The devil step and a strange slope

Introduction

One of the properties of the chaotic theory is considered as non-differentiability of functions which appear in that theory. The Cantor function $C(x)$ has such a property. Namely this function has a strange property in the following sense, and is sometimes called the Devil’s step function.

(C-1) $C(x)$ is a monotone function of $[0,1]$ onto itself,
(C-2) $C'(x) = 0$ for almost all $x$ in $[0,1]$,
(C-3) $C(x)$ jumps on a set whose measure is 0.

In the present note, we exhibit another strange function $h(x)$, whose property is

(h-1) $h(x)$ is a strictly monotone function of $[0,1]$ onto itself,
(h-2) $h'(x) = 0$ for almost all $x$ in $[0,1]$,
(h-3) $h(x)$ jumps on a set whose measure is 0 but which is dense in $[0,1]$.

The function $h(x)$ appears as a topological conjugacy between two tent maps on $[0,1]$. The following are graphs of two functions.

![Graphs of $C(x)$ and $h(x)$](image)

Tent maps are typical examples in the chaotic theory and the first author has shown convergent theorems about the Perron-Frobenius operator associated with chaotic map in the context of the theory of functional analysis (cf.[1],[3],[4]). After that, the role of topological conjugacy in the convergence theorem turned out to be clear and now we can show that every topological conjugacy between different two tent maps has no absolute-continuity on any interval in $[0,1]$. 
In Section 1, we state three propositions concerning the uniform convergency of a orbit of probability density function by tent maps in the context of $L^1([0,1])$-space, the existence of topological conjugacies between two tent maps and non-absolute-continuity of those topological conjugacies.

In Section 2, we show our main result concerning the derivative $h'$ of topological conjugacy $h$ mentioned above and the detail of non-absolute-continuity of $h$ with respect to the Lebesgue measure. Namely we show that

(1) $h'(x) = 0$ or $\infty$ if there exists a differential coefficient $h'(x)$.
(2) $h'(x) = 0$ on a dense set $E_0$ with $\mu(E_0) = 1$,
(3) $h'(x) = \infty$ on a dense set $E_\infty$ with $\mu(E_\infty) = 0$,
(4) $\mu(h(E_0)) = 0$ and $\mu(h(E_\infty)) = 1$.

In our discussion, two sequences play an important role; one is the sequence denoting the orbit (itinerary) of a point under the tent map and the other is the sequence given by the infinite binary expansion of a number. The role of the former is very similar to that in the kneading theory (cf.[5]). In our theory, we show that the value $h'(x)$ is deeply related to the orbit of $x$. We here note that an element of $x$ in $[0,1]$ is called a real number or a point by considering situation of discussion and that the symbol $\mathbb{N}$ means the set of all positive integers.

1. Convergence theorem for the Perron-Frobenius operator

In this paper, a unimodal map means a continuous map $\varphi : [0, 1] \to [0, 1]$ defined by

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{if } x \in [0, c), \\ \varphi_2(x) & \text{if } x \in [c, 1], \end{cases}$$

where (1) $0 < c < 1$, (2) $\varphi_1$ and $\varphi_2$ are monotonically increasing and monotonically decreasing respectively, (3) $\varphi$ and $\varphi_i^{-1} (i = 1, 2)$ are absolutely continuous.

The map $\varphi$ canonically induces an isometric operator $T_{\varphi}$ on $L^\infty([0,1])$. Namely

$$(T_{\varphi}g)(x) = g(\varphi(x)), \quad (g \in L^\infty([0,1])).$$

We denote by $A_{\varphi}$ the dual operator $T_{\varphi}^*$ with respect to a duality between two Banach spaces $(L^1([0,1]), L^\infty([0,1]))$ with the usual dual relation $<\cdot, \cdot>$:

$$< f, g > = \int_{[0,1]} f(x)g(x)d\mu, \quad (f \in L^1([0,1]), g \in L^\infty([0,1]),$$

where $\mu$ is the Lebesgue measure on $[0,1]$. Thus we have

$$\int_{[0,1]} (A_{\varphi}f)(x)g(x)d\mu = < A_{\varphi}f, g >= < f, T_{\varphi}g > = \int_{[0,1]} f(x)g(\varphi(x))d\mu.$$
Hence, it follows that
\[(A_{\varphi}f)(x) = \frac{d\mu \circ \varphi^{-1}}{d\mu}(x)f(\varphi^{-1}(x)) + \frac{d\mu \circ \varphi^{-1}}{d\mu}(x)f(\varphi^{-1}(x)),\]
where \(\frac{d\mu \circ \varphi^{-1}}{d\mu}\) are the Radon-Nikodym derivatives of the measure \((\mu \circ \varphi^{-1})\) with respect to \(\mu\), \((i = 1, 2)\), where \((\mu \circ \varphi^{-1})(E) = \mu(\varphi^{-1}(E))\) for each measurable set \(E\) in \([0, 1]\). Moreover we note that \(A_{\varphi}\) is a bounded linear operator on \(L^{1}([0, 1])\) and is called the Perron-Frobenius operator associated with \(\varphi\).

In the following, we mention a convergence theorem of the iterations \(\{A_{\varphi}^{n}\}\) in the case where \(\varphi\) is so-called a chaotic map. First, let us consider the logistic map \(\lambda\) defined by \(\lambda(x) = 4x(1 - \tau)\).

Second we consider the (generalized) tent maps \(\tau_{c}\) \((0 < c < 1)\) defined by
\[\tau_{c}(x) = \begin{cases} \frac{1}{c}x & \text{if } x \in [0, c), \\ \frac{1}{c-1}(x-1) & \text{if } x \in [c, 1]. \end{cases}\]

The Perron-Frobenius operator \(A_{\tau_{c}}\) associated with \(\tau_{c}\) is easily calculated as follows:
\[(A_{\tau_{c}}f)(x) = cf(cx) + (1-c)f(1-(1-c)x).\]

In particular, in the case of \(c = 1/2\), we write \(\tau = \tau_{1/2}\) and it follows that
\[(A_{\tau}f)(x) = \frac{1}{2} \left\{ f \left( \frac{x}{2} \right) + f \left( 1 - \frac{x}{2} \right) \right\}.\]

In both cases where \(\varphi = \lambda\) and \(\varphi = \tau_{c}\), we have the following convergence theorem:
\[
\lim_{n \to \infty} \|A_{\varphi}^{n}f - e\|_{1} = 0, \quad \lim_{n \to \infty} \|A_{\varphi}^{n}f - g\|_{1} = 0
\]
for any probability density function \(f\) on \([0, 1]\), where \(e(x) = \frac{1}{\pi \sqrt{x(1-x)}}\) and \(g(x) = \chi_{[0,1]}(x) = 1\). Needless to say, \(\chi_{E}\) means the characteristic function of \(E\). These results are derived by convergence theorems obtained by the first author (cf.[4:Theorem 2.4, Corollary 2.5, Example 2.9, Example 2.19]), in which proofs are given in the context of operator algebras. In the present note, we need a concrete and precise discussion on convergence theorem in order to analyze differentiability of the functions which are our main object. Here, we give only statements of the convergence theorems in the context of \(L^{1}([0, 1])\)-space.
Proposition 1.1. Suppose $c \in (0, 1)$. Then it follows that
\[
\lim_{n \to \infty} \|A_{\tau_c}^n f - \chi_{[0,1]}\|_1 = 0
\]
for any probability density function $f$ in $L^1([0,1])$.

Here we discuss the conjugate relation between two tent maps. Two unimodal maps $\psi$ and $\varphi$ are said to be topologically conjugate if there exists a homeomorphism $h$ of $[0,1]$ onto itself such that $\varphi = h \circ \psi \circ h^{-1}$.

The homeomorphism $h$ is called a topological conjugacy (cf. [2: Definition 7.4]). In the case where $h$ and $h^{-1}$ are absolutely continuous, we can define an isometric operator $U_h$ on $L^1([0,1])$ as follows:
\[
(U_h f)(x) = \frac{d\mu \circ h^{-1}}{d\mu}(x) f(h^{-1}(x))
\]
and the equation $A_{\varphi} = U_h^{-1} A_{\psi} U_h$ holds. Moreover we have the following lemma.

Lemma 1.2. Let $\varphi$ and $\psi$ be topologically conjugate unimodal maps on $[0,1]$ with $\varphi = h \circ \psi \circ h^{-1}$. If $h$ and $h^{-1}$ are absolutely continuous, then the following conditions (A) and (B) are equivalent.

(A) $\lim_{n \to \infty} \|A_{\psi}^n f - e\|_1 = 0$ for any probability density function $f$ in $L^1([0,1])$.

(B) $\lim_{n \to \infty} \|A_{\varphi}^n f - g\|_1 = 0$ for any probability density function $f$ in $L^1([0,1])$, where $g = U_h^{-1} e$.

Remark 1.3. The logistic map $\lambda$ is topologically conjugate to the tent map $\tau$ with $h(x) = \sin^2(\pi x / 2)$. Namely $\lambda = h \circ \tau \circ h^{-1}$ and $\frac{d\mu \circ h^{-1}}{d\mu}(x) = \frac{1}{\pi \sqrt{x(1-x)}}$.
Since $A_{\tau_c}^n f$ converges to $\chi_{[0,1]}$, it follows that $A_{\varphi}^n f$ converges to the function $g(x) = \frac{1}{\pi \sqrt{x(1-x)}}$.

In addition to $\tau$ and $\lambda$, there exists a topological conjugacy $h$ between $\tau_c$ and $\tau_d$. This is well-known and a general form in those kind of theorems concerning topological conjugacies is given by [5: Theorem 3.1 of Chapter II]. Here we note the statement of existence of $h$. 


Proposition 1.4. For any c and d in (0, 1), two tent maps \(\tau_c\) and \(\tau_d\) are topologically conjugate.

As mentioned above, \(\tau_c\) and \(\tau_d\) are topologically conjugate with a unique topological conjugacy \(h\). Now suppose that \(h\) and \(h^{-1}\) are absolutely continuous. Then, by Proposition 1.1, we have

\[
\lim_{n \to \infty} A^n_{\tau_c} = \chi_{[0,1]} = \lim_{n \to \infty} A^n_{\tau_d}
\]

for any probability density function \(f\) in \(L^1([0,1])\). Hence, by Lemma 1.2, we have

\[
\chi_{[0,1]} = U^{-1}_h \chi_{[0,1]} = \frac{d\mu \circ h^{-1}}{d\mu}.
\]

This implies that \(\frac{d\mu \circ h^{-1}}{d\mu}(x) = 1\) for almost all \(x\) in [0, 1]. Thus \(h^{-1}(x) = h(x) = x\) for all \(x\) in [0, 1]. Namely \(c = d\). There we obtained the following proposition.

Proposition 1.5. Let \(c\) and \(d\) be in (0, 1) and \(h\) the topological conjugacy between two tent maps \(\tau_c\) and \(\tau_d\), \((\tau_c = h \circ \tau_d \circ h^{-1})\). If \(c \neq d\), then \(h\) and \(h^{-1}\) are not absolutely continuous on [0, 1].

In the following section, we show the property of non-absolute-continuity of \(h\).

2. Property of the topological conjugacy between two tent maps \(\tau\) and \(\tau_c\)

First we note that throughout this section \(\tau\) denotes the tent map and \(h\) the topological conjugacy between \(\tau\) and \(\tau_c\). Namely, \(\tau\) and \(h\) mean

\[
\tau = \tau_{\frac{1}{2}}\quad \text{and} \quad \tau_c = h \circ \tau \circ h^{-1}.
\]

The graph of \(y = \tau(x)\), \(y = \tau_c(x)\) and that of \(y = h(x)\), \(y = h^{-1}(x)\) in the case \(c = 1/4\) are shown at Graph[A] and Graph[B].
In the case \( c = 1/2 \), the topological conjugacy \( h \) is of course the identity map \( h(x) = x \) and \( h'(x) = 1 \). Now we start calculations of coefficients \( h'(x) \)'s and, needless to say, we note that \( h'(0) \) and \( h'(1) \) mean
\[
h'(0) = \lim_{\epsilon \to 0^+}(h(\epsilon) - h(0))/\epsilon \quad \text{and} \quad h'(1) = \lim_{\epsilon \to 0^+}(h(1) - h(1 - \epsilon))/\epsilon.
\]

Lemma 2.1. Let \( x \) be in \([0, 1]\).

(1) Suppose that there exist a sequences \( \{y_i\}_{i=1}^{\infty} \) of points in \([0, 1]\) and a sequence \( \{n(i)\}_{i=1}^{\infty} \) of positive integers satisfying the following conditions.

1. \( y_1 < y_2 < \cdots < y_i < \cdots < x \) and \( \lim_{i \to \infty} y_i = x \).
2. \( n(1) < n(2) < \cdots < n(i) < \cdots \) and there exists a positive integer \( K \) such that \( n(i+1) - n(i) \leq K \) for all \( i \).
3. \( |x - y_i| = 1/2^{n(i)} \) for all \( i \).

Then it follows that \( f_-(x) = \omega_- \).

(2) Suppose that there exist a sequences \( \{z_i\}_{i=1}^{\infty} \) of points in \([0, 1]\) and a sequence \( \{m(i)\}_{i=1}^{\infty} \) of integers satisfying the following conditions.

1. \( z_1 > z_2 > \cdots > z_i > \cdots > x \) and \( \lim_{i \to \infty} z_i = x \).
2. \( m(1) < m(2) < \cdots < m(i) < \cdots \) and there exists an integer \( L \) such that \( m(i+1) - m(i) \leq L \) for all \( i \).
3. \( |z_i - x| = 1/2^{m(i)} \) for all \( i \).

Then it follows that \( h_+(x) = \omega_+ \).

The following is a key lemma in our calculation of coefficients.

Lemma 2.2. Suppose \( c \in (0, 1) \). Then the following equation holds.

\[
h(x) = \begin{cases} 
  ch(\tau(x)) & \text{if } x \in [0, 1/2), \\
  (c-1)h(\tau(x)) + 1 & \text{if } x \in [1/2, 1].
\end{cases}
\]

In the following, we define two cardinal numbers related to the orbit of a point under the tent map \( \tau \), which play an important role in this paper.

\[
N_0(x, n) = \# \{ i | \tau^i(x) \in [0, 1/2), 0 \leq i \leq n \},
\]

\[
N_1(x, n) = \# \{ i | \tau^i(x) \in [1/2, 1], 0 \leq i \leq n \},
\]

where \( \# \) means the number of a set. Then we have \( N_0(x, n) + N_1(x, n) = n + 1 \) and, in case of no confusion, we use the notation \( N_0(n) \) and \( N_1(n) \) instead of \( N_0(x, n) \) and \( N_1(x, n) \).
Lemma 2.3. Suppose $c \in (0, 1)$. Let $x$ be in $[0, 1]$. Then, for each positive integer $n$, it follows that
\[ h(x) = \alpha_n h(\tau^n(x)) + \beta_n, \]
where $\alpha_n = c^{N_0(n-1)}(c - 1)^{N_1(n-1)}$ and $\beta_n$ is a real number.

The following is the first calculation of $h'(x)$.

Theorem 2.4. (1) Suppose $c \in (0, 1/2)$. Then $h'(0) = h'(1) = 0$.
(2) Suppose $c \in (1/2, 1)$. Then $h'(0) = h'(1) = \infty$.

Here we divide the real all numbers in $[0, 1]$ into two families of real numbers. One is the set of those real numbers in $[0, 1]$ which are fractions of the form $q/2^p$ and is denoted by $F_2$. The other one is the complement of $F_2$ in $[0, 1]$ and is denoted by $NF_2$. Concerning points in $F_2$, we have the following theorem.

Theorem 2.5. Let $x$ be a point in $F_2$.
(1) Suppose $c \in (0, 1/2)$. Then $h'(x) = 0$.
(2) Suppose $c \in (1/2, 1)$. Then $h'(x) = \infty$.

Even in calculation of $f'(x)$ for $x$ in $NF_2$, we need a sequence in $[0, 1]$ which converges to $x$. Moreover we need two sequences consisting of 0 and 1 associated with a point in $NF_2$. First we note that, for $x$ in $NF_2$, the following expression denotes the infinite binary expansion of $x$:
\[ x = \sum_{n=1}^{\infty} \frac{b[x]_n}{2^n}, \]
where each $b[x]_n$ is in $\{0, 1\}$, and the sequence $\{b[x]_n\}_{n=1}^{\infty}$ is denoted by $B(x)$.

Next, for $x$ in $[0, 1]$, we define a sequence consisting of 0 and 1, which is associated with the orbit of $x$ for $\tau$. We set $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1]$. For $x \in NF_2$, we denote by $O(x) = (x_n)_{n=0}^{\infty}$ the sequence defined by
\[ x_n = i \text{ if } \tau^n(x) \text{ is in } I_i. \]

This means that
\[ x \in I_{x_0}, \tau(x) \in I_{x_1}, \ldots, \tau^n(x) \in I_{x_n}, \ldots. \]

Setting $i_j = x_j + 1$ ($j = 0, \ldots, n - 1$), we can see that the above relation is equivalent to
\[ x \in \tau_i^{-1}(I_{x_1}), \tau(x) \in \tau_i^{-1}(I_{x_2}), \ldots, \tau^n(x) \in \tau_i^{-1}(I_{x_n}), \ldots. \]
and this is written by
\[ x \in \tau_{i_{0}}^{-1}(I_{x_{1}}), \ldots, x \in \tau_{i_{n-1}}^{-1}(I_{x_{n}}), \ldots. \]

Now we set
\[ K_{n} = \tau_{i_{0}}^{-1} \circ \tau_{i_{1}}^{-1} \circ \cdots \circ \tau_{i_{n-1}}^{-1}(I_{x_{n}}) \subset I, \quad (n = 1, 2, \ldots). \]

Then \( K_{n} \) is an open or half open interval and it follows that
1. \( x \in K_{n} \),
2. \( I \supset K_{1} \supset K_{2} \supset \cdots \supset K_{n} \supset \cdots \),
3. The length of \( K_{n} \) is \( 1/2^{n+1} \).

Thus it follows that \( x \in \bigcup_{n=1}^{\infty} K_{n} = \{x\} \). Therefore the map
\[ x \rightarrow O(x) = (x_{n})_{n=0}^{\infty} \]
is an injective map of \([0, 1]\) into the infinite product \( \prod_{n=0}^{\infty}\{0, 1\} \). We here remark that this map is not surjective. Indeed, the sequence
\[ (0, 1, 0, 0, 0, \ldots) \]
does not correspond to \( O(x) \) for any \( x \in [0, 1] \), although
\[ (1, 1, 0, 0, 0, \ldots) = O(1/2). \]

Now we note a relationship between two sequences \( O(x) = \{x_{n}\}_{n=0}^{\infty} \) and \( B(x) = \{b[x]_{n}\}_{n=1}^{\infty} \) for a point \( x \) in \( NF_{2} \).

(R1) \( x_{0} = b[x]_{1} \) and \( x_{n} = b(x^{n}(x))_{1} \) for \( n \geq 1 \).

(R2) For \( x = \sum_{n=1}^{\infty} b[x]_{n}/2^{n} \), it follows that
\[ \tau(x) = \begin{cases} \sum_{n=1}^{\infty} b[x]_{n+1}/2^{n} & \text{if } x \in I_{0}, \\ \sum_{n=1}^{\infty} (1 - b[x]_{n+1})/2^{n} & \text{if } x \in I_{1}. \end{cases} \]

In the following, we note some relationships between the periodicity of a point in \([0, 1]\) under \( \tau \) and two sequences \( O(x), B(x) \).

**Proposition 2.6.** Let \( x \) be a point in \([0, 1]\). Then we have the following.

1. \( x \) is a periodic point under \( \tau \) with period \( p \) if and only if \( O(x) \) is a periodic sequence with period \( p \).
2. If \( x \) is in \( NF_{2} \) and a periodic point under \( \tau \) with period \( p \), then \( B(x) \) is a periodic sequence with period \( 2p \).
(3) If $x$ is in $NF_2$ and $B(x)$ is a periodic sequence with period $p$, then there exists a positive integer $i$ such that $\tau^i(x)$ is a periodic point under $\tau$ with period $q \leq p$.

In order to show the difference between $O(x)$ and $B(x)$, we give an example of a periodic point $x = 2/3$ with period 1.

$O(2/3) = (1, 1, 1, 1, \ldots)$,
$B(2/3) = (1, 0, 1, 0, 1, \ldots)$.

Moreover, before giving a sequence which converges to $x$, we note that, if $x$ is in $NF_2$, $B(x)$ is uniquely determined and satisfies the following property.

Property (NF$_2$): For any positive integer $N$, there exists positive integers $m, n \geq N$ such that $b[x]_m = 0$ and $b[x]_n = 1$

For $x = \sum_{n=1}^{\infty} \frac{b[x]_n}{2^n}$, we define $x(k)$ by

$$x(k) = \frac{b[x]_n}{2^n} + \frac{1 - b[x]_k}{2^k} + \sum_{n=k+1}^{\infty} \frac{b[x]_n}{2^n}.$$ 

Then we have

$$x(k) = \begin{cases} x + (1/2^k) & \text{if } b[x]_k = 0, \\ x - (1/2^k) & \text{if } b[x]_k = 1. \end{cases}$$

Thus $|x(k) - x| = 1/2^k$ and $O(x(k))$ is given as follows.

**Lemma 2.7.** Let $x$ be a point in $NF_2$ with $O(x) = (x_n)_{n=0}^{\infty}$. Then it follows that

$$O(x(k)) = (x_0, \ldots, x_{k-2}, z_{k-1}, z_k, x_{k+1}, \ldots),$$

where $|z_i - x_i| = 1$ for $i = k - 1, k$.

Hereafter, in the calculation of $f'(x)$, we use following two notations.

$$H_k(x) = h(x(k)) - h(x) \quad \text{and} \quad D_k(x) = \frac{h(x(k)) - h(x)}{x(k) - x}.$$ 

Then $D_k(x) = |2^k H_k(x)|$. Now we introduce a concept concerning the behavior of the orbit of a point in $NF_2$. By virtue of the definition of $NF_2$, the set 

$\{\ell \geq 1 | b[x]_n = b[x]_{n+\ell}\}$ is not empty. For a point $x$ in $NF_2$, we set

$$p(n) = \min \{\ell \geq 1 | b[x]_n = b[x]_{n+\ell}\}.$$ 

We say that the sequence $B(x) = \{b[x]_n\}_{n=1}^{\infty}$ is quasi-periodic if there exists a positive integer $K$ such that $p(n) \leq K$ for all $n$. Of course, if $B(x)$ is periodic with period $p$, then it is quasi-periodic with $K = p$. Moreover, it follows that, if
a point $x$ is periodic point or eventually periodic under $\tau$, $B(x)$ is quasi-periodic. For a point $x$ in $NF_2$ such that $B(x)$ is quasi-periodic, we have some lemmas.

Lemma 2.8. Let $x$ be a point in $NF_2$ such that $B(x)$ is quasi-periodic. Then there exists a limit $f'(x) = \omega$ if and only if there exists a limit $\omega = \lim_{n \to \infty} D_k$.

Lemma 2.9. Let $x$ be a point in $NF_2$ with $O(x) = (x_n)_{n=0}^{\infty}$. Then it follows that

Case 1. $H_k(x) = \alpha_{k-1} \cdot ((1 - 2c)h(\tau^{k+1}(x)) + c)$ if $(x_{k-1}, x_k) = (0, 0)$,
Case 2. $H_k(x) = \alpha_{k-1} \cdot (1 - c)$ if $(x_{k-1}, x_k) = (0, 1)$,
Case 3. $H_k(x) = \alpha_{k-1} \cdot (c - 1)$ if $(x_{k-1}, x_k) = (1, 0)$,
Case 4. $H_k(x) = \alpha_{k-1} \cdot ((2c - 1)h(\tau^{k+1}(x)) - c)$ if $(x_{k-1}, x_k) = (1, 1)$.

Lemma 2.10. Let $x$ be a point in $NF_2$.

(1) Suppose $c \in (0, 1/2)$. Then we have the following inequality.

\[ 2^k c |\alpha_{k-1}| \leq D_k(x) \leq 2^k (1 - c) |\alpha_{k-1}| \]

(2) Suppose $c \in (1/2, 1)$. Then we have the following inequality.

\[ 2^k (1 - c) |\alpha_{k-1}| \leq D_k(x) \leq 2^k c |\alpha_{k-1}| \]

Using Lemma 2.3, we express Lemma 2.10 as follows.

Lemma 2.11. Let $x$ be a point in $NF_2$.

(1) Suppose $c \in (0, 1/2)$. Then we have the following inequality.

\[ 2^k c^{N_0 k - 2 + 1} (1 - c)^{N_1 k - 2} \leq D_k(x) \leq 2^k c^{N_0 k - 2} (1 - c)^{N_1 k - 2} + 1 \]

(2) Suppose $c \in (1/2, 1)$. Then we have the following inequality.

\[ 2^k c^{N_0 k - 2 + 1} (1 - c)^{N_1 k - 2} \leq D_k(x) \leq 2^k c^{N_0 k - 2 + 1} (1 - c)^{N_1 k - 2} \]

Lemma 2.11 immediately implies the following.

Lemma 2.12. Suppose $c \in (0, 1/2) \cup (1/2, 1)$. Let $x$ be a point in $NF_2$ such that $B(x)$ is quasi-periodic. Then we have the following.

(a) $\lim_{k \to \infty} 2^k c^{N_0 k} (1 - c)^{N_1 k} = 0$ if and only if $\lim_{k \to \infty} D_k(x) = 0$.

(b) $\lim_{k \to \infty} 2^k c^{N_0 k} (1 - c)^{N_1 k} = \infty$ if and only if $\lim_{k \to \infty} D_k(x) = \infty$.

We have discussed the existence of $h'(x)$ and now we show that the possibility of taking the value of $f'(x)$ is only 0 and $\infty$. 
Lemma 2.13. Let $x$ be a point in $NF_2$. If there exists $\omega = \lim_{k \to \infty} D_k(x)$, then $\omega = 0$ or $\omega = \infty$.

Immediately, by Theorem 2.5 and Lemma 2.13, we have the following proposition.

Proposition 2.14. If there exists $f'(x)$, then $f'(x) = 0$ or $f'(x) = \infty$.

By Lemma 2.9, 2.12 and Proposition 2.14, we have the following proposition.

Proposition 2.15. Let $x$ be a point in $NF_2$ such that $B(x)$ is quasi-periodic. Then it follows that

$$h'(x) = \begin{cases} 
0 & \text{if } \lim_{n \to \infty} 2^n c^{N_0(n)}(1-c)^{N_1(n)} = 0 \\
\infty & \text{if } \lim_{n \to \infty} 2^n c^{N_0(n)}(1-c)^{N_1(n)} = \infty \\
does not exist & \text{otherwise.}
\end{cases} \cdots (1)$$

Now let $x$ be a periodic point with period $p$. Then $O(x) = (x_n)_{n=0}^{\infty}$ is a periodic sequence with period $p$. Here we set $r_p(x) = 2^p c^{N_0(p-1)}(1-c)^{N_1(p-1)}$. Then, since $N_i(mp-1) = m N_i(p-1) (i = 0, 1)$, it follows that $r_{mp}(x) = (r_p(x))^m$. Hence we have the following.

Proposition 2.16. If $x$ is a periodic point in $[0, 1]$ with period $p$ under $\tau$, then it follows that

$$h'(x) = \begin{cases} 
0 & \text{if } r_p(x) < 1, \\
\infty & \text{if } r_p(x) > 1.
\end{cases}$$

Using Proposition 2.16, we have the following proposition.

Proposition 2.17. Suppose $c \in (0, 1/2) \cup (1/2, 1)$. Let $E_0 = \{x \in [0, 1] | h'(x) = 0\}$ and $E_\infty = \{x \in [0, 1] | h'(x) = \infty\}$. Then

1. $E_0$ and $E_\infty$ are dense in $[0, 1]$,
2. $\mu(E_0) = 1$ and $\mu(E_\infty) = 0$,
3. $\mu(h(E_0)) = 0$ and $\mu(h(E_\infty)) = 1$.

Remark 2.18. We note that $r(x) \neq 1$ does not necessarily hold though it holds if $c$ is a rational number. Indeed, it does not hold if $c = \frac{3-\sqrt{5}}{4}$. In the following, we show that $h'(x)$ does not exist for a $\tau$-periodic point with period 3.

Example 2.19. Let $c = \frac{3-\sqrt{5}}{4} \in (0, 1/2)$ and $x = 2/7$. Then $8c(c-1)^2 = 1$ and $x$ is a $\tau$-periodic point with period 3 with the following orbit:

$$O(x) = (0, 1, 1) \quad \text{and} \quad B(x) = (0, 1, 0).$$
Thus \( r_3(2/7) = 2^3c(c-1)^2 = 1 \). Here we calculate \( D_k(x)'s \). Using \( 8c(1-c)^2 = 1 \), we have
\[
\alpha_{3\ell} = \{c(1-c)^2\}^\ell = 1/2^{3\ell}, \\
\alpha_{3\ell+1} = (c-1)\alpha_{3\ell} = (c-1)/2^{3\ell}, \\
\alpha_{3\ell+2} = (c-1)\alpha_{3\ell+1} = (c-1)^2/2^{3\ell}.
\]
Thus, since \((x_{3(\ell-1)+2}, x_{3\ell}) = (1,0)\), we have
\[
D_{3\ell}(x) = 2^{3\ell}\vert H_{3\ell}\vert = 2^{3\ell}\vert \alpha_{3\ell-1}(c-1)\vert = 2^{3\ell}(1-c)^3/2^{3\ell-1} = 8(1-c)^3 - 2 + \sqrt{5}.
\]
Moreover, since \((x_{3\ell}, x_{3\ell+1}) = (0,1)\), we have
\[
D_{3\ell+1}(x) = 2^{3\ell+1}\vert H_{3\ell+1}\vert = 2^{3\ell+1}\vert \alpha_{3\ell}(1-c)\vert = 2^{3\ell+1}(1-c)/2^{3\ell} = 2(1-c) = \frac{1+\sqrt{5}}{2}.
\]
Therefore the sequence \( \{D_k(x)\}_{k=1}^\infty \) does not converge, that is, \( h'(x) \) does not exist.

Now we reach at our conclusion.

Theorem 2.20. Suppose \( c \in (0,1/2) \cup (1/2,1) \). Then
\[
h'(x) = \begin{cases} 
0 & \text{if } x \in E_0, \\
\infty & \text{if } x \in E_\infty, \\
\text{does not exist} & \text{if } x \in F,
\end{cases}
\]
where \( \{E_0, E_\infty, F\} \) are measurable sets satisfying the following conditions.
  (1) \( \{E_0, E_\infty, F\} \) are mutually disjoint,
  (2) \( \mu(E_0) = 1, \mu(E_\infty) = 0 \) and \( \mu(F) = 0 \),
  (3) \( E_0 \) and \( E_\infty \) are dense in \([0,1]\),
  (4) \( \mu(h(E_0)) = 0, \mu(h(E_\infty)) = 1 \) and \( \mu(h(F)) = 0 \).
Moreover it follows that
  (5) the set \( F_2 \) is included in \( E_0 \) (resp. \( E_\infty \)) if \( c \in (0,1/2) \) (resp. \( c \in (1/2,1) \)).

Finally we note that the condition of quasi-periodicity in Lemma 2.8 may be deleted, though we cannot prove the lemma without that condition in the present paper and have no counter example.

References


