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<thead>
<tr>
<th>Title</th>
<th>On integral bases of real octic 2-elementary abelian extensions (Algebraic Number Theory and Related Topics)</th>
</tr>
</thead>
<tbody>
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Kyoto University
On integral bases of real octic 2-elementary abelian extensions
(実 8 次 2 基本アーベル拡大体の整数基について)

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Abstract. Let $K$ be an abelian field whose Galois group is 2-elementary abelian over the rationals $Q$. If an octic field $K$ is monogenic and a quadratic subfield with odd discriminant and a quartic subfield of $K$ are linearly disjoint, then $K$ coincides with the field $Q(\sqrt{-1}, \sqrt{2}, \sqrt{-3})$, namely $K$ is equal to the cyclotomic field $Q(\zeta_{24})$ [MN]. In this article, we explain how to prove that all the real octic fields $K$ are non-monogenic, that is, the rings $Z_K$ of integers in $K$ do not have any power integral basis. Finally, we propose a few problems on the evaluation on the field index of $K$ and the non-essential factor (außerwesentliche Diskriminantenanteiler) of $K$.

§1. Introduction

Let $K$ be an algebraic number field over the rationals $Q$. We denote the ring of integers in $K$ by $Z_K$. When $Z_K = Z[\alpha]$ for some element $\alpha$ of $Z_K$, it is said that $\alpha$ generates a power integral basis of the ring $Z_K$ or simply $Z_K$ has a power integral basis. The field $K$ is called monogenic if $Z_K$ has a power integral basis. It is known as a problem of Hasse to characterize whether a field $K$ is monogenic or not [Gy]. In this article, we consider the fields $K$ whose Galois groups are 2-elementary abelian. Since the field $K$ for $[K : Q] \geq 16$

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On integral bases of real octic 2-elementary abelian extensions

is non-monogenic, i.e., the ring $Z_K$ of integers in $K$ has no power integral basis by virtue of the decomposition theory of a prime number ([Lemma 1, SN], [MNS], [Wa]) and by the works of K. S. Williams, M.-N. Gras and F. Tanoé for Dirichlet fields $K$, ([Wi], [GT]) it is enough for us to investigate the octic 2-elementary abelian fields. Let $k$ and $L$ be a quadratic subfield of odd discriminant and a quartic subfield of $K$, respectively. If $k$ and $L$ are linearly disjoint, then such an octic field $K = kL$ is non-monogenic except for the cyclotomic field $Q(\zeta_{24})$ of conductor 24 [MN]. In this paper, we will show an integral basis of the ring $Z_K$ over the ring $Z$ of rational integers in an octic field $K$ [Theorem 1]. Next, being based on the linear equations

$$a_1E_{i1} + a_2E_{i2} + a_3E_{i3} = 0 \quad (1 \leq i \leq 7)$$

with suitable factors $a_{ij}$ of the field discriminant $D_K$, where $(a_{ij}, D_i) = 1$ and units $E_{ij}$ as coefficients of valables $a_{ij}$ in each quadratic subfield $k_j = Q(\sqrt{D_j})$ [Proposition 2], we can prove that all the real 2-elementary abelian fields $K$ of degree 8 have no power integral basis [Theorem 2].

§2. Integral bases

We determine explicit integral bases of some octic fields $K$ whose Galois groups are 2-elementary abelian. We denote the Galois group

$$\langle \tau, \sigma, \rho \mid \tau : \sqrt{mn} \mapsto -\sqrt{mn}, \sigma : \sqrt{dn} \mapsto -\sqrt{dn}, \rho : \sqrt{d_1m_1n_1\ell} \mapsto -\sqrt{d_1m_1n_1\ell} \rangle$$

of $K/Q$ by $G$.

The following lemma and proposition are available to deduce the type of 2-elementary abelian extension fields $K$ which would have power integral bases.

Lemma 1([SN]). Let $\ell$ be a prime number and let $F/Q$ be a Galois extension of degree $n = efg$ with ramification index $e$ and the relative degree $f$ with respect to $\ell$. If one of the following conditions is satisfied, then $Z_F$ has no power integral basis, i.e., $F$ is non-monogenic;

1. $ef\ell < n$ if $f = 1$; or
2. $ef\ell \leq n + e - 1$ if $f \geq 2$.

Proposition 1([MN]). Let $a_1, a_2, \cdots, a_r$ be square free rational integers and $F$ be the field $Q(\sqrt{a_1}, \sqrt{a_2}, \cdots, \sqrt{a_r})$ of degree $2^r$, $r \geq 4$. Then $F$ is non-monogenic.

Proof. Without loss of generality, we may assume that there exists at most two generators $\sqrt{a_1}, \sqrt{a_2}$ of $F$ with $a_j \not\equiv 1 \pmod{4}$ ($1 \leq j \leq 2$). Then the ramification index $e$ of the prime
Kyoung Ho Park, Toru Nakahara and Yasuo Motoda

is at most $2^2$. Since the Galois group $G = Gal(F/\mathbb{Q})$ is 2-elementary, the relative degree $f$ of the prime 2 is at most 2, because the inertia subgroup of $G$ is cyclic. In Lemma 1 let $\ell$ be equal to 2. Then we can deduce $e\ell^j \leq 2^2 \cdot 2^1 < 2^r$ if $f = 1$ and $e\ell^j \leq 2^2 \cdot 2^2 \leq 2^r + e - 1$ if $f = 2$. Thus $F$ is non-monogenic.

By the proof of Proposition 1, if an octic field $K$ is monogenic, it is sufficient to consider that $K$ contains two quadratic subfields of even discriminant and one of odd discriminant.

The main theorem is based on the following theorem, which is an extension of a result of the case of quartic fields $[M_1, M_2, Wi]$.

**Theorem 1 ([PMN]).** Let $K$ be an octic field $\mathbb{Q}(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1m_1n_1\ell})$ with $d = d_1d_2$, $m = m_1m_2$, $n = n_1n_2$, $mn \equiv 3$, $dn \equiv 2$, $d_1n_1n_1\ell \equiv 1$, $d_2 \equiv 2 \pmod{4}$, $d_1, m_1, n_1 \geq 1$ and $d_1m_1n_1\ell$ is square free. Let $D_K$ be the field discriminant of the octic field $K$. Then we have $D_K = 2^{12}(dmn\ell)^4$ and an integral basis of $K$ is:

$$\mathcal{O}_K = \mathbb{Z}[1, \sqrt{mn}, \sqrt{dn}, \frac{\sqrt{dm} + \sqrt{dn}}{2}, \frac{1 + \sqrt{d_1m_1n_1\ell}}{2}, \frac{\sqrt{dm} + \sqrt{dn} + e_1\sqrt{d_2m_2n_2\ell} + e_2\sqrt{d_2m_2n_2\ell}}{4}]$$

where $e_i = \pm 1$ ($i = 1, 2$), $e_1 \equiv d_1n_1, e_2 \equiv d_1n_1 \pmod{4}$.

**§ 3. Non-monogenic field**

It is known that in the case of $d_1m_1n_1 = 1$ that is, there exist a quartic subfield $L$ and a quadratic $k$ of $K$ with $(D_L, D_k) = 1$, the fields $K$ are non-monogenic except for the cyclotomic field $\mathbb{Q}(\zeta_{24})$ of conductor 24 [MN], where $D_F$ means the discriminant of an algebraic number field $F$ over $\mathbb{Q}$. From now on, we consider the case of $d_1m_1n_1 \geq 1$ and as an application of Theorem 1, we can slightly generalize Proposition 5 in [MN], whose proof was done using the relative different with respect to $K$ over a suitable quadratic subfield. We assume that $K$ is monogenic.

Let

$$\xi = b_1\sqrt{mn} + b_2\sqrt{dn} + b_3\frac{\sqrt{dm} + \sqrt{dn}}{2} + b_4\frac{1 + \sqrt{d_1m_1n_1\ell}}{2} + b_5\frac{\sqrt{dm} + \sqrt{d_1m_2n_2\ell}}{2}$$

$$+ b_6\frac{\sqrt{dn} + \sqrt{d_2m_1n_2\ell}}{2} + b_7\frac{\sqrt{dm} + \sqrt{dn} + e_1\sqrt{d_2m_2n_1\ell} + e_2\sqrt{d_2m_1n_2\ell}}{4}$$

be a generator of a power integral basis of $\mathcal{O}_K$. Now we calculate a factor $(\xi - \xi^\sigma)(\xi - \xi^\rho)^\rho$
of the discriminant \( d_{K/Q}(\xi) = \Delta^2 \left[ 1, \xi, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6, \xi^7 \right] \) of a number \( \xi; \)

\[
\begin{align*}
(\xi - \xi^\sigma)(\xi - \xi^\rho)^p &= \left\{ \left( 2b_2 + b_3 + b_6 + \frac{b_7}{2} \right) \sqrt{dn} + \left( b_3 + \frac{b_7}{2} \right) \sqrt{dm} + \left( b_6 + \frac{b_7 e_2}{2} \right) \sqrt{d_2 m_1 n_2 \ell} + \frac{b_7 e_1 \sqrt{d_2 m_2 n_1 \ell}}{2} \right\} \\
&\times \left\{ \left( 2b_2 + b_3 + b_6 + \frac{b_7}{2} \right) \sqrt{dn} + \left( b_3 + \frac{b_7}{2} \right) \sqrt{dm} - \left( b_6 + \frac{b_7 e_2}{2} \right) \sqrt{d_2 m_1 n_2 \ell} - \frac{b_7 e_1 \sqrt{d_2 m_2 n_1 \ell}}{2} \right\} \\
&= \left\{ \left( 2b_2 + b_3 + b_6 + \frac{b_7}{2} \right) \sqrt{dn} + \left( b_3 + \frac{b_7}{2} \right) \sqrt{dm} \right\}^2 - \left\{ \left( b_6 + \frac{b_7 e_2}{2} \right) \sqrt{d_2 m_1 n_2 \ell} + \frac{b_7 e_1 \sqrt{d_2 m_2 n_1 \ell}}{2} \right\}^2 \\
&= \left\{ \left( 2b_2 + b_3 + b_6 \right)^2 + \left( 2b_2 b_7 + b_3 b_7 + b_6 b_7 \right) + \frac{b_7^2}{4} \right\} dn^\ell - \left\{ b_3^2 + b_3 b_7 + \frac{b_7^2}{4} \right\} dm \\
&- \left\{ b_6^2 + b_6 b_7 + \frac{b_7^2}{4} \right\} \ell m_1 n_2 + \frac{b_7^2}{4} \right\} dm \\
&- \left( b_6^2 + b_6 b_7 + \frac{b_7^2}{4} \right) \ell d_1 m_2 n_2 - \frac{b_7^2}{4} \right\} dm \\
&+ \left\{ \left( 2b_2 b_3 + 2b_3 b_7 + 2b_3 b_7 + 2b_3 b_7 + \frac{b_7^2}{2} \right) d - \left( b_6 b_7 e_1 \ell + \frac{b_7^2 e_1 \ell}{2} \right) \ell \right\} \sqrt{mn},
\end{align*}
\]

namely, this factor is an integer of the quadratic field \( k_1 = \mathbb{Q}(\sqrt{mn}) \) of the fixed field by the subgroup \( \langle \sigma, \rho \rangle \) in \( G. \) Then we denote it by \( \eta_1 = B + C(\sqrt{mn}). \) Thus we obtain

\[
\begin{align*}
B/d_2 &\equiv \left\{ b_3^2 + b_6^2 + b_3 b_7 + \frac{b_7^2}{4} \right\} d_1 n + \left( b_3^2 + b_3 b_7 + \frac{b_7^2}{4} \right) d_1 m \\
&- \left( b_6^2 + b_6 b_7 + \frac{b_7^2}{4} \right) m_1 n_2 \ell - \frac{b_7^2}{4} \right\} dm \\
&\equiv \frac{b_7^2}{4} \left( d_1 (m + n) - (m_1 n_2 + m_2 n_1) \ell \right) \\
&\equiv \frac{\left\{ d_1 (m + n) - (d_1 n + 4k + d_1 m + 4k) \right\}}{4} \equiv 0 \pmod{2},
\end{align*}
\]

by \( d_1 m_1 n_1 \ell \equiv 1 + 4k \pmod{8} \) and \( m + n \equiv 0 \pmod{4} \), since \( m_1 n_2 \ell \cdot 1 \equiv d_1 m_2^2 n_1 n_2 \ell^2 + 4m_1 n_2 \ell k \equiv d_1 n + 4k \pmod{8} \) and \( m_2 n_1 \ell \cdot 1 \equiv d_1 m_1 m_2 n_2 \ell^2 + 4m_2 n_1 \ell k \equiv d_1 m + 4k \pmod{8} \).

\[
\begin{align*}
C/d_2 &\equiv \left( b_6 b_7 + \frac{b_7^2}{2} \right) d_1 - \left( b_6 b_7 e_1 \ell + \frac{b_7^2 e_1 \ell}{2} \right) \\
&\equiv b_6 b_7 \left( d_1 - e_1 \ell \right) + \frac{b_7^2}{2} \left( d_1 - e_2 e_1 \ell \right) \equiv 0 \pmod{2},
\end{align*}
\]

by \( e_1 \equiv d_1 m_1, \ e_2 \equiv d_1 n_1 \pmod{4} \), since \( d_1 - e_2 e_1 \ell \equiv d_1 - d_1^2 m_1 n_1 \ell \equiv d_1 (1 - d_1 m_1 n_1 \ell) \equiv 0 \pmod{4} \). So we can write \( \eta_1 = (\xi - \xi^\sigma)(\xi - \xi^\rho)^p = 2d_2 E_1 \) for an integer \( E_1 = B_1 + C_1 \sqrt{mn} \) in \( k_1 = \mathbb{Q}(\sqrt{mn}) \). By the same computation, we obtain \( \eta_2 = (\xi - \xi^\rho)(\xi - \xi^\rho)^p = \ell E_2, \)

\[
\eta_3 = (\xi - \xi^\rho)(\xi - \xi^\rho)^p = d_1 E_3 \text{ for units } E_j \text{ in } k_1(j = 2, 3). \] By the assumption that \( Z_K \) is generated by \( \xi \), we have

\[
d_{K/Q}(\xi) = \pm N_K(\mathfrak{a}(\xi)) = \pm D_K,
\]
$V(\alpha)$, $N_{K}(\alpha)$ and $N_{K}(a)$ means the different of a number, norm of $\alpha$ and an ideal $a$ with respect to $K/Q$, respectively [Wa]. Then, because $\eta_{1j}$ is a partial factor of $d_{K/Q}(\xi)$, the integers $E_{j}$ should be units in $k_{1} = Q(\sqrt{mn})$. Here the following is our basic identity:

$$(\xi - \xi^\sigma)(\xi - \xi^\rho) - (\xi - \xi^\rho)(\xi - \xi^\sigma) - (\xi - \xi^{\sigma\rho})(\xi - \xi^{\sigma\rho}) = 0$$

for $(\xi - \xi^\sigma)(\xi - \xi^\rho) = \eta_{11}$, $(\xi - \xi^\rho)(\xi - \xi^\sigma) = \eta_{12}$ and $(\xi - \xi^{\sigma\rho})(\xi - \xi^{\sigma\rho}) = \eta_{13}$. Then we have the equation

$$2d_{2}E_{1} - \ell E_{2} - d_{1}E_{3} = 0 \quad \text{in } k_{1} = Q(\sqrt{D_{1}}), \quad D_{1} = m_{1} \cdot 2m_{2} \cdot n_{1} \cdot 2n_{2},$$

where $E_{1}$, $E_{2}$ and $E_{3}$ are units in $k_{1}$.

In the same way, we obtain seven equations corresponding to each of the seven quadratic subfields $k_{i}$ of $K$.

**Proposition 2.** If $K = Q(\sqrt{mn}, \sqrt{dn}, \sqrt{d_{1}m_{1}n_{1}P})$ is monogenic, then the following simultaneous equations hold:

1. $\ell E_{11} + 2d_{2}E_{12} + d_{1}E_{13} = 0 \quad \text{in } k_{1} = Q(\sqrt{D_{1}}), \quad D_{1} = m_{1} \cdot 2m_{2} \cdot n_{1} \cdot 2n_{2},$
2. $\ell E_{21} + 2m_{2}E_{22} + m_{1}E_{3} = 0 \quad \text{in } k_{2} = Q(\sqrt{D_{2}}), \quad D_{2} = d_{1} \cdot 2d_{2} \cdot n_{1} \cdot 2n_{2},$
3. $\ell E_{31} + 2n_{2}E_{32} + n_{1}E_{33} = 0 \quad \text{in } k_{3} = Q(\sqrt{D_{3}}), \quad D_{3} = d_{1} \cdot 2d_{2} \cdot m_{1} \cdot 2m_{2},$
4. $2d_{2}E_{14} + 2m_{2}E_{42} + 2n_{2}E_{43} = 0 \quad \text{in } k_{4} = Q(\sqrt{D_{4}}), \quad D_{4} = d_{1} \cdot m_{1} \cdot n_{1} \cdot P,$
5. $2d_{2}E_{51} + m_{1}E_{52} + n_{1}E_{63} = 0 \quad \text{in } k_{5} = Q(\sqrt{D_{5}}), \quad D_{5} = 2d_{2} \cdot m_{1} \cdot 2n_{2} \cdot \ell,$
6. $d_{1}E_{61} + 2m_{2}E_{62} + n_{1}E_{63} = 0 \quad \text{in } k_{6} = Q(\sqrt{D_{6}}), \quad D_{6} = d_{1} \cdot 2d_{2} \cdot m_{1} \cdot 2n_{2} \cdot \ell,$
7. $d_{1}E_{71} + m_{1}E_{72} + 2n_{2}E_{73} = 0 \quad \text{in } k_{7} = Q(\sqrt{D_{7}}), \quad D_{7} = 2d_{2} \cdot 2m_{2} \cdot n_{1} \cdot \ell,$

where each $E_{ij}$ is a unit in the corresponding quadratic subfield $k_{i}$ of $K$ and each $D_{i}$ the field discriminant of $k_{i}$, respectively.

For the case of a real quadratic field, the following lemma holds:

**Lemma 2.** Let $E_{j}$ be a power $\epsilon_{0}^j = \frac{u_{j} + v_{j} \sqrt{D}}{2}$ of the fundamental unit $\epsilon_{0} = \frac{u + v \sqrt{D}}{2} > 1$ in a real quadratic field $Q(\sqrt{D})$ with the field discriminant $D$ and $\overline{\alpha} = \alpha^{\gamma}$ for $\alpha$ in $Q(\sqrt{D})$ and $\gamma(\neq I)$ in $Gal(Q(\sqrt{D})/Q)$. Let

$$\begin{cases}
    a + bE_{j} + cE_{k} = 0, \\
    a + b\overline{E_{j}} + c\overline{E_{k}} = 0
\end{cases} \quad (*)$$

for $abc \neq 0$. Denote the matrix

$$\begin{pmatrix}
    1 & E_{j} & E_{k} \\
    1 & \overline{E_{j}} & \overline{E_{k}}
\end{pmatrix}$$
On integral bases of real octic 2-elementary abelian extensions

attached to the equation (*) by $A$ and the rank of $A$ by $r_D$. Then we have a solution $(a, b, c)$ of rational integers:

$$\begin{aligned}
\left\{ \begin{array}{ll}
a \pm b \pm c = 0 & \text{for } r_D = 1, \\
a = \frac{b}{u_kv_j - u_jv_k} = \frac{c}{-2v_j} & \text{for } r_D = 2
\end{array} \right.
\end{aligned}$$

with $E_i = \frac{u_i + v_i\sqrt{D}}{2}$.

**Proof.** This lemma means that the integral solutions should be on the plane for the rank $r_D = 1$ of the coefficient matrix $A$ and on the line i.e. the intersection of two planes for $r_D = 2$, respectively.

First, we consider the case of $r_D = 1$, then for

$$\begin{aligned}
E_i &= \frac{u_i + v_i\sqrt{D}}{2}, \\
\overline{E_i} &= \frac{u_i - v_i\sqrt{D}}{2},
\end{aligned}$$

$E_i, \overline{E_i}$ should be a rational number. Then we have $E_j = u_j = \pm 1$ and $E_k = u_k = \pm 1$. Hence $a \pm b \pm c = 0$. Second, we assume $r_D = 2$. Then we have

$$\begin{aligned}
a : b : c &= \left| \begin{array}{ccc}
E_j & E_k & 1 \\
\overline{E_j} & \overline{E_k} & 1
\end{array} \right| = u_kv_j - u_jv_k : 2v_k : -2v_j.
\end{aligned}$$

Hence

$$\begin{aligned}
a &= \frac{b}{2v_k} = \frac{c}{-2v_j}.
\end{aligned}$$

In the case of any octic field $Q(\sqrt{m_1m_2n_1n_2}, \sqrt{d_1d_2n_1n_2}, \sqrt{d_1m_1n_1\ell})$, by the following lemma, we can deduce to evaluate the rank $r_D$ of a quadratic field $Q(\sqrt{D})$ for a few cases with respect to the order of values $d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell$ in the set of seven parameters.

**Lemma 3.** Let denote the set $\{d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell\}$ by $D$. Then it holds that:

1. For one parameter $s$ in $D$, there exist only four quadratic subfields $k_j$ whose discriminants $D_j$ are divisible by $s$.

2. For two parameters $s, t$ in $D$, there exist only two quadratic subfields $k_j$ whose discriminants $D_j$ are divisible by $st$.

3. Let $s, t, u$ be three parameters in $D$, such that $stu$ is a divisor of the field discriminant of $D_j$ of $k_j$. Then there exists only one quadratic subfield $k_j$ whose discriminant $D_j$ is divisible by $stu$. 

\square
Proof. (1) We can confirm the claim (1) for each of \( \binom{\#D}{1} = 7 \) parameter in \( D \) from seven equations in Proposition 2, such that there exist just four fields \( k_1, k_3, k_4, k_5 \) whose discriminant is divisible by \( m_1 \).

(2) We can do the claim (2) of \( \binom{\#D}{2} = 21 \) pairs of parameters in \( D \) by the same way as in (1). For instance, there exist just two fields \( k_3, k_7 \) whose discriminants are divisible by \( d_2m_2 \).

(3) We assume that \( D_i = stu \) and \( D_j = stub \). Then we have \( D_iD_j = (stu)^2ab \). However, the quadratic subfield \( Q(\sqrt{ab}) \) does not coincide with any \( k_j (1 \leq j \leq 7) \).

Remark 1. We can confirm that the number of triplets \( (s, t, u) \) within the order of parameters in \( D \) is equal to \( 28 = 7 \times 1 \times \binom{4}{3} < \binom{\#D}{3} = 35 \) such that each of \( s tu \) is a divisor of the field discriminant \( D_j \) of \( k_j \).

Next, we prepare the key lemma for the proof of Theorem 2.

Lemma 4. For the set \( D = \{a, b, c, d, e, f, g\} \) of seven positive rational integers, assume that \( a > b \geq c > \max\{d, e, f, g\} \) and \( d > f \) or \( a > b > c \geq \max\{d, e, f, g\} \) and \( d > f \). Then

(1) For the field \( Q(\sqrt{bcs}) \), where \( s, t \in D \setminus \{a, b, c\} \) and units \( E_i \) in \( Q(\sqrt{bcs}) \), the rank \( \rho_{bcs} \) of the equations
\[
\begin{align*}
a + uE_j + vE_k &= 0, \\
a + u\overline{E_j} + v\overline{E_k} &= 0,
\end{align*}
\]
with \( \{u, v\} = D \setminus \{a, b, c, s, t\} \) is equal to 1.

(2) For the field \( Q(\sqrt{astu}) \), where \( s, t, u \in D \setminus \{a, b, c\} \) and units \( E_i \) in \( Q(\sqrt{astu}) \), the rank \( \rho_{astu} \) of the equations
\[
\begin{align*}
b + cE_j + vE_k &= 0, \\
b + c\overline{E_j} + v\overline{E_k} &= 0,
\end{align*}
\]
with \( \{v\} = D \setminus \{a, b, c, s, t, u\} \) is equal to 1.

Sketch of Idea. Our idea for the proof of this lemma is as follows. For the quadratic subfield \( k \) including the coefficients of the simultaneous equation (\( \ast \)), if the field discriminant \( D_k \) is divisible by the biggest parameter (case (1)) or the second and the third ones (case (2)), since the fundamental unit (\( > 1 \)) of \( k \) is relatively big, the ratios for the line in Lemma 2 would not be permitted. Thus the ranks of the coefficient matrix for both cases should be equal to one, respectively, namely any integral solution of (\( \ast \)) lies on the plane \([PMN]\). \( \square \)
Finally, we show the following main theorem, which is a generalization of a prototype[PMN].

**Theorem 2.** Let $K = Q(\sqrt{a_1}, \cdots, \sqrt{a_r})$ be the 2-elementary abelian extensions over $Q$ whose degree $2^e$ is greater than 8 or real octic ones for square free integers $a_1, \cdots, a_r$. Then the fields $K$ are non-monogenic.

**Sketch of Proof.** By Proposition 1, it is enough to consider an octic field $K$. Let $(2) = \mathcal{L}_1^{e} \cdots \mathcal{L}_g^{e}$ be the prime ideal decomposition of a rational prime 2 in $K$. For the ramification index of 2, if $e \leq 1$, then by Lemma 1 and the relative degree $f$ of a prime 2 is at most 2, we have $1 \cdot 2^e < 8$ or $1 \cdot 2^e \leq 8 + 1 - 1$ for $e = 1$ and $2 \cdot 2^e \leq 8$ or $2 \cdot 2^e \leq 8 + 2 - 1$ for $e = 2$, namely $K$ is non-monogenic. Then in the case of $e \geq 3$, we can deduce that the type of an octic field is $K = Q(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$, where $a_1 = mn \equiv 3$, $a_2 = dn \equiv 2$, $a_3 = d_1m_1n_1\ell \equiv 1$ (mod 4), for $d = d_1d_2$, $m = m_1m_2$, $n = n_1n_2$ and $dmn\ell$ is square free. Put $D = \{d_1, d_2, m_1, 2m_2, n_1, 2n_2, \ell\}$. We denote again by $\{a, b, c, d, e, f, g\}$ any transposition on the seven parameters in $D$. Without loss of generality, we may assume that $a > b > c \geq \max\{d, e, f, g\}$. Using Lemma 4, it is enough for us to consider the following two cases.

Case (I). The field $K$ includes $k_{j_1} = Q(\sqrt{abct})$ for some $t \in D\{a, b, c\}$, for instance, $t = d$.

Case (II). The field $K$ does not include the field $Q(\sqrt{abc}s)$ for any $s \in D\{a, b, c\}$.

In the case (I), we can deduce that the four parameters $a, b, c, d$ with $c \geq d$ must lie on suitable two planes and in the case (II), $a, b, e, g$ with $e > g$ do on four planes, respectively. However, the order of the parameters would be destroyed. Then we can prove that any real octic fields $K$ does not have a power integral basis[PMN].

**Remark 2.** Recently, in [PNM] we proved that all the 2-elementary abelian fields $K$ with degree $[K : Q] \geq 8$ are non-monogenic except for the field $Q(\sqrt{-1}, \sqrt{2}, \sqrt{-3}) = Q(\zeta_{24})$.

**Problem.** For a primitive element $\xi$ in $K$, let $\text{Ind}(\xi)$, $\hat{m}(K)$ and $m(K)$ be the index $\sqrt{\frac{d_K(\xi)}{D_K}}$ of an element $\xi$, the minimum index $\min_{\xi \in K}\text{Ind}(\xi)$ of $K$ and the field index $\gcd_{\xi \in K}\text{Ind}(\xi)$ of $K$, respectively. Let the fields $K$ run through all the real octic fields whose Galois groups are 2-elementary abelian. Then evaluate the values of

$$\inf_{K} \hat{m}(K) \quad \text{and} \quad \inf_{K} m(K),$$
respectively.

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References

[GT] M.-N. GRAS and F. TANOÉ, 


