ a \in F : a(D(A)) \subset D(A) \}$,

- $B(A) := \{ a \in I(A) : [A, a] \text{ extends to an element } \delta_A(a) \in F \}.$

Inductively, one obtains:

- $\Psi_0^V := F$, with semi-norms $q_{0,j} := q_j$ for $j \in \mathbb{N}$,

- $\Psi_1^V := \bigcap_{A \in V} B(A)$,

- $\Psi_k^V := \{ a \in \Psi_{k-1}^V : \delta_A a \in \Psi_{k-1}^V \text{ for all } A \in V \}$ where $k \geq 2$,

- $\Psi_{\infty}^V := \bigcap_{k \in \mathbb{N}} \Psi_k^V$.

This process leads to a decreasing scale of algebras in $F$:

$$F = \Psi_0^V \supset \cdots \supset \Psi_n^V \supset \Psi_{n+1}^V \supset \cdots \supset \Psi_{\infty}^V := \bigcap_{k \in \mathbb{N}} \Psi_k^V. \quad (2.2)$$

For $n \geq 1$, we inductively define a system $(q_{n,j})_{j \in \mathbb{N}}$ (resp. $(q_{n,j})_{j,n \in \mathbb{N}}$) of norms on $\Psi_n^V$ (resp. on $\Psi_{\infty}^V$) by:

$$g_{n,j}(a) := q_{n-1,j}(a) + \sum_{A \in V} q_{n-1,j}(\delta_A a). \quad (2.3)$$
According to [12], $\Psi_{n}^{\mathcal{V}}$ is a sub-multiplicative $\Psi_0$-algebra in $\mathcal{F}$. In the case where each $A \in \mathcal{V}$ is symmetric we replace $B(A)$ by:

$$B^*(A) := \{ a \in B(A) : a^* \in B(A) \}.$$ 

Then the algebras $\Psi_{n}^{\mathcal{V}}$ are symmetric and $\Psi_{n}^{\mathcal{V}}$ is a $\Psi^*$-algebra in $\mathcal{L}(H)$. Let $D \subset H$ be a core for $\mathcal{V}$, i.e. the inclusion $D \hookrightarrow \mathcal{D}(A)$ is dense with respect to the graph norm for all $A \in \mathcal{V}$. Then it was shown in [2], [3]:

**Proposition 2.1** Assume that $a \in \mathcal{F}$ and property $(E_k)$ holds for $k \in \mathbb{N} \cup \{\infty\}$:

$(E_k)$: $D$ is invariant under all $A \in \mathcal{V}$ and $a \in \mathcal{F}$. Moreover, assume that for any system

$$A \subset S_k(\mathcal{V}) := \{ [A_1, \ldots, A_j] : \text{where } A_l \in \mathcal{V} \text{ and } 1 \leq l \leq j \leq k \}.$$ 

$\text{ad}[A](a) : H \supset D \to H$ has a continuous extensions to $C(A, a) \in \mathcal{F}$.

Then $a \in \Psi_{k}^{\mathcal{V}}$ and $C(A, a)$ is a bounded extension of $\text{ad}[A](a) : H \subset \mathcal{D}(A) \to H$ to $H$ for any operator $A \in \mathcal{V}$.

The (locally) spectral invariance of $A \subset B$ is preserved under projections $p = p^2 \in A$. It is readily verified that $A_p := p A p$ is (locally) spectral invariant in $B_p := p B p$. If in addition $B$ is a $C^*$-algebra, $A$ is symmetric in $B$ and $p = p^*$, then $A_p$ is symmetric and spectral invariant in $B_p$.

With (2.2) and an orthogonal projection $p \in \Psi_{n}^{\mathcal{V}}$, $n \in \mathbb{N} \cup \{\infty\}$ from $H$ onto a closed subspace $H_0 \subset H$ there is a scale of projected algebras in $\mathcal{L}(H_0)$:

$$\mathcal{L}(H_0) \supset \mathcal{F}_p = \Psi_{0p}^{\mathcal{V}} \supset \cdots \supset \Psi_{n-1p}^{\mathcal{V}} \supset \Psi_{np}^{\mathcal{V}}.$$ \hspace{1cm} (2.4)

It can be shown that (2.4) arises by commutator methods with a system $\mathcal{V}_p$ of closed operators on $H_0$ where $\mathcal{D}(A_p) := p [\mathcal{D}(A)]$ and

$$\mathcal{V}_p := \{ A_p := p A p : H_0 \supset \mathcal{D}(A_p) \to H_0 : A \in \mathcal{V} \}.$$ 

Defining (2.4) by commutator conditions with respect to $\mathcal{V}_p$ only requires that $p \in \Psi_{1}^{\mathcal{V}}$. Thus this method gives a natural extension of (2.4) to an infinite scale for $n \in \mathbb{N}$.

There is a corresponding scale of $\mathcal{V}$-Sobolev spaces in $H$:

- $\mathcal{H}_{0}^{\mathcal{V}} := H$ with the norm $p_0 := \| \cdot \|_H$.
- $\mathcal{H}_{1}^{\mathcal{V}} := \bigcap_{A \in \mathcal{V}} \mathcal{D}(A)$.
- $\mathcal{H}_{k}^{\mathcal{V}} := \{ x \in \mathcal{H}_{k-1}^{\mathcal{V}} : Ax \in \mathcal{H}_{k-1}^{\mathcal{V}} \text{ for all } A \in \mathcal{V} \}, \quad k \geq 2$.
- $\mathcal{H}_{\infty}^{\mathcal{V}} := \bigcap_{k \in \mathbb{N}} \mathcal{H}_{k}^{\mathcal{V}}$. 
We endow $\mathcal{H}_V^k$ with the norm
\[ p_k(x) := p_{k-1}(x) + \sum_{A \in \mathcal{V}} p_{k-1}(Ax), \quad x \in \mathcal{H}_V^k. \]

Let the topology of $\mathcal{H}^\infty_V$ be defined by the system of norms $(p_k)_{k \in \mathbb{N}_0}$. It can be shown that $(\mathcal{H}_V^k, p_k)$ is a Banach space and $(\mathcal{H}^\infty_V, (p_k)_{k \in \mathbb{N}})$ turns into a Fréchet space. Moreover, each $A \in \mathcal{V}$ induces a bounded operator $A_k: \mathcal{H}_V^k \to \mathcal{H}_V^{k-1}$. For $n \in \mathbb{N} \cup \{\infty\}$ it was shown in [12] that all maps
\[ \Psi^*_k \times \mathcal{H}_V^k \longrightarrow \mathcal{H}_V^k: (a, x) \mapsto a(x) \]
are bilinear and continuous. The following result on regularity was proved in [13]:

**Theorem 2.2** Let $A \in \Psi^*_\infty$ be a Fredholm operator and $u \in H$ with $Au = f \in \mathcal{H}_V^k$ for some $k \in \mathbb{N} \cup \{\infty\}$. Then it follows that $u \in \mathcal{H}_V^k$.

### 3 On the Segal-Bargmann Projection

Throughout this paper we write $\langle x, y \rangle := x_1 \bar{y}_1 + \cdots x_n \bar{y}_n$ for the Hermitian inner product on $\mathbb{C}^n$ and $|x| := \sqrt{\langle x, x \rangle}$. For $c > 0$ and the Lebesgue measure $\mu$ let us denote by $\mu_c$ the Gaussian measure on $\mathbb{C}^n$ given by:
\[ d\mu_c = c^n \pi^{-n} \exp(-c|x|^2) dv. \]

With $\mu := \mu_1$ let $H^2(\mathbb{C}^n, \mu)$ be the *Segal-Bargmann space* of $\mu$-square integrable entire functions on $\mathbb{C}^n$. We denote by $P$ the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto $H^2(\mathbb{C}^n, \mu)$. The reproducing kernel $K$ (resp. the normalized kernel $k$) corresponding to $H^2(\mathbb{C}^n, \mu)$ are known to be:

(a) $K(y, x) := \exp(\langle y, x \rangle)$,

(b) $k_x(y) := K(y, x) \| K(\cdot, x) \|^{-1} = \exp(\langle y, x \rangle - \frac{1}{2} |x|^2)$

where $\| \cdot \|$ denotes the $L^2(\mathbb{C}^n, \mu)$-norm. For $z, w \in \mathbb{C}^n$ we write $\tau_w(z) := z + w$ for the shift by $w$. Consider the space of measurable symbols on $\mathbb{C}^n$ given by:
\[ T(\mathbb{C}^n) := \{ g : g \circ \tau_x \in L^2(\mathbb{C}^n, \mu) \text{ for all } x \in \mathbb{C}^n \}. \]

For $g \in T(\mathbb{C}^n)$ and with the natural domain of definition
\[ D(T_g) := \{ f \in H^2(\mathbb{C}^n, \mu) : gf \in L^2(\mathbb{C}^n, \mu) \} \]
the *Toeplitz operator* $T_g$ on $H^2(\mathbb{C}^n, \mu)$ is densely defined by:
\[ T_g : D(T_g) \ni f \mapsto P(fg). \]

If $g$ has *polynomial growth at infinity* we can determine an invariant subspace for $T_g$:

We inductively define a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_1 := \frac{1}{4}$ and $a_{n+1} := \lfloor 4 \cdot (1 - a_n) \rfloor^{-1}$ for all $n \geq 2$. It can be checked that:
W. BAUER

(a) \( a_n < \frac{1}{2} \), \( \forall n \in \mathbb{N} \),
(b) \((a_n)_{n \in \mathbb{N}}\) is strictly increasing,
(c) \( \lim_{n \to \infty} a_n = \frac{1}{2} \).

Let \( \mathcal{P}[\mathbb{C}^n] \) be the space of all polynomials on \( \mathbb{C}^n \) in the variables \( z := (z_1, \ldots, z_n) \) and \( \bar{z} := (\bar{z}_1, \ldots, \bar{z}_n) \). We write \( \mathcal{P}_a[\mathbb{C}^n] \) for all analytic polynomials and set:

\[
L_{\text{exp}}(\mathbb{C}^n) := \{ f \in L^2(\mathbb{C}^n, \mu) : \exists c < \frac{1}{2}, 0 < D \text{ s.t. } |f(z)| \leq D \exp(c|z|^2) \text{ a.e.} \}.
\]

Because of \( \mathcal{P}[\mathbb{C}^n] \subset L_{\text{exp}}(\mathbb{C}^n) \) it follows that \( L_{\text{exp}}(\mathbb{C}^n) \) is dense in \( L^2(\mathbb{C}^n, \mu) \).

Consider the symbols having polynomial growth at \( \infty \):

\[
\text{Po}_1(\mathbb{C}^n) := \{ f : \exists j \in \mathbb{N} \text{ s.t. } |f(z)|(1+|z|^2)^{-j} \in L^\infty(\mathbb{C}^n) \}.
\]

**Proposition 3.1** It holds \( P[L_{\text{exp}}(\mathbb{C}^n)] \subset H_{\text{exp}}(\mathbb{C}^n) \) and for \( f \) in \( \text{Pol}(\mathbb{C}^n) \):

\[
T_f[H_{\text{exp}}(\mathbb{C}^n)] \subset H_{\text{exp}}(\mathbb{C}^n) \subset D(T_f) \tag{3.2}
\]

**Proof:** It is obvious that \( H_{\text{exp}}(\mathbb{C}^n) \subset D(T_f) \). Because the multiplication by \( f \) clearly maps \( H_{\text{exp}}(\mathbb{C}^n) \) into \( L_{\text{exp}}(\mathbb{C}^n) \) it is sufficient to prove the first assertion of Proposition 3.1. For \( g \in L_{\text{exp}}(\mathbb{C}^n) \) there are \( c < \frac{1}{2} \) and \( D > 0 \) such that a.e.:

\[
|g(z)| \leq D \exp(c|z|^2).
\]

By (a), (b) and (c) and with \((a_n)_{n \in \mathbb{N}}\) above we can choose \( n_0 \in \mathbb{N} \) with \( c < a_{n_0} < \frac{1}{2} \).

Using the transformation formula and the reproducing property of \( K \) we obtain:

\[
\begin{align*}
| [Pg](z) | & \leq \int_{\mathbb{C}^n} |g \exp\{ \langle z, \cdot \rangle \}| d\mu \\
& \leq D \pi^{-n} \int_{\mathbb{C}^n} \exp\left\{ \text{Re}\langle z, \cdot \rangle - \left[ 1 - a_{n_0} \right] |\cdot|^2 \right\} d\mu \\
& = D (1 - a_{n_0})^{-n} \int_{\mathbb{C}^n} \exp\left\{ 2\text{Re}\left( 2^{-1} (1 - a_{n_0})^{-\frac{1}{2}} z, \cdot \right) \right\} d\mu \\
& = D (1 - a_{n_0})^{-n} \exp\left\{ \frac{4}{a_{n_0} + 1} |z|^2 \right\}.
\end{align*}
\]

From (a) above we conclude that \( Pg \in H_{\text{exp}}(\mathbb{C}^n) \).

Hence all finite products of Toeplitz operators with symbols in \( \text{Pol}(\mathbb{C}^n) \) are well-defined on the dense subspace \( H_{\text{exp}}(\mathbb{C}^n) \) of \( H^2(\mathbb{C}^n, \mu) \). In particular, all iterated commutators of \( P \) and multiplication operators \( M_f \) with \( f \in \text{Pol}(\mathbb{C}^n) \) can be considered as elements in \( L(L_{\text{exp}}(\mathbb{C}^n)) \). In fact, they can be written as integral operators and a standard application of the Schur test leads to a criterion for the boundedness.
Lemma 3.1 Let \( L : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \) be a measurable function such that:
\[
|L(x, y)| \leq |F(x-y)| \exp \left\{ \text{Re} \langle x, y \rangle \right\}
\]
where \( F \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}}) \). Then the integral operator \( A \) on \( L^2(\mathbb{C}^n, \mu) \) defined by
\[
[Af](z) := \int_{\mathbb{C}^n} L(z, \cdot) f d\mu
\]
is bounded on \( L^2(\mathbb{C}^n, \mu) \) with \( \|A\| \leq 2^n \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})} \).

Proof: With \( p := q = \exp(\frac{1}{2} |\cdot|^2) \) on \( \mathbb{C}^n \) it follows that:
\[
\int_{\mathbb{C}^n} |L(\cdot, y)| p d\mu \leq \frac{1}{\pi^n} \int_{\mathbb{C}^n} |F(\cdot-y)| \exp \left\{ \text{Re} \langle \cdot, y \rangle - \frac{1}{2} |\cdot|^2 \right\} dv \\
= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |F| \exp \left\{ \text{Re} \langle \cdot+y, y \rangle - \frac{1}{2} |\cdot+y|^2 \right\} dv \\
= 2^n p(y) \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}.
\]
Similarly, we get \( \int |L(x, \cdot)| p d\mu \leq 2^n p(x) \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})} \).
Applying the Schur test we obtain the desired result. \( \square \)

Consider the subspace \( \text{SP}_{\text{Lip}}(\mathbb{C}^n) \) of \( \text{Pol}(\mathbb{C}^n) \) defined by:
\[
\text{SP}_{\text{Lip}}(\mathbb{C}^n) := \{ f \in \text{Pol}(\mathbb{C}^n) : \exists c, D > 0 \text{ s.t. } |f(z) - f(w)| \leq D \exp(c |z-w|) \}.
\]

As an application of Lemma (3.1) we can prove:

Proposition 3.2 Let \( m \in \mathbb{N} \) and \( S_m := \{M_{f_1}, \ldots, M_{f_m}\} \) with \( f_j \in \text{SP}_{\text{Lip}}(\mathbb{C}^n) \). Then the commutator \( \text{ad}[\mathcal{S}_m](P) \in L(L_{\text{exp}}(\mathbb{C}^n)) \) has a continuous extension to \( L^2(\mathbb{C}^n, \mu) \).

Proof: It is easy to check that the commutator \( \text{ad}[\mathcal{S}_m](P) \) can be written as an integral operator on \( L^2(\mathbb{C}^n, \mu) \) with kernel:
\[
K_m(z, u) = \exp \left( \langle z, u \rangle \right) \prod_{j=1}^m \{ f_j(z) - f_j(u) \}. \tag{3.3}
\]

By (3.3) and our assumptions on \( f_j \in \mathcal{S}_m \) we can choose \( c, D > 0 \) such that
\[
|K_m(z, u)| \leq D \exp \left( c |z-u| + \text{Re} \langle z, u \rangle \right).
\]

Because of \( F := D \exp(c |\cdot|) \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}}) \) Lemma 3.1 implies the assertion. \( \square \)

We remark that by (3.3) the maps \( \text{ad}[\mathcal{S}_m](P) \) are invariant under permutations of the system \( \mathcal{S}_m \). Now, we can prove the boundedness of a class of iterated commutators.
Corollary 3.1 Let $g \in L^\infty(\mathbb{C}^n)$ and $S_m := \{M_{f_1}, \cdots, M_{f_m}\}$ with $f_j \in \text{SP}_{Lip}(\mathbb{C}^n)$. Then the commutator

$$\text{ad} [S_m] \left( \left[ P, M_g \right] \right) \in L( L_{\exp}(\mathbb{C}^n) )$$

has a bounded extensions $A(S_m, g)$ to $L^2(\mathbb{C}^n, \mu)$ and (3.4) below is continuous between Banach spaces:

$$L^\infty(\mathbb{C}^n) \ni g \mapsto A(S_m, g) \in L( L^2(\mathbb{C}^n, \mu) ) .$$

(3.4)

Proof: It can be checked by induction or our remark following Proposition 3.2 that:

$$\text{ad} [S_m] \left( \left[ P, M_g \right] \right) = \left[ \text{ad} [S_m] (P), M_g \right] \in L( L_{\exp}(\mathbb{C}^n) ) .$$

Because $M_g$ is bounded and $\text{ad} [S_m] (P)$ has a bounded extension to $L^2(\mathbb{C}^n, \mu)$ by Proposition 3.2 we conclude the desired result. \qed

Given a finite set $X := \{X_1, \cdots, X_n\} \subset L(L^2(\mathbb{C}^n, \mu))$ we denote by $A(X)$ the algebra generated by $X$. Moreover, we write:

$$A_P(X) := P A(X) P := \{ PAP : A \in A(X) \} .$$

for the corresponding projected algebra in $L( H^2(\mathbb{C}^n, \mu) )$. By Proposition 3.1 and for all $m \geq 1$ it follows that the commutator:

$$\text{ad} [S_{m-1}] \left( \left[ P, M_{f_m} \right] \right) = -\text{ad} [S_m] (P)$$

can be regarded as bounded operators on $L^2(\mathbb{C}^n, \mu)$.

Proposition 3.3 Let $g \in L^\infty(\mathbb{C}^n)$ and $T_m := \{T_{f_1}, \cdots, T_{f_m}\}$ with $f_j \in \text{SP}_{Lip}(\mathbb{C}^n)$. Then

$$\text{ad} [T_m] (T_g) \in L( H_{\exp}(\mathbb{C}^n) )$$

is well-defined. More precisely, with $S_m := \{M_{f_1}, \cdots, M_{f_m}\}$ it holds:

$$\text{ad} [T_m] (T_g) \in A_P \left\{ \text{ad} [N] (P), M_g : \text{with } N \subset S_m \right\} \quad (3.5)$$

and $\text{ad} [T_m] (T_g)$ has a bounded extension $C(T_m, g)$ to $H^2(\mathbb{C}^n, \mu)$. Moreover, the symbols map

$$L^\infty(\mathbb{C}^n) \ni g \mapsto C(T_m, g) \in L( H^2(\mathbb{C}^n, \mu) ) \quad (3.6)$$

is continuous between Banach spaces.

Proof: By Proposition 3.1 the iterated commutators $\text{ad} [T_m] (T_g)$ are well-defined. It is a straightforward computation that:

$$\text{ad} [T_1] (T_g) = P \left[ \left[ P, M_{f_1} \right], \left[ P, M_g \right] \right] P$$
which proves (3.5) in the case $m = 1$. By induction assume $\text{ad} [T_j] (T_g)$ has the form:

$$\text{ad} [T_j] (T_g) = \sum_{l \in I} P A_l M_g B_l P$$

(3.7)

where $I$ is a finite index set, $I$ the identity operator and

$$A_l, B_l \in \mathcal{A}(S_j) := \mathcal{A} \{ \text{ad} [\mathcal{N}] (P), I : \text{with } \mathcal{N} \subset S_j \}.$$  

(3.8)

Then it follows that:

$$\text{ad} [T_{j+1}] (T_g) = \sum_{l \in I} [T_{f_{j+1}}, P A_l M_g B_l P].$$

To prove (3.7) in the case $j + 1$ it is sufficient to show for all $l \in I$ the existence of a finite set $\tilde{I} \subset \mathbb{N}$ and operators $C_k, D_k \in \mathcal{A}(S_{j+1})$ such that

$$[T_{f_{j+1}}, P A_l M_g B_l P] = \sum_{k \in \tilde{I}} P C_k M_g D_k P.$$  

(3.9)

Note that (3.9) follows from $T_{f_{j+1}} P A_l M_g B_l P = P M_{f_{j+1}} P A_l M_g B_l P$ and

$$[M_{f_{j+1}}, Q] \in \mathcal{A}(S_{j+1})$$

for $Q \in \{ P, A_l, B_l \}$. The continuity of (3.6) is a direct consequence of (3.7). \qed

As an immediate consequence of Proposition 3.2 we remark:

**Lemma 3.2** Let $f \in \text{SP}_{Lip}(\mathbb{C}^n)$ and $\mathcal{D}(T_f)$ as in (3.1). Then the Toeplitz operator $T_f$ is densely defined and closed on $\mathcal{D}(T_f)$.

**Proof:** Because of $f \in T(\mathbb{C}^n)$ it follows that $T_f$ is densely defined. Moreover,

$$M_f = T_f + [M_f, P] : \mathcal{D}(T_f) \subset H^2(\mathbb{C}^n, \mu) \rightarrow L^2(\mathbb{C}^n, \mu).$$  

(3.10)

Proposition 3.2 with $j = 1$ shows that the commutator $[M_f, P]$ has a continuous extension to $H^2(\mathbb{C}^n, \mu)$. Choose a sequence $(h_n)_{n \in \mathbb{N}} \subset \mathcal{D}(T_f)$ such that:

(i) $\lim_{n \to \infty} h_n = h \in H^2(\mathbb{C}^n, \mu),$

(ii) $\lim_{n \to \infty} T_f h_n = g \in H^2(\mathbb{C}^n, \mu).$

Then we conclude from the continuity of $[M_f, P]$ and (3.10) that

$$f h = \lim_{n \to \infty} f h_n \in L^2(\mathbb{C}^n, \mu)$$

Hence $h \in \mathcal{D}(T_f)$ and $g = \lim_{n \to \infty} P(f h_n) = T_f h$. \qed
Let $\mathcal{T}_m := \{T_{f_1}, \cdots, T_{f_m}\}$ be a system of Toeplitz operators where $f_j \in \text{SpLip}(\mathbb{C}^n)$ for $j = 1, \cdots, n$. From Lemma 3.2 it follows that the domains $\mathcal{D}(T_{f_j})$ are closed with respect to the graph norm $\| \cdot \|_{\text{gr}} := \| \cdot \| + \| T_{f_j} \cdot \|$. Consider $D_j \subset H^2(\mathbb{C}^n, \mu)$ defined by:

$$D_j := \| \cdot \|_{\text{gr}} - \text{closure of } H_{\exp}(\mathbb{C}^n) \text{ in } \mathcal{D}(T_{f_j})$$

If we consider $T_{f_j}$ as a closed operator on $D_j$ we can define a scale of algebras (2.2) by commutator methods with the system $\mathcal{S}_m$. By Lemma 2.1 with $D := H_{\exp}(\mathbb{C}^n)$ our result in Proposition 3.3 can be formulated as follows:

**Theorem 3.1** The symbol map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi_\infty^{\mathcal{S}_m}$ is well-defined and continuous.

4. **Toeplitz $\Psi^*$-algebras via the Segal-Bargmann representation**

There is a unitary representation of the Heisenberg group $\mathbb{H}_n$ in $\mathcal{L}(L^2(\mathbb{C}, \mu))$. By identifying $\mathbb{H}_n$ with $\mathbb{C}^n \times \mathbb{R}$ the group law is given by, [10]:

$$(z, t) \ast (w, s) := (z + w, t + s + 2^{-1} \text{Im}(w, z)).$$

For $z \in \mathbb{C}^n$ and $f \in L^2(\mathbb{C}^n, \mu)$ we define the operator $W_z f := k_z \cdot f \circ \tau_- z$. It follows by an easy calculation:

**Lemma 4.1** $H^2(\mathbb{C}^n, \mu)$ is an invariant subspace for all $W_z$ where $z \in \mathbb{C}^n$. Moreover,

1. $W_z$ is unitary with $W_z^* = W_{-z} = W_z^{-1}$,
2. The commutator $[P] W_z$ vanishes,
3. For $z, w \in \mathbb{C}^n : W_z W_w = \exp(i \text{Im}(w, z)) W_{z+w}$.

By Lemma 4.1 a unitary representation $\tilde{\rho} : \mathbb{H}_n \rightarrow \mathcal{L}(L^2(\mathbb{C}^n, \mu))$ of $\mathbb{H}_n$ is given by:

$$\tilde{\rho}(z, t) := e^{it} W_{\frac{z}{\sqrt{2}}}.$$

Moreover, the restriction of $\tilde{\rho}(z, t)$ to $H^2(\mathbb{C}^n, \mu)$ gives rise to a unitary representation $\rho$ of $\mathbb{H}_n$ in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$. It is well-known that $\rho$ is irreducible and strongly continuous and it is referred to as Segal-Bargmann representation, c.f. [10].

For any $A \in B := \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ we define the map:

$$\Phi_A : \mathbb{H}_n \rightarrow B$$

$$(z, t) \mapsto \rho(z, t) A \rho(z, t)^{-1} = W_{\frac{z}{\sqrt{2}}} A W_{\frac{z}{\sqrt{2}}}.$$ (4.1)
In particular, note that for $f \in L^\infty(\mathbb{C}^n)$
$$
\Phi_{T_f}(z, t) = T_{f \circ \tau_{-z, t}}.
$$

For $k \in \mathbb{N} \cup \{\infty\}$ we consider the $C^k$-elements
$$
\Psi^k := \{ A \in \mathcal{B} : \Phi_A \in C^k(\mathbb{H}_n, \mathcal{B}) \}
$$
defined via $\rho$. To any $z \in \mathbb{C}^n$ we associate $\varphi_A^z : \mathbb{R} \to \mathcal{B}$ by $\varphi_A^z(s) := W_{sz}AW_{-sz}$. According to (4.1) it follows that:
$$
\Psi^k = \bigcap_{z \in \mathbb{C}^n} \Psi^{k, z}
$$
where $\Psi^{k, z} := \{ A \in \mathcal{B} : \varphi_A^z \in C^k(\mathbb{R}, \mathcal{B}) \}$.

Here we characterize the $C^k$-Toeplitz operators (i.e. the Toeplitz operators $T_f \in \Psi^k$) in terms of their symbols. We use a characterization of $\Psi^\infty$ by commutator conditions and apply our results of the previous section.

For all $z \in \mathbb{C}^n$ the map $(W_{sz})_{s \in \mathbb{R}} \subset \mathcal{B}$ defines a strongly continuous unitary group. By $V_z$ we denote its infinitesimal generator with domain of definition:
$$
D(V_z) := \{ h \in H^2(\mathbb{C}^n, \mu) : V_zh := \lim_{s \to 0} s^{-1}(W_{sz} - I)h \text{ exists} \}.
$$

By Stone's Theorem $iV_z$ is selfadjoint and associated to $V_z := [iV_z]$ there is a scale:
$$
\mathcal{B} := \Psi_0^\infty \supset \cdots \Psi_n^\infty \supset \Psi_{n+1}^\infty \supset \cdots \supset \Psi_{\infty}^\infty := \bigcap_{k\in \mathbb{N}} \Psi_k^\infty
$$
of algebras in $\mathcal{B}$ defined by commutator methods with $V_z$ as it was described in (2.2) of section 2.1. In particular, $\Psi_\infty^\infty$ is a $\Psi^*$-algebra and it is well-known that (4.2) and (4.3) are related as follows, see [16]:

**Proposition 4.1** For $z \in \mathbb{C}^n$ let $V_z := [iV_z]$ then:

(i) $\Psi^{k, z} \subset \Psi_k^V$ for $k \in \mathbb{N},$

(ii) $\Psi_k^V \subset \Psi_{k+1}^\infty$ for $k \in \mathbb{N}_0$ and $\Psi_{\infty}^\infty = \Psi_\infty^V$.

Using the fact that convergence in $H^2(\mathbb{C}^n, \mu)$ implies uniformly compact convergence on $\mathbb{C}^n$ we can calculate $V_z$ explicitly. Let $h \in D(V_z)$ and $w \in \mathbb{C}^n$:
$$
[V_z h](w) = \frac{d}{ds} [k_{sz}(w) h(w - sz)]_{s=0} = \left\{ \langle w, z \rangle - \sum_{j=1}^{n} z_j \frac{\partial}{\partial w_j} \right\} h(w). \quad (4.4)
$$

It easily can be seen that all the monomials $m_\alpha(z) := z^\alpha$ for $\alpha \in \mathbb{N}_0^n$ are contained in the domain $D(V_z)$. Moreover, from the standard identities $M_{w_j} := T_{w_j}$ and $\frac{\partial}{\partial w_j} := T_{\overline{w_j}}$ it follows that the restriction of $V_z$ to $\mathbb{P}_a[\mathbb{C}^n]$ coincides with an unbounded Toeplitz operator:
$$
V_z p := T_{\langle \cdot, z \rangle - \langle z, \cdot \rangle p} = 2i \, T_{\text{Im} \langle \cdot, z \rangle \, p}, \quad p \in \mathbb{P}_a[\mathbb{C}^n].
$$
In the following we write:

\[ g_z := 2i \text{Im} \langle \cdot, z \rangle \]

for the symbol of the Toeplitz operator appearing above. Consider the space \( \mathcal{D}(T_{g_z}) \) with the graph norm \( \| \cdot \|_{gr} := \| \cdot \| + \| T_{g_z} \cdot \| \). By Lemma 3.2 it follows that \( (\mathcal{D}(T_{g_z}), \| \cdot \|_{gr}) \) is a Banach space containing \( \mathbb{P}_a[\mathbb{C}^n] \) and \( H_{exp}(\mathbb{C}^n) \).

**Lemma 4.2** For all \( z \in \mathbb{C}^n \) the embedding \( \mathbb{P}_a[\mathbb{C}^n] \hookrightarrow H_{exp}(\mathbb{C}^n) \) is dense with respect to the graph norm topology. Moreover,

\[ H_{exp}(\mathbb{C}^n) \subset \mathcal{D}(V^z) \cap \mathcal{D}(T_{g_z}) \]  \( (4.5) \)

and the restrictions of \( V^z \) and \( T_{g_z} \) to \( H_{exp}(\mathbb{C}^n) \) coincide.

**Proof:** For \( f \in H_{exp}(\mathbb{C}^n) \) we can choose \( c_1 \in (0, \frac{1}{2}) \) and \( D_1 > 0 \) such that:

\[ |f(w)| \leq D_1 \exp(c_1 |w|^2) \]

for all \( z \in \mathbb{C}^n \). Hence, \( f \in L^2(\mathbb{C}^n, \mu_{r}) \) for all \( r \in (2c_1, 1) \). Fix \( c_2, c_3 \) with \( 2c_1 < c_2 < c_3 < 1 \) and choose \( D_2 > 0 \) with

\[ |w|^2 \leq D_2 \exp(\left| c_3 - c_2 \right| |w|^2) \]

for all \( w \in \mathbb{C}^n \). Then we obtain for all \( p \in \mathbb{P}_a[\mathbb{C}^n] \):

\[ \| T_{g_z} (f - p) \|^2 \leq \| g_z (f - p) \|^2 \]

\[ \leq 2 |z|^2 \int_{\mathbb{C}^n} \| f - p \|^2 \, d\mu \]

\[ \leq 2D_2 |z|^2 r^{-n} \| f - p \|^2_{L^2(\mathbb{C}^n, \mu_{r})} < \infty \]

where \( r = 1 - c_3 + c_2 \in (2c_1, 1) \). Because \( \mathbb{P}_a[\mathbb{C}^n] \) is dense in \( L^2(\mathbb{C}^n, \mu_{r}) \cap \mathcal{H}(\mathbb{C}^n) \) for all \( r > 0 \) the first assertion follows.

Now, \( (4.5) \) immediately can be derived from \( T_{g_z} p = V^z p \) for \( p \in \mathbb{P}_a[\mathbb{C}^n] \) and the density result above which implies that:

\[ H_{exp}(\mathbb{C}^n) \subset \text{closure}( \mathbb{P}_a[\mathbb{C}^n], \| \cdot \|_{gr} ) \subset \mathcal{D}(V^z) \cap \mathcal{D}(T_{g_z}) \]  \( (4.6) \)

Finally, we apply the continuity of \( V^z, T_{g_z} : (\mathbb{P}_a[\mathbb{C}^n], \| \cdot \|_{gr}) \to H^2(\mathbb{C}^n, \mu) \).

For \( z \in \mathbb{C}^n \) we denote by \( \tilde{V}^z \) the infinitesimal generator of \( (W_{sz})_{s \in \mathbb{R}} \) considered as strongly continuous group of unitary operators on \( L^2(\mathbb{C}^n, \mu) \). Let \( \mathcal{D}(\tilde{V}^z) \) be its domain of definition, then \( V^z \) can be obtained by restricting \( \tilde{V}^z \) to \( \mathcal{D}(V^z) \). For \( f \in \text{SP}_{\text{Lip}}(\mathbb{C}^n) \) and \( r \in \mathbb{N} \) we write

\[ \mathcal{A}_r(f) := \mathcal{A}( [M_{f_{1}}, \ldots, M_{f_{r}}] ) \subset \mathcal{L}(L^2(\mathbb{C}^n, \mu)) \]

where the algebra on the right hand side was defined in (3.8) of Proposition 3.3.
**Lemma 4.3** The domain $\mathcal{D}(\tilde{V}^{z})$ is invariant under $A \in \mathcal{A}_{r}(f)$ where $f$ is a linear function on $\mathbb{C}^{n}$. Moreover, the commutator $[A, \tilde{V}^{z}]$ vanishes as an operator on $\mathcal{D}(\tilde{V}^{z})$.

**Proof:** It is sufficient to show that for all $j \in \mathbb{N}$ the space $\mathcal{D}(\tilde{V}^{z})$ is invariant under the operators

$$a_{j}(f) := \text{ad}^{j}[M_{f}](P).$$

Note that $\mathbb{L}_{\exp}(\mathbb{C}^{n})$ is an invariant under $W_{z}$ and it holds $W_{-z}M_{f}W_{z} = M_{f_{0}r_{z}}$. Because $W_{z}$ commutes with $P$ it follows that:

$$W_{-z}a_{j}(f)W_{z} = \text{ad}^{j}[M_{f_{0}r_{z}}](P) = a_{j}(f).$$

We have used the linearity of $f$ for the second equality. Hence, the commutator $[A, W_{z}]$ vanishes as an operator on $\mathcal{D}(\tilde{V}^{z})$.

Fix $h \in \mathcal{D}(\tilde{V}^{z})$ and $A \in \mathcal{A}_{r}(f)$, then:

$$\frac{1}{s}\{W_{sz} - I\} A h = A\frac{1}{s}\{W_{sz} - I\} h \rightarrow A\tilde{V}^{z}h$$

as $s$ tends to 0. It follows that $Ah \in \mathcal{D}(\tilde{V}^{z})$ with $\tilde{V}^{z}Ah = A\tilde{V}^{z}h$. □

**Remark 4.1** Let $W$ be any subspace of $\mathcal{H} := \mathcal{H}^{2}(\mathbb{C}^{n}, \mu)$ such that $\mathbb{L}_{\exp}(\mathbb{C}^{n}) \subset W$. Consider the operators:

$$\mathcal{O}_{W} := \{ A \in \mathcal{L}(W, \mathcal{H}) : \mathbb{L}_{\exp}(\mathbb{C}^{n}) \text{ is an invariant space for } A \}.$$ 

Let $A \in \mathcal{O}_{W}$ and assume there is $A^{*} \in \mathcal{O}_{W}$ with $\langle Af, g \rangle = \langle f, A^{*}g \rangle$ for all $f, g \in W$. Because of $K(\cdot, \lambda) \in \mathbb{L}_{\exp}(\mathbb{C}^{n})$ for all $\lambda \in \mathbb{C}^{n}$ it follows that $A$ can be written as an integral operator with kernel:

$$K_{A}(\cdot, z)(w) = \overline{A^{*}K(\cdot, z)(w)}. \quad (4.6)$$

In particular, $A$ completely is determined by the restriction of $A^{*}$ to $\mathbb{L}_{\exp}(\mathbb{C}^{n})$. Assume that $A$ has a continuous extensions $\bar{A}$ from $\mathbb{L}_{\exp}(\mathbb{C}^{n})$ to $\mathcal{H}^{2}(\mathbb{C}^{n}, \mu)$. Fix $g \in \mathcal{H}^{2}(\mathbb{C}^{n}, \mu)$ and a sequence $(g_{n})_{n} \subset \mathbb{L}_{\exp}(\mathbb{C}^{n})$ with $g = \lim_{n \rightarrow \infty} g_{n}$. Then it follows for $z \in \mathbb{C}^{n}$:

$$[\bar{A}g](z) = \lim_{n \rightarrow \infty} \langle Ag_{n}, K(\cdot, z) \rangle = \lim_{n \rightarrow \infty} \langle g_{n}, A^{*}K(\cdot, z) \rangle = \langle g, A^{*}K(\cdot, z) \rangle$$

and $\bar{A}$ is given by the same integral formula. In particular, $A$ has a (unique) extension from $W$ to $\mathcal{H}^{2}(\mathbb{C}^{n}, \mu)$.

Let $h \in L^{\infty}(\mathbb{C}^{n})$ and $f : \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a linear function. We write $C_{j}(f, h)$ for the continuous extensions of the commutators

$$\text{ad}^{j}[T_{f}](T_{h}) \in \mathcal{L}(\mathcal{H}_{\exp}(\mathbb{C}^{n}))$$

to $\mathcal{H}^{2}(\mathbb{C}^{n}, \mu)$, (note that $f \in \text{SP}_{\text{Lip}}(\mathbb{C}^{n})$ and Proposition 3.3).
Corollary 4.1 Let $h \in L^\infty(\mathbb{C}^n)$. Assume that $\mathcal{D}(\tilde{V}^z)$ is invariant under the multiplication operator $M_h$. Then $\mathcal{D}(V^z)$ is invariant under $C_j(f, h)$ for all $j \in \mathbb{N}$.

Proof: According to (3.7) there is a finite index set $\mathcal{I}$ and $A_i, B_i \in A_j(f)$ such that

$$\text{ad}^j [T_f](T_h) = \sum_{i \in \mathcal{I}} P A_i M_h B_i P.$$ 

Due to our assumption on $h$ and by Lemma 4.3 the assertion follows. \qed

Now, we can proof our main result on the smoothness of Toeplitz operators with respect to the Segal-Bargmann representation $\rho$ of the Heisenberg group:

Theorem 4.1 Let $\mathcal{S}_s := \mathcal{S} \cap \overline{\mathcal{S}}$ where $\overline{\mathcal{S}} = \{ \overline{h} : h \in \mathcal{S} \}$ and $S := \{ h \in L^\infty(\mathbb{C}^n) : \text{s. t. } \mathcal{D}(\tilde{V}^z) \text{ is invariant under } M_h \text{ for all } z \in \mathbb{C}^n \}$.

Then the symbol map into the $\Psi^*$-algebra $\Psi^\infty$ given by:

$$S \ni h \mapsto T_h \in \Psi^\infty$$

is well-defined and continuous if $\mathcal{S}_s$ carries the $L^\infty(\mathbb{C}^n)$-topology.

Proof: Using our notation in (4.2) and (4.3) we must show that $T_h \in \Psi^\infty$, for all complex directions $z \in \mathbb{C}^n$ and $V^z := [iV^z]$:

$\mathcal{D}(V^z)$ is invariant under $T_q$ for $q \in \{ h, \overline{h} \} \subset \mathcal{S}_s$ and by Lemma 4.2 it follows that the commutators $A_{1} := [iV^z, T_q]$ and $[T_{ig_{z}}, T_q]$ coincide on $\mathcal{H}_{\text{exp}}(\mathbb{C}^n)$. Because $iV^z$ is self-adjoint we can define $A_{1}^{*} := [T_q, iV^z]$ and $W := \mathcal{D}(V^z)$ in Remark 4.1. The operator $[T_{ig_{z}}, T_q]$ has a bounded extension $C_{1}(ig_{z}, q)$ from $\mathcal{H}_{\text{exp}}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. We conclude from Remark 4.1 that $C_{1}(ig_{z}, q)$ is an extension of $A_{1}$ from $W$ to $H^2(\mathbb{C}^n, \mu)$ and $T_q \in \Psi^\infty$. By induction we must prove for $j \in \mathbb{N}$:

1. The domain of definition $\mathcal{D}(V^z)$ is invariant under $C_{j}(ig_{z}, q)$,

2. The commutators $A_{j+1} := [iV^z, C_{j}(ig_{z}, q)]$ have the bounded extension $C_{j+1}(ig_{z}, q)$ from $\mathcal{D}(V^z)$ to $H^2(\mathbb{C}^n, \mu)$.

Assertion (1) is a direct consequence of Corollary 4.1 and (2) can be derived from Remark 4.1 with $A_{j+1}^{*} := [C_{j}(ig_{z}, q)^*, iV^z]$ on $W := \mathcal{D}(V^z)$ and the fact that $A_{j+1}$ has the continuous extension $C_{j+1}(ig_{z}, q)$ from $\mathcal{H}_{\text{exp}}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. The continuity of the symbols map follows from (2.3) together with the continuity of (3.6) in Proposition 3.3. \qed

---

3Note that by Corollary 4.1 and the identity $C_{j}(ig_{z}, q)^* = (-1)^j C_{j}(ig_{z}, \overline{q})$ the commutator $A_{j+1}^{*}$ is well-defined on $\mathcal{D}(V^z)$. 
5 Examples and Applications

Let $A$ denote the subalgebra of $\mathcal{L}(L^{2}(\mathbb{C}^{n}, \mu))$ of all multiplication operators with bounded symbols $h \in L^{\infty}(\mathbb{C}^{n})$. For $z \in \mathbb{C}^{n}$ and with $\tilde{V}^{z} := [i\tilde{V}^{z}]$ there is a scale of algebras arising by commutator methods:

$$A \supset \Psi_{1}^{\tilde{V}^{z}} \supset \cdots \supset \Psi_{n}^{\tilde{V}^{z}} = \bigcap_{n \in \mathbb{N}} \Psi_{n}^{\tilde{V}^{z}}.$$  \hspace{1cm} (5.1)

In general, the inclusions above will be proper. As an immediate consequence of Theorem 4.1 it follows for the projected scale of vector spaces:

$$A_{P} \supset \Psi_{1}^{\tilde{V}^{z}} \supset \cdots \supset \Psi_{nP}^{\tilde{V}^{z}} = \Psi_{n+1_{P}}^{\overline{\mathcal{V}^{z}}}.$$  \hspace{1cm} (5.2)

Here $A_{P} \subset \mathcal{L}(H^{2}(\mathbb{C}^{n}, \mu))$ is the space of Toeplitz operators with bounded measurable symbols. By passing from (5.1) to the scale (5.2) the underlying $C^{k}$-structure is lost.

We give an example of a class of bounded functions $g$ such that $D(\tilde{V}^{z})$ is an invariant subspace for $M_{g}$ and $M_{\mathbf{g}}$ for all $z \in \mathbb{C}^{n}$.

Example 5.1 Denote by $C_{c}^{\infty}(\mathbb{C}^{n})$ the space of compactly supported smooth functions. For $z = (z_{1}, \cdots, z_{n}) \in \mathbb{C}^{n}$ we write $z_{j} := x_{j} + iy_{j}$ and with $\alpha, \beta \in \mathbb{N}_{0}^{n}$:

$$z^{\alpha,\beta} := x^{\alpha}y^{\beta}, \quad \partial^{\alpha,\beta} := \frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\beta}}{\partial y^{\beta}}.$$  \hspace{1cm} (5.3)

Fix $h \in D(\tilde{V}^{z})$ and $z \in \mathbb{C}^{n}$. For $g \in C_{c}^{\infty}(\mathbb{C}^{n})$ (real valued) and $s \neq 0$ we write:

$$\frac{1}{s} [W_{sz} - I] M_{g} h = \frac{1}{s} [M_{g_{\tau - sz}} - M_{g}] W_{sz} h + M_{g} \frac{1}{s} [W_{sz} - I] h.$$  \hspace{1cm} (5.4)

The second term converges in $L^{2}(\mathbb{C}^{n}, \mu)$ as $s \to 0$. Consider the smooth and compactly supported function $dg(z, \cdot) := -\left\langle \mathrm{grad} g(\cdot), z \right\rangle_{\mathbb{R}^{2n}}$. Then:

$$C_{s,z} := \left\| \frac{1}{s} [M_{g_{\tau - sz}} - M_{g}] - M_{dg(z, \cdot)} \right\| \leq \sum_{|\alpha| + |\beta| = 2} \frac{|s|}{(\alpha + \beta)!} \left\| \partial^{\alpha,\beta} g \right\|_{\infty} |z^{\alpha,\beta}|.$$  \hspace{1cm} (5.5)

Hence $\lim_{s \to 0} C_{s,z} = 0$ and the right hand side of

$$\left\| \frac{1}{s} [M_{g_{\tau - sz}} - M_{g}] W_{sz} h - M_{dg(z, \cdot)} h \right\| \leq C_{s,z} \left\| h \right\| + \left\| dg(z, \cdot) \right\|_{\infty} \left\| (W_{sz} - I) h \right\|$$

tends to 0 as $s \to 0$. It follows $gh \in D(V^{z})$. With our notation of Theorem 4.1 we conclude that $C_{c}^{\infty}(\mathbb{C}^{n}) \subset S_{s}$. By the continuity of

$$L^{\infty}(\mathbb{C}^{n}) \subset S_{s} \ni h \mapsto T_{h} \in \Psi^{\infty}$$

and the fact that $C_{c}^{\infty}(\mathbb{C}^{n})$ is uniformly dense in the space $C_{0}(\mathbb{C}^{n})$ of all continuous functions vanishing at infinity it follows that $\{ T_{h} : h \in C_{0}(\mathbb{C}^{n}) \} \subset \Psi^{\infty}$.
In our second example we construct a compact operator $A \in \mathcal{B} := \mathcal{L}(H^2(\mathbb{C}; \mu))$ which is not contained in $\Psi^{1,\ast}$ for any $z \in \mathbb{C}$ (with our notation in (4.2)). As a consequence and using Example 5.1 $A$ is not limit point of finite sums of finite products of Toeplitz operators with symbols in $C_0(\mathbb{C})$ and with respect to the Fréchet topology of $\Psi^{\infty,\ast}$. However, since $A$ is compact it can be approximated by Toeplitz operators with smooth and compactly supported symbols in the topology of $\mathcal{B}$, c.f. [8].

Example 5.2 For $j \in \mathbb{N}_0$ let $P_j \in \mathcal{B}$ be the rank one projection onto $\text{span}\{m_j := z^j\}$. With a sequence $a := (a_n)_{n \in \mathbb{N}}$ tending to zero consider the compact diagonal operator:

$$A := \sum_{j \in \mathbb{N}} a_j P_j \in \mathcal{B}.$$ 

With $z \in \mathbb{C}$, $|z| = 1$ and $g_z := 2i\text{Im}(\cdot, z)$ we compute $[T_{g_z}, A] m_j = [V^z, A] m_j$ explicitly for all $j \in \mathbb{N}$. By (4.4) one obtains that:

$$[T_{g_z}, A] m_j = a_j T_{g_z} m_j - A[\bar{z} m_{j+1} - j z m_{j-1}]$$

$$= a_j (\bar{z} m_{j+1} - j z m_{j-1}) - (a_{j+1} \bar{z} m_{j+1} - j a_{j-1} z m_{j-1})$$

$$= (a_j - a_{j+1}) \bar{z} m_{j+1} - j z (a_j - a_{j-1}) m_{j-1}.$$ 

With $e_j := (j!)^{-\frac{1}{2}}z^j$ we have $(e_j, e_l)_2 = \delta_{i,j}$ for all $j, l \in \mathbb{N}$. Hence it follows that

$$\| [T_{g_z}, A] e_j \|_2^2 = (j+1) |a_j - a_{j+1}|^2 + j |a_j - a_{j-1}|^2.$$ 

(5.4)

We choose $a$ such that the right hand side of (5.4) tends to infinity for $j \to \infty$. This can be done by the choice of an oscillating sequence $a_j := (-1)^j j^{-\frac{1}{4}}$. Then it follows

$$(j+1) |a_j - a_{j+1}|^2 = (j+1) |j^{-\frac{1}{4}} + (j+1)^{-\frac{1}{4}}|^2 \geq \sqrt{j+1}$$

and so the right hand side of (5.4) is unbounded for $j \to \infty$. Hence $[T_{g_z}, A]$ has no bounded extension to $H^2(\mathbb{C}; \mu)$ and $A \notin \Psi^{1,\ast}$ by Proposition 4.1.

Let $\beta : L^2(\mathbb{R}^n) \to H^2(\mathbb{C}^n; \mu)$ denote the Bargmann isometrice, c.f. [10]. Our results on Toeplitz operators on $H^2(\mathbb{C}^n, \mu)$ can be used in the analysis of a class of Gabor-Daubechies windowed localization operators $L_h := \beta^{-1} T_h \beta$ on $L^2(\mathbb{R}^n)$ where $h \in L^\infty(\mathbb{C}^n)$, c.f. [9]. It was remarked in [14] the operator $L_h$ can be considered as a pseudodifferential operator $W_\sigma(h)$ in Weyl quantization with Weyl symbol $\sigma(h)$ on $\mathbb{R}^{2n}$. Via the identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ the correspondence between $h$ and $\sigma(h)$ can be expressed in terms of the heat equation on $\mathbb{R}^{2n}$. More precisely, $\sigma(h)$ is a solution with initial data $h$ at a fixed time $t_0 > 0$. In the next example we describe how the operators introduced in the previous sections transform under $\beta$, c.f. [10].

×Here the window is a Hermite function on $\mathbb{R}^n$.
Example 5.3 For $u \in L^2(\mathbb{R}^n)$ it is well-known that $\beta u$ can be expressed by the integral:

$$[\beta u](z) = (2\pi)^{-\frac{n}{4}} \int_{\mathbb{R}^n} u(x) \exp \left\{ \langle x, z \rangle - \frac{1}{4} |x|^2 - \frac{1}{2} \langle z, \overline{z} \rangle \right\} dx.$$ 

Fix $a = p + iq \in \mathbb{C}^n$, then it can be checked that $W_a \in \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ transform as:

$$B_a u := [\beta^{-1} W_a \beta](u) = u(\cdot - 2p) \exp \{ iq(p - \cdot) \}.$$ 

In particular, in the case $q = 0$ the unitary operator $B_a$ is a usual shift in direction $2p$. For $j = 1, \cdots, n$ it is readily verified that $T_{z_j}$ and $T_{\overline{z}_j}$ transform in the following way:

(i) $\beta^{-1} T_{z_j} \beta = \frac{1}{2} x_j - \partial_{x_j}$,

(ii) $\beta^{-1} T_{\overline{z}_j} \beta = \frac{1}{2} x_j + \partial_{x_j}$.

From (i), (ii) and for $\alpha \in \mathbb{N}_0^n$ one obtains the identity:

$$\beta \partial_x^\alpha = (-1)^{|\alpha|} T_{i\text{Im} z_1}^\alpha \cdots T_{i\text{Im} z_n}^\alpha \beta = (-1)^{|\alpha|} T_{i\text{Im} z}^\alpha \beta.$$ 

Let $g \in D(\mathbb{R}^n)$ be a test function and fix $f \in H_{\exp}(\mathbb{C}^n)$. It follows that:

$$\langle \beta^{-1} f, \partial_x^\alpha g \rangle_{L^2(\mathbb{R}^n)} = \langle f, \beta \partial_x^\alpha g \rangle = \langle \beta^{-1} T_{i\text{Im} z_1}^\alpha \cdots T_{i\text{Im} z_n}^\alpha f, g \rangle_{L^2(\mathbb{R}^n)}.$$ 

Here we have used the fact that $H_{\exp}(\mathbb{C}^n)$ is invariant under all unbounded Toeplitz operators $T_{i\text{Im} z_j}$ which was proved in Proposition 3.1. It follows that:

$$D := \beta^{-1}[H_{\exp}(\mathbb{C}^n)] \subset H^\infty(\mathbb{R}^n) = \cap_{k \in \mathbb{N}} H^k(\mathbb{R}^n)$$ 

where $H^s(\mathbb{R}^n)$ denotes the $k$-th Sobolev space. Hence, for $\alpha, \beta \in \mathbb{N}_0^n$ the restriction of (2.1) in Theorem 2.1 to $D$:

$$\text{ad}[-iz]^\alpha \text{ad}[i\partial_x]^\beta (B) : D \to D$$

(5.5) is well-defined for any $B \in L(D)$. With the choice $h \in L^\infty(\mathbb{C}^n)$ and $L_h := \beta^{-1} T_h \beta \in L(D)$ we obtain by conjugating (5.5) with $\beta$ and using (i), (ii) above:

$$\text{ad}[iT_{2\text{Re} z}]^\alpha \text{ad}[T_{i\text{Im} z}]^\beta (T_h) : H_{\exp}(\mathbb{C}^n) \to H_{\exp}(\mathbb{C}^n).$$

(5.6)

It follows by Proposition 3.3 that the operators in (5.6) have bounded extensions to $H^2(\mathbb{C}^n, \mu)$ and so (5.5) can be extended continuously to $L^2(\mathbb{R}^n)$. Hence we have proved a weaker version of the defining property (2.1) for $\Psi_{\rho, \delta}$ in Theorem 2.1.

Since the Gaussian measure $\mu$ is invariant under unitary transformations of $\mathbb{C}^n$, there is a natural group representation of $U_n$ in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$ generating $\Psi^*$-algebras of smooth elements. As a final example we want to remark:
Example 5.4 Let $A \in \mathbb{R}^{n \times n}$ be self-adjoint and consider the unitary group:

$$\mathbb{R} \ni t \mapsto e^{itA} \in U_n.$$  

The group of unitary composition operators $C_t f := f \circ e^{itA}$ on $H^2(\mathbb{C}^n, \mu)$ can be shown to be strongly continuous, cf. [3]. The restriction of the infinitesimal generator $L_A$ of $(C_t)_{t \in \mathbb{R}}$ to $P_a[\mathbb{C}^n]$ coincides with an (unbounded) Toeplitz operator. More precisely, it was shown in [3] that:

$$L_{Ap} = [T_{<Az,z>} - n \cdot \text{trace}(A)]p, \quad p \in P_a[\mathbb{C}^n].$$

Hence, in general the symbol of $L_A$ regarded as a Toeplitz operator is a polynomial of degree 2, which is not globally lipschitz continuous on $\mathbb{C}^n$. Proposition 3.3 cannot be applied in this situation and the smoothness of a Toeplitz operator $T_f$ with bounded symbols $f$ with respect to $(C_t)_t$ requires further assumption on the symbol $f$. For a more detailed calculation we refer to [3].

Acknowledgment: The author wishes to express his thanks to Professor B. Gramsch for many hints and explanations concerning the theory of spectral invariant Fréchet algebras.

References


TOEPLITZ $\Psi^*$-ALGEBRAS


