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Kyoto University
Uniform non-$\ell_1^n$-ness of direct sums of Banach spaces

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Abstract. This is a résumé of some recent results on the uniform non-$\ell_1^n$-ness of direct sums of Banach spaces. In particular we present those for the $\ell_1$- and $\ell_\infty$-sums as well.

1. Introduction

Since it was introduced in [24], the $\psi$-direct sum of Banach spaces have attracted a good deal of attention ([5, 6, 7, 13, 14, 19, 20, 17, 16, etc.]; see also [22, 23]). The aim of this note is to present a sequence of recent results on the uniform non-$\ell_1^n$-ness of direct sums of Banach spaces. Our starting point is Theorem 1 below concerning the uniform non-squareness by the authors ([14]). To treat the uniform non-$\ell_1^n$-ness is much more complicated than expected. The results presented here is almost taken from the recent paper of the present authors [16].

Let $\Psi$ be the family of all convex (continuous) functions $\psi$ on $[0, 1]$ satisfying

$$\psi(0) = \psi(1) = 1 \text{ and } \max\{1 - t, t,\} \leq \psi(t) \leq 1 \quad (0 \leq t \leq 1).$$

(1)

For any $\psi \in \Psi$ define

$$\|(z, w)\|_\psi = \left\{ \begin{array}{ll}
(|z| + |w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z, w) \neq (0,0), \\
0 & \text{if } (z, w) = (0,0).
\end{array} \right.$$  

(2)
Then \( || \cdot || = || \cdot ||_\psi \) is an absolute normalized norm on \( \mathbb{C}^2 \) (that is, \( ||(z, w)|| = ||(|z|, |w|)|| \) and \( ||(1, 0)|| = ||(0, 1)|| = 1 \) and satisfies
\[
\psi(t) = ||(1-t, t)|| \quad (0 \leq t \leq 1).
\] (3)

Conversely for any absolute normalized norm \( || \cdot || \) on \( \mathbb{C}^2 \) define a convex function \( \psi \in \Psi \) by (3). Then \( || \cdot || = || \cdot ||_\psi \).

The \( \ell_p \)-norms \( || \cdot ||_p \) are such examples and for all absolute normalized norms \( || \cdot || \) on \( \mathbb{C}^2 \) we have
\[
|| \cdot ||_\infty \leq || \cdot || \leq || \cdot ||_1 \quad (4)
\]
([2]). By (3) the convex functions corresponding to the \( \ell_p \)-norms are given by
\[
\psi_p(t) := \begin{cases} 
((1-t)^p + t^p)^{1/p} & \text{if } 1 \leq p < \infty, \\
\max\{1-t, t\} & \text{if } p = \infty.
\end{cases}
\] (5)

Let \( X \) and \( Y \) be Banach spaces and let \( \psi \in \Psi \). The \( \psi \)-direct sum \( X \oplus_{\psi} Y \) of \( X \) and \( Y \) is the direct sum \( X \oplus Y \) equipped with the norm
\[
||(x, y)||_\psi = ||(||x||, ||y||)||_\psi,
\] (6)
where the \( ||(\cdot, \cdot)||_\psi \) term in the right hand side is the absolute normalized norm on \( \mathbb{C}^2 \) corresponding to the convex function \( \psi \) ([24, 13]; see [21] for several examples). This extends the notion of the \( \ell_p \)-sum \( X \oplus_p Y \).

A Banach space \( X \) is said to be uniformly non-\( \ell^*_1 \) (cf. [1, 18]) provided there exists \( \epsilon \) \((0 < \epsilon < 1)\) such that for any \( x_1, \ldots, x_n \in S_X \), the unit sphere of \( X \), there exists an \( n \)-tuple of signs \( \theta = (\theta_j) \) for which
\[
\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq n(1 - \epsilon).
\] (7)
We may take \( x_1, \ldots, x_n \) from the unit ball \( B_X \) of \( X \) in the definition. In case of \( n = 2 \) \( X \) is called uniformly non-square ([12]; cf. [1, 18]).

As is well known ([3, 11]), if \( X \) is uniformly non-\( \ell^*_1 \), then \( X \) is uniformly non-\( \ell^*_1 \) for every \( n \in \mathbb{N} \).

2. Uniform non-\( \ell^*_1 \)-ness of \( X \oplus_{\psi} Y \), \( \psi \neq \psi_1, \psi_\infty \)

The following result by the authors [14] is our starting point.
**Theorem 1** (Kato-Saito-Tamura [14]). Let $X$ and $Y$ be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.

(i) $X \oplus_{\psi} Y$ is uniformly non-square.

(ii) $X$ and $Y$ are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.

To treat the uniform non-$\ell_1^n$-ness is much more complicated than expected. Indeed we need to prepare several lemmas, though we skip to mention them.

**Theorem 2.** Let $X$ and $Y$ be Banach spaces and let $\psi \in \Psi$, $\psi \neq \psi_1, \psi_\infty$. Then the following are equivalent.

(i) $X \oplus_{\psi} Y$ is uniformly non-$\ell_1^n$.

(ii) $X$ and $Y$ are uniformly non-$\ell_1^n$.

Theorem 2 does not answer the following question: Let $X$ and $Y$ be uniformly non-$\ell_1^n$. Is it possible for $X \oplus_{\psi} Y$ to be uniformly non-$\ell_1^n$ with $\psi = \psi_1$ or $\psi = \psi_\infty$?

The next theorem will give an answer.

**Theorem 3.** Let $X$ and $Y$ be Banach spaces and let $\psi \in \Psi$. Assume that neither $X$ nor $Y$ is uniformly non-$\ell_1^{n-1}$. Then the following are equivalent.

(i) $X \oplus_{\psi} Y$ is uniformly non-$\ell_1^n$.

(ii) $X$ and $Y$ are uniformly non-$\ell_1^n$ and $\psi \neq \psi_1, \psi_\infty$.

Theorem 3 includes Theorem 1 as the case $n = 2$.

**Remark 1.** In Theorem 3 we can not remove the condition that neither $X$ nor $Y$ is uniformly non-$\ell_1^{n-1}$ ([16, Section 6]).

**3. The $\ell_1$- and $\ell_\infty$-sums**

**Theorem 4.** Let $X$ and $Y$ be Banach spaces. The following are equivalent.

(i) $X \oplus_1 Y$ is uniformly non-$\ell_1^n$.

(ii) There exist positive integers $n_1$ and $n_2$ with $n_1 + n_2 = n - 1$ such that $X$ is uniformly non-$\ell_1^{n_1+1}$ and $Y$ is uniformly non-$\ell_1^{n_2+1}$.

According to Theorem 1 the uniform non-squareness of $X$ and $Y$ is not inherited to the $\ell_1$-sum $X \oplus_1 Y$, whereas we have the following result as the case $n = 3$ of Theorem 4.
Theorem 5. Let $X$ and $Y$ be Banach spaces. Then the following are equivalent.

(i) $X \oplus_1 Y$ is uniformly non-$\ell_1^3$.
(ii) $X$ and $Y$ are uniformly non-square.

For the $\ell_\infty$-sum we obtain the following.

Theorem 6. Let $X_1, \ldots, X_m$ be uniformly non-square Banach spaces. Then $(X_1 \oplus \cdots \oplus X_m)_\infty$ is uniformly non-$\ell_1^n$ if and only if $m < 2^{n-1}$.

According to Theorem 5 the $\ell_1$-sum $X \oplus_1 Y$ is uniformly non-$\ell_1^3$ if and only if $X$ and $Y$ are uniformly non-square. On the other hand for the $\ell_\infty$-sum, by Theorem 6, if $X$ and $Y$ are uniformly non-square, then $X \oplus_\infty Y$ is uniformly non-$\ell_1^3$, whereas the converse is not true ([16, Remark 5.5]). Instead we obtain the following result which is interesting in contrast with the $\ell_1$-sum case.

Theorem 7. Let $X$, $Y$ and $Z$ be Banach spaces. Then the following are equivalent.

(i) $(X \oplus Y \oplus Z)_\infty$ is uniformly non-$\ell_1^3$.
(ii) $X$, $Y$ and $Z$ are uniformly non-square.

References


[16] M. Kato, K.-S. Saito and T. Tamura, Uniform non-$\ell_1^n$-ness of $\psi$-direct sums of Banach spaces $X \oplus_{\psi} Y$, submitted.


