数理解析研究所講究録
1520 巻 2006 年 104-108

Uniform non-ℓ_1^n-ness of direct sums of Banach spaces

九州工業大学・工学部　加藤幹雄 (Mikio Kato)
Department of Mathematics, Kyushu Institute of Technology
e-mail: katom@tobata.isc.kyutech.ac.jp

新潟大学・理学部　斎藤吉助 (Kichi-Suke Saito)
Department of Mathematics, Faculty of Science, Niigata University
e-mail: saito@math.sc.niigata-u.ac.jp

千葉大学大学院・人文社会科学研究科　田村高幸 (Takayuki Tamura)
Graduate School of Humanities and Social Sciences, Chiba University
e-mail: tamura@le.chiba-u.ac.jp

Abstract. This is a résumé of some recent results on the uniform non-ℓ_1^n-ness of direct sums of Banach spaces. In particular we present those for the ℓ_1- and ℓ_∞-sums as well.

1. Introduction

Since it was introduced in [24], the ψ-direct sum of Banach spaces have attracted a good deal of attention ([5, 6, 7, 13, 14, 19, 20, 17, 16, etc.]; see also [22, 23]). The aim of this note is to present a sequence of recent results on the uniform non-ℓ_1^n-ness of direct sums of Banach spaces. Our starting point is Theorem 1 below concerning the uniform non-squareness by the authors ([14]). To treat the uniform non-ℓ_1^n-ness is much more complicated than expected. The results presented here is almost taken from the recent paper of the present authors [16].

Let Ψ be the family of all convex (continuous) functions ψ on $[0,1]$ satisfying

$$\psi(0)=\psi(1)=1 \quad \text{and} \quad \max\{1-t,t\}\leq\psi(t)\leq 1 \quad (0\leq t\leq 1). \tag{1}$$

For any $\psi\in\Psi$ define

$$\|(z,w)\|_\psi=\begin{cases}(|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w)\neq(0,0),\\ 0 & \text{if } (z,w)=(0,0).\end{cases} \tag{2}$$

Then $\|\cdot\| = \|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ (that is, $\|(z,w)\| = \|(|z|,|w|)\|$ and $\|(1,0)\| = \|(0,1)\| = 1$) and satisfies

$$\psi(t) = \|(1-t,t)\| \quad (0 \le t \le 1). \tag{3}$$

Conversely for any absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$ define a convex function $\psi \in \Psi$ by (3). Then $\|\cdot\| = \|\cdot\|_\psi$.

The ℓ_p-norms $\|\cdot\|_p$ are such examples and for all absolute normalized norms $\|\cdot\|$ on $\mathbb{C}^2$ we have

$$\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1 \tag{4}$$

([2]). By (3) the convex functions corresponding to the ℓ_p-norms are given by

$$\psi_p(t) := \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases} \tag{5}$$

Let X and Y be Banach spaces and let $\psi \in \Psi$. The ψ-direct sum $X \oplus_\psi Y$ of X and Y is the direct sum $X \oplus Y$ equipped with the norm

$$\|(x,\, y)\|_\psi = \|(\|x\|,\, \|y\|)\|_\psi, \tag{6}$$

where the $\|(\cdot,\,\cdot)\|_\psi$ term in the right hand side is the absolute normalized norm on $\mathbb{C}^2$ corresponding to the convex function ψ ([24, 13]; see [21] for several examples). This extends the notion of the ℓ_p-sum $X \oplus_p Y$.

A Banach space X is said to be *uniformly non-ℓ_1^n* (cf. [1, 18]) provided there exists ϵ $(0 < \epsilon < 1)$ such that for any $x_1, \cdots, x_n \in S_X$, the unit sphere of X, there exists an n-tuple of signs $\theta = (\theta_j)$ for which

$$\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \le n(1-\epsilon). \tag{7}$$

We may take $x_1, \cdots, x_n$ from the unit ball B_X of X in the definition. In case of $n = 2$ X is called *uniformly non-square* ([12]; cf. [1, 18]).

As is well known ([3, 11]), *if X is uniformly non-ℓ_1^n, then X is uniformly non-ℓ_1^{n+1} for every $n \in \mathbb{N}$.*

2. Uniform non-ℓ_1^n-ness of $X \oplus_\psi Y$, $\psi \ne \psi_1, \psi_\infty$

The following result by the authors [14] is our starting point.

Theorem 1 (Kato-Saito-Tamura [14]). *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.*

(i) *$X \oplus_\psi Y$ is uniformly non-square.*

(ii) *X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.*

To treat the uniform non-ℓ_1^n-ness is much more complicated than expected. Indeed we need to prepare several lemmas, though we skip to mention them.

Theorem 2. *Let X and Y be Banach spaces and let $\psi \in \Psi, \psi \neq \psi_1, \psi_\infty$. Then the following are equivalent.*

(i) *$X \oplus_\psi Y$ is uniformly non-ℓ_1^n.*

(ii) *X and Y are uniformly non-ℓ_1^n.*

Theorem 2 does not answer the following question: Let X and Y be uniformly non-ℓ_1^n. Is it possible for $X \oplus_\psi Y$ to be uniformly non-ℓ_1^n with $\psi = \psi_1$ or $\psi = \psi_\infty$? The next theorem will give an answer.

Theorem 3. *Let X and Y be Banach spaces and let $\psi \in \Psi$. Assume that neither X nor Y is uniformly non-ℓ_1^{n-1}. Then the following are equivalent.*

(i) *$X \oplus_\psi Y$ is uniformly non-ℓ_1^n.*

(ii) *X and Y are uniformly non-ℓ_1^n and $\psi \neq \psi_1, \psi_\infty$.*

Theorem 3 includes Theorem 1 as the case $n = 2$.

Remark 1. In Theorem 3 we can not remove the condition that neither X nor Y is uniformly non-ℓ_1^{n-1} ([16, Section 6]).

3. The ℓ_1- and ℓ_∞-sums

Theorem 4. *Let X and Y be Banach spaces. The following are equivalent.*

(i) *$X \oplus_1 Y$ is uniformly non-ℓ_1^n.*

(ii) *There exist positive integers n_1 and n_2 with $n_1 + n_2 = n - 1$ such that X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$.*

According to Theorem 1 the uniform non-squareness of X and Y is not inherited to the ℓ_1-sum $X \oplus_1 Y$, whereas we have the following result as the case $n = 3$ of Theorem 4.

Theorem 5. *Let X and Y be Banach spaces. Then the following are equivalent.*

(i) *$X \oplus_1 Y$ is uniformly non-ℓ_1^3.*

(ii) *X and Y are uniformly non-square.*

For the ℓ_∞-sum we obtain the following.

Theorem 6. *Let $X_1, \dots, X_m$ be uniformly non-square Banach spaces. Then $(X_1 \oplus \cdots \oplus X_m)_\infty$ is uniformly non-ℓ_1^n if and only if $m < 2^{n-1}$.*

According to Theorem 5 *the ℓ_1-sum $X \oplus_1 Y$ is uniformly non-ℓ_1^3 if and only if X and Y are uniformly non-square.* On the other hand for the ℓ_∞-sum, by Theorem 6, *if X and Y are uniformly non-square, then $X \oplus_\infty Y$ is uniformly non-ℓ_1^3, whereas the converse is not true* ([16, Remark 5.5]). Instead we obtain the following result which is interesting in contrast with the ℓ_1-sum case.

Theorem 7. *Let X, Y and Z be Banach spaces. Then the following are equivalent.*

(i) *$(X \oplus Y \oplus Z)_\infty$ is uniformly non-ℓ_1^3.*

(ii) *X, Y and Z are uniformly non-square.*

References

[1] B. Beauzamy, Introduction to Banach Spaces and their Geometry, 2nd ed., North-Holland, 1985.

[2] F. F. Bonsall and J. Duncan, Numerical Ranges II, London Math. Soc. Lecture Note Ser. **10**, 1973.

[3] D. R. Brown, B-convexity in Banach spaces, Doctoral dissertation, Ohaio State University, 1970.

[4] M. Denker and H. Hudzik, Uniformly non-$l_n^{(1)}$ Musielak-Orlicz sequence spaces, Proc. Indian Acad. Sci. Math. Sci. **101** (1991), 71–86

[5] S. Dhompongsa, A. Kaewkhao and S. Saejung, Uniform smoothness and U-convexity of ψ-direct sums, J. Nonlinear Convex Anal. **6** (2005), 327–338.

[6] S. Dhompongsa, A. Kaewcharoen and A. Kaewkhao, Fixed point property of direct sums, Nonlinear Anal., to appear.

[7] P. N. Dowling, On convexity properties of ψ-direct sums of Banach spaces, J. Math. Anal. Appl. **288** (2003), 540–543.

[8] P. N. Dowling and B. Turett, Complex strict convexity of absolute norms on $\mathbb{C}^n$ and direct sums of Banach spaces, to appear in J. Math. Anal. Appl.

[9] D. P. Giesy, On a convexity condition in normed linear spaces. Trans. Amer. Math. Soc. **125** (1966), 114–146.

[10] D. P. Giesy and R. C. James, Uniformly non-$l_n^{(1)}$ and B-convex spaces, Studia Math. **48** (1973), 61–69.

[11] H. Hudzik, Uniformly non-$l_n^{(1)}$ Orlicz spaces with Luxemburg norm, Studia Math. **81** (1985), 271–284.

[12] C. James, Uniformly non-square Banach spaces, Ann. of Math. **80** (1964), 542–550.

[13] M. Kato, K.-S. Saito and T. Tamura, On ψ-direct sums of Banach spaces and convexity, J. Aust. Math. Soc. **75** (2003), 413–422.

[14] M. Kato, K.-S. Saito and T. Tamura, Uniform non-squareness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$, Math. Inequal. Appl. **7** (2004), 429–437.

[15] M. Kato, K.-S. Saito and T. Tamura, Sharp triangle inequality and its reverse in Banach spaces, to appear in Math. Inequal. Appl.

[16] M. Kato, K.-S. Saito and T. Tamura, Uniform non-ℓ_1^n-ness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$, submitted.

[17] M. Kato and T. Tamura, Weak nearly uniform smoothness and worth property of ψ-direct sums of Banach spaces $X \oplus_\psi Y$, Comment. Math. Prace Mat. **46** (2006), 113-129.

[18] R. E. Megginson, An Introduction to Banach Space Theory, Springer, 1998.

[19] K. Mitani, K.-S. Saito and T. Suzuki, Smoothness of absolute norms on $\mathbb{C}^n$, J. Convex Anal. **10** (2003), 89–107.

[20] K. Mitani, S. Oshiro and K.-S. Saito, Smoothness of ψ-direct sums of Banach spaces, Math. Inequal. Appl. **8** (2005), 147–157.

[21] K.-S. Saito and M. Kato, Uniform convexity of ψ-direct sums of Banach spaces, J. Math. Anal. Appl. **277** (2003), 1–11

[22] K.-S. Saito, M. Kato and Y. Takahashi, Von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$, J. Math. Anal. Appl. **244** (2000), 515–532.

[23] K.-S. Saito, M. Kato and Y. Takahashi, On absolute norms on $\mathbb{C}^n$, J. Math. Anal. Appl. **252** (2000), 879–905.

[24] Y. Takahashi, M. Kato and K.-S. Saito, Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces, J. Inequal. Appl. **7** (2002), 179–186.