On integral bases of real octic 2-elementary abelian extensions
(実 8次 2-基本アーベル拡大体の整数基について)

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Abstract. Let $K$ be an abelian field whose Galois group is 2-elementary abelian over the rationals $Q$. If an octic field $K$ is monogenic and a quadratic subfield with odd discriminant and a quartic subfield of $K$ are linearly disjoint, then $K$ coincides with the field $Q(\sqrt{-1}, \sqrt{2}, \sqrt{-3})$, namely $K$ is equal to the cyclotomic field $Q(\zeta_{24})$ [MN]. In this article, we explain how to prove that all the real octic fields $K$ are non-monogenic, that is, the rings $Z_K$ of integers in $K$ do not have any power integral basis. Finally, we propose a few problems on the evaluation on the field index of $K$ and the non-essential factor (außerwesentliche Diskriminanteiler) of $K$.

§1. Introduction

Let $K$ be an algebraic number field over the rationals $Q$. We denote the ring of integers in $K$ by $Z_K$. When $Z_K = \mathbb{Z}[\alpha]$ for some element $\alpha$ of $Z_K$, it is said that $\alpha$ generates a power integral basis of the ring $Z_K$ or simply $Z_K$ has a power integral basis. The field $K$ is called monogenic if $Z_K$ has a power integral basis. It is known as a problem of Hasse to characterize whether a field $K$ is monogenic or not [Gy]. In this article, we consider the fields $K$ whose Galois groups are 2-elementary abelian. Since the field $K$ for $[K:Q] \geq 16$

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On integral bases of real octic 2-elementary abelian extensions

is non-monogenic, i.e., the ring \( Z_K \) of integers in \( K \) has no power integral basis by virtue of the decomposition theory of a prime number ([Lemma 1, SN], [MNS], [Wa]) and by the works of K. S. Williams, M.-N. Gras and F. Tanoé for Dirichlet fields \( K, ([Wi], [GT]) \) it is enough for us to investigate the octic 2-elementary abelian fields. Let \( k \) and \( L \) be a quadratic subfield of odd discriminant and a quartic subfield of \( K \), respectively. If \( k \) and \( L \) are linearly disjoint, then such an octic field \( K = kL \) is non-monogenic except for the cyclotomic field \( Q(\zeta_{24}) \) of conductor 24 [MN]. In this paper, we will show an integral basis of the ring \( Z_K \) over the ring \( Z \) of rational integers in an octic field \( K \) [Theorem 1].

Next, being based on the linear equations

\[
a_{i1}E_{i1} + a_{i2}E_{i2} + a_{i3}E_{i3} = 0 \quad (1 \leq i \leq 7)
\]

with suitable factors \( a_{ij} \) of the field discriminant \( D_K \), where \( (a_{ij}, D_i) = 1 \) and units \( E_{ij} \) as coefficients of valuations \( a_{ij} \) in each quadratic subfield \( k_j = Q(\sqrt{D_j}) \) [Proposition 2], we can prove that all the real 2-elementary abelian fields \( K \) of degree 8 have no power integral basis [Theorem 2].

§2. Integral bases

We determine explicit integral bases of some octic fields \( K \) whose Galois groups are 2-elementary abelian. We denote the Galois group

\[
\langle \tau, \sigma, \rho \mid \tau: \sqrt{mn} \mapsto -\sqrt{mn}, \sigma: \sqrt{dn} \mapsto -\sqrt{dn}, \rho: \sqrt{d_1m_1n_1\ell} \mapsto -\sqrt{d_1m_1n_1\ell} \rangle
\]

of \( K/Q \) by \( G \).

The following lemma and proposition are available to deduce the type of 2-elementary abelian extension fields \( K \) which would have power integral bases.

**Lemma 1** ([SN]). Let \( \ell \) be a prime number and let \( F/Q \) be a Galois extension of degree \( n = efg \) with ramification index \( e \) and the relative degree \( f \) with respect to \( \ell \). If one of the following conditions is satisfied, then \( Z_F \) has no power integral basis, i.e., \( F \) is non-monogenic:

1. \( e\ell^f < n \) if \( f = 1 \);
   or
2. \( e\ell^f \leq n + e - 1 \) if \( f \geq 2 \).

**Proposition 1** ([MN]). Let \( a_1, a_2, \ldots, a_r \) be square free rational integers and \( F \) be the field \( Q(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_r}) \) of degree \( 2^r, r \geq 4 \). Then \( F \) is non-monogenic.

**Proof.** Without loss of generality, we may assume that there exists at most two generators \( \sqrt{a_1}, \sqrt{a_2} \) of \( F \) with \( a_j \not\equiv 1 \pmod{4} \) \((1 \leq j \leq 2) \). Then the ramification index \( e \) of the prime
Kyoung Ho Park, Toru Nakahara and Yasuo Motoda

is at most $2^2$. Since the Galois group $G = Gal(F/\mathbb{Q})$ is 2-elementary, the relative degree $f$ of the prime 2 is at most 2, because the inertia subgroup of $G$ is cyclic. In Lemma 1 let $\ell$ be equal to 2. Then we can deduce $e\ell^{j} \leq 2^{2} \cdot 2^{1} < 2^{r}$ if $f = 1$ and $e\ell^{j} \leq 2^{2} \cdot 2^{2} \leq 2^{f} + e - 1$ if $f = 2$. Thus $F$ is non-monogenic. □

By the proof of Proposition 1, if an octic field $K$ is monogenic, it is sufficient to consider that $K$ contains two quadratic subfields of even discriminant and one of odd discriminant.

The main theorem is based on the following theorem, which is an extension of a result of the case of quartic fields $[M_1, M_2, Wi]$.

**Theorem 1 ([PMN]).** Let $K$ be an octic field $\mathbb{Q}(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1m_1n_1\ell})$ with $d = d_1d_2, m = m_1m_2, n = n_1n_2, mn \equiv 3, dn \equiv 2, d_1m_1n_1\ell \equiv 1, d_2 \equiv 2 \pmod{4}, d_1, m_1, n_1 \geq 1$ and $dmnl$ is square free. Let $D_K$ be the field discriminant of the octic field $K$. Then we have $D_K = 2^{12}(dmnP)^{4}$ and an integral basis of $K$ is:

$$Z_K = \mathbb{Z} \left[ 1, \sqrt{m}, \sqrt{d_1m_1n_1\ell}, \sqrt{dm} + \sqrt{dn}, 1 + \sqrt{d_1m_1n_1\ell}, \sqrt{mn} + \sqrt{d_1m_2n_2\ell}, \sqrt{dm} + \sqrt{dn} + e_1\sqrt{d_2m_2n_1\ell} + e_2\sqrt{d_2m_1n_2\ell} \right],$$

where $e_i = \pm 1 \ (i = 1, 2), e_1 \equiv d_1m_1, e_2 \equiv d_1n_1 \pmod{4}$.

§3. Non-monogenic field

It is known that in the case of $d_1m_1n_1 = 1$ that is, there exist a quartic subfield $L$ and a quadratic $k$ of $K$ with $(D_L, D_k) = 1$, the fields $K$ are non-monogenic except for the cyclotomic field $\mathbb{Q}(\zeta_{24})$ of conductor 24 $[MN]$, where $D_F$ means the discriminant of an algebraic number field $F$ over $\mathbb{Q}$. From now on, we consider the case of $d_1m_1n_1 \geq 1$ and as an application of Theorem 1, we can slightly generalize Proposition 5 in $[MN]$, whose proof was done using the relative different with respect to $K$ over a suitable quadratic subfield. We assume that $K$ is monogenic.

Let

$$\xi = b_1\sqrt{mn} + b_2\sqrt{dn} + b_3\sqrt{dm} + b_4\sqrt{dn} + b_5\sqrt{mn} + \sqrt{d_1m_2n_2\ell}$$

$$+ b_6\sqrt{dm} + \sqrt{d_2m_1n_2\ell} + b_7\sqrt{dm} + \sqrt{dn} + e_1\sqrt{d_2m_2n_1\ell} + e_2\sqrt{d_2m_1n_2\ell}$$

be a generator of a power integral basis of $Z_K$. Now we calculate a factor $(\xi - \xi^{\sigma})(\xi - \xi^{\sigma})^{\rho}$.
of the discriminant $d_{K/Q}(\xi) = \Delta^2 \left[ 1, \xi, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6, \xi^7 \right]$ of a number $\xi$;

$$(\xi - \xi^\sigma)(\xi - \xi^\rho)^p$$

$$= \left\{ (2b_2 + b_3 + b_6 + \frac{b_7}{2}) \sqrt{dn} + (b_3 + \frac{b_7}{2}) \sqrt{dm} + (b_6 + \frac{b_7\sigma}{2}) \sqrt{d_{2}m_{1}n_{2}\ell} + \frac{b_7e_1 \sqrt{d_{2}m_{2}n_{1}\ell}}{2} \right\}$$

$$\times \left\{ (2b_2 + b_3 + b_6 + \frac{b_7}{2}) \sqrt{dn} + (b_3 + \frac{b_7}{2}) \sqrt{dm} - (b_6 + \frac{b_7\rho}{2}) \sqrt{d_{2}m_{1}n_{2}\ell} - \frac{b_7e_1 \sqrt{d_{2}m_{2}n_{1}\ell}}{2} \right\}$$

$$= \left\{ (2b_2 + b_3 + b_6 + \frac{b_7}{2}) \sqrt{dn} + (b_3 + \frac{b_7}{2}) \sqrt{dm} \right\}^2 - \left\{ (b_6 + \frac{b_7\rho}{2}) \sqrt{d_{2}m_{1}n_{2}\ell} + \frac{b_7e_1 \sqrt{d_{2}m_{2}n_{1}\ell}}{2} \right\}^2$$

$$= \left\{ (2b_2 + b_3 + b_6)^2 + (2b_2b_7 + b_3b_7 + b_6b_7) + \frac{b_7^2}{4} \right\} dn\ell - (b_3^2 + b_3b_7 + \frac{b_7^2}{4}) dm$$

$$- (b_6^2 + b_6b_7 + \frac{b_7^2}{4}) d_{2}m_{1}n_{2}\ell - \frac{b_7^2m_{2}n_{1}\ell}{4}$$

$$\equiv \left\{ d_{1}(m+n) - (d_{1}n + 4k + d_{1}m + 4k) \right\} \equiv 0 \pmod{2},$$

by $d_{1}m_{1}n_{1}\ell \equiv 1 + 4k \pmod{8}$ and $m + n \equiv 0 \pmod{4}$, since $m_{1}n_{2}\ell \cdot 1 \equiv d_{1}m_{1}^2n_{1}n_{2}\ell^2 + 4m_{1}n_{2}\ell k \equiv d_{1}n + 4k \pmod{8}$ and $m_{2}n_{1}\ell \cdot 1 \equiv d_{1}m_{1}m_{2}n_{1}^2\ell^2 + 4m_{2}n_{1}k \equiv d_{1}m + 4k \pmod{8}.$

$$C/d_{2} \equiv (b_6b_7 + \frac{b_7^2}{2}) d_{1} - (b_6b_7\sigma + \frac{b_7^2e_1\ell}{2})$$

$$\equiv b_6b_7(d_{1} - e_{1}\ell) + \frac{b_7^2}{2} (d_{1} - e_{2}e_1\ell) \equiv 0 \pmod{2},$$

by $e_{1} \equiv d_{1}m_{1}, e_{2} \equiv d_{1}n_{1} \pmod{4}$, since $d_{1} - e_{2}e_1\ell \equiv d_{1} - d_{1}^2m_{1}n_{1}\ell \equiv d_{1}(1 - d_{1}m_{1}\ell) \equiv 0 \pmod{4}.$ So we can write $\eta_{11} = (\xi - \xi^\sigma)(\xi - \xi^\rho)^p = 2d_{2}E_{1}$ for an integer $E_{1} = B_{1} + C_{1}\sqrt{mn}$ in $k_{1} = \mathbb{Q}(\sqrt{mn})$. By the same computation, we obtain $\eta_{12} = (\xi - \xi^\rho)(\xi - \xi^\rho)^p = \ell E_{2}$, $\eta_{13} = (\xi - \xi^\rho)(\xi - \xi^\rho)^p = d_{1}E_{3}$ for units $E_{j}$ in $k_{1}(j = 2, 3).$ By the assumption that $Z_{K}$ is generated by $\xi$, we have

$$d_{K/Q}(\xi) = \pm N_{K}(\varpi(\xi)) = \pm D_{K},$$
where $d(\alpha), N_K(\alpha)$ and $N_K(a)$ means the different of a number, norm of $\alpha$ and an ideal $a$ with respect to $K/Q$, respectively [Wa]. Then, because $\eta_{1j}$ is a partial factor of $d_{K/Q}(\xi)$, the integers $E_j$ should be units in $k_1 = Q(\sqrt{mn})$. Here the following is our basic identity:

$$(\xi - \xi^\sigma)(\xi - \xi^\sigma)^\rho - (\xi - \xi^\rho)(\xi - \xi^\rho)^\sigma - (\xi - \xi^{\sigma\rho})(\xi - \xi^{\sigma\rho})^\rho = 0$$

for $(\xi - \xi^\sigma)(\xi - \xi^\sigma)^\rho = \eta_{11}, (\xi - \xi^\rho)(\xi - \xi^\rho)^\sigma = \eta_{12}$ and $(\xi - \xi^{\sigma\rho})(\xi - \xi^{\sigma\rho})^\rho = \eta_{13}$. Then we have the equation

$$2d_2E_1 - \ell E_2 - d_1E_3 = 0 \quad \text{in} \quad k_1 = Q(\sqrt{D_1}), \quad D_1 = m_1 \cdot 2m_2 \cdot n_1 \cdot 2n_2,$$

where $E_1, E_2$ and $E_3$ are units in $k_1$.

In the same way, we obtain seven equations corresponding to each of the seven quadratic subfields $k_i$ of $K$.

**Proposition 2.** If $K = Q(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1m_1n_1P})$ is monogenic, then the following simultaneous equations hold:

1. $\ell E_{11} + 2d_2E_{12} + d_1E_{13} = 0 \quad \text{in} \quad k_1 = Q(\sqrt{D_1}), \quad D_1 = m_1 \cdot 2m_2 \cdot n_1 \cdot 2n_2,$
2. $\ell E_{21} + 2m_2E_{22} + m_1E_{23} = 0 \quad \text{in} \quad k_2 = Q(\sqrt{D_2}), \quad D_2 = d_1 \cdot 2d_2 \cdot n_1 \cdot 2n_2,$
3. $\ell E_{31} + 2n_2E_{32} + n_1E_{33} = 0 \quad \text{in} \quad k_3 = Q(\sqrt{D_3}), \quad D_3 = d_1 \cdot 2d_2 \cdot m_1 \cdot 2m_2,$
4. $2d_2E_{41} + 2m_2E_{42} + 2n_2E_{43} = 0 \quad \text{in} \quad k_4 = Q(\sqrt{D_4}), \quad D_4 = d_1 \cdot m_1 \cdot n_1 \cdot \ell,$
5. $2d_2E_{51} + m_1E_{52} + n_1E_{53} = 0 \quad \text{in} \quad k_5 = Q(\sqrt{D_5}), \quad D_5 = d_1 \cdot 2m_2 \cdot 2n_2 \cdot \ell,$
6. $d_1E_{61} + 2m_2E_{62} + n_1E_{63} = 0 \quad \text{in} \quad k_6 = Q(\sqrt{D_6}), \quad D_6 = 2d_2 \cdot m_1 \cdot 2n_2 \cdot \ell,$
7. $d_1E_{71} + m_1E_{72} + 2n_2E_{73} = 0 \quad \text{in} \quad k_7 = Q(\sqrt{D_7}), \quad D_7 = 2d_2 \cdot 2m_2 \cdot n_1 \cdot \ell,$

where each $E_{ij}$ is a unit in the corresponding quadratic subfield $k_i$ of $K$ and each $D_i$ the field discriminant of $k_i$, respectively.

For the case of a real quadratic field, the following lemma holds:

**Lemma 2.** Let $E_j$ be a power $\epsilon_{0^j} = \frac{u_j + v_j\sqrt{D}}{2}$ of the fundamental unit $\epsilon_0 = \frac{u + v\sqrt{D}}{2} > 1$ in a real quadratic field $Q(\sqrt{D})$ with the field discriminant $D$ and $\overline{\alpha} = \alpha^\gamma$ for $\alpha$ in $Q(\sqrt{D})$ and $\gamma(\neq I)$ in $Gal(Q(\sqrt{D})/Q)$. Let

$$\begin{cases}
    a + bE_j + cE_k = 0, \\
    a + b\overline{E_j} + c\overline{E_k} = 0
\end{cases} \quad (*)$$

for $abc \neq 0$. Denote the matrix

$$
\begin{pmatrix}
    1 & E_j & E_k \\
    1 & \overline{E_j} & \overline{E_k}
\end{pmatrix}
$$
On integral bases of real octic 2-elementary abelian extensions

attached to the the equation (*) by $A$ and the rank of $A$ by $r_D$. Then we have a solution $(a, b, c)$ of rational integers:

\[
\begin{align*}
    a \pm b \pm c &= 0 & \text{for } & r_D = 1, \\
    \frac{a}{u_kv_j - u_jv_k} &= \frac{b}{2v_k} = \frac{c}{-2v_j} & \text{for } & r_D = 2
\end{align*}
\]

with $E_i = \frac{u_i + v_i\sqrt{D}}{2}$.

**Proof.** This lemma means that the integral solutions should be on the plane for the rank $r_D = 1$ of the coefficient matrix $A$ and on the line i.e. the intersection of two planes for $r_D = 2$, respectively.

First, we consider the case of $r_D = 1$, then for

\[
\begin{align*}
    E_i &= \frac{u_i + u_i\sqrt{D}}{2}, \\
    E_i &= \frac{u_i - u_i\sqrt{D}}{2},
\end{align*}
\]

$E_i, \overline{E}_i$ should be a rational number. Then we have $E_j = u_j = \pm 1$ and $E_k = u_k = \pm 1$. Hence $a \pm b \pm c = 0$. Second, we assume $r_D = 2$. Then we have

\[
a : b : c = \left| \begin{array}{ccc} E_j & E_k & 1 \\ \overline{E}_j & \overline{E}_k & 1 \end{array} \right| = 1 \\
\text{and} \\
\frac{a}{u_kv_j - u_jv_k} = \frac{b}{2v_k} = \frac{c}{-2v_j}.
\]

Hence

\[
\frac{a}{u_kv_j - u_jv_k} = \frac{b}{2v_k} = \frac{c}{-2v_j}.
\]

In the case of any octic field $Q(\sqrt{m_1m_2n_1n_2}, \sqrt{d_1d_2n_1n_2}, \sqrt{d_1m_1n_1\ell})$, by the following lemma, we can deduce to evaluate the rank $r_D$ of a quadratic field $Q(\sqrt{D})$ for a few cases with respect to the order of values $d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell$ in the set of seven parameters.

**Lemma 3.** Let denote the set $\{d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell\}$ by $D$. Then it holds that:

1. For one parameter $s$ in $D$, there exist only four quadratic subfields $k_j$ whose discriminants $D_j$ are divisible by $s$.
2. For two parameters $s, t$ in $D$, there exist only two quadratic subfields $k_j$ whose discriminants $D_j$ are divisible by $st$.
3. Let $s, t, u$ be three parameters in $D$, such that $stu$ is a divisor of the field discriminant of $D_j$ of $k_j$. Then there exists only one quadratic subfield $k_j$ whose discriminant $D_j$ is divisible by $stu$. 
Proof. (1) We can confirm the claim (1) for each of \( \binom{\#D}{1} = 7 \) parameter in \( D \) from seven equations in Proposition 2, such that there exist just four fields \( k_1, k_3, k_4, k_6 \) whose discriminant is divisible by \( m_1 \).

(2) We can do the claim (2) of \( \binom{\#D}{2} = 21 \) pairs of parameters in \( D \) by the same way as in (1). For instance, there exist just two fields \( k_3, k_7 \) whose discriminants are divisible by \( d_2m_2 \).

(3) We assume that \( D_i = stub \) and \( D_j = stub \). Then we have \( D_iD_j = (stu)^2ab \). However, the quadratic subfield \( Q(\sqrt{ab}) \) does not coincide with any \( k_j (1 \leq j \leq 7) \). \( \square \)

Remark 1. We can confirm that the number of triplets \((s, t, u)\) within the order of parameters in \( D \) is equal to \( 28 = 7 \times 1 \times \binom{4}{3} < \binom{\#D}{3} = 35 \) such that each of \( stu \) is a divisor of the field discriminant \( D_j \) of \( k_j \).

Next, we prepare the key lemma for the proof of Theorem 2.

Lemma 4. For the set \( D = \{a, b, c, d, e, f, g\} \) of seven positive rational integers, assume that \( a > b \geq c \geq \max\{d, e, f, g\} \) and \( d > f \) or \( a > b > c \geq \max\{d, e, f, g\} \) and \( d > f \). Then

(1) For the field \( Q(\sqrt{bcst}) \), where \( s, t \in D \setminus \{a, b, c\} \) and units \( E_i \) in \( Q(\sqrt{bcst}) \), the rank \( r_{bcst} \) of the equations
\[
\begin{align*}
&\left\{ a + uE_j + vE_k = 0, \\
&\quad a + u\overline{E}_j + v\overline{E}_k = 0,
\end{align*}
\]
with \( \{u, v\} = D \setminus \{a, b, c, s, t\} \) is equal to 1.

(2) For the field \( Q(\sqrt{astu}) \), where \( s, t, u \in D \setminus \{a, b, c\} \) and units \( E_i \) in \( Q(\sqrt{astu}) \), the rank \( r_{astu} \) of the equations
\[
\begin{align*}
&\left\{ b + cE_j + vE_k = 0, \\
&\quad b + c\overline{E}_j + v\overline{E}_k = 0,
\end{align*}
\]
with \( \{v\} = D \setminus \{a, b, c, s, t, u\} \) is equal to 1.

Sketch of Idea. Our idea for the proof of this lemma is as follows. For the quadratic subfield \( k \) including the coefficients of the simultaneous equation \((*)\), if the field discriminant \( D_k \) is divisible by the biggest parameter (case (1)) or the second and the third ones (case (2)), since the fundamental unit (> 1) of \( k \) is relatively big, the ratios for the line in Lemma 2 would not be permitted. Thus the ranks of the coefficient matrix for both cases should be equal to one, respectively, namely any integral solution of \((*)\) lies on the plane \([PMN]\). \( \square \)
Finally, we show the following main theorem, which is a generalization of a prototype[PMN].

**Theorem 2.** Let $K = Q(\sqrt{a_1}, \ldots, \sqrt{a_r})$ be the 2-elementary abelian extensions over $Q$ whose degree $2^e$ is greater than 8 or real octic ones for square free integers $a_1, \ldots, a_r$. Then the fields $K$ are non-monogenic.

**Sketch of Proof.** By Proposition 1, it is enough to consider an octic field $K$. Let $(2) = \mathfrak{L}_1 \cdots \mathfrak{L}_g$ be the prime ideal decomposition of a rational prime 2 in $K$. For the ramification index of 2, if $e \leq 1$, then by Lemma 1 and the relative degree $f$ of a prime 2 is at most 2, we have $1 \cdot 2^1 < 8$ or $1 \cdot 2^2 \leq 8 + 1 - 1$ for $e = 1$ and $2 \cdot 2^1 \leq 8$ or $2 \cdot 2^2 \leq 8 + 2 - 1$ for $e = 2$, namely $K$ is non-monogenic. Then in the case of $e \geq 3$, we can deduce that the type of an octic field $K$ is $K = Q(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$, where $a_q = mn \equiv 3, a_2 = dn \equiv 2, a_3 = d_1m_1n_1 \ell \equiv 1 \pmod{4}$, for $d = d_1d_2, m = m_1m_2, n = n_1n_2$ and $dmn\ell$ is square free. Put $D = \{d_1, 2d_2, m_1, 2m_2, n_1, 2m_2, \ell\}$. We denote again by $\{a, b, c, d, e, f, g\}$ any transposition on the seven parameters in $D$. Without loss of generality, we may assume that $a > b > c \geq \max\{d, e, f, g\}$. Using Lemma 4, it is enough for us to consider the following two cases.

Case (I). The field $K$ includes $k_{ji} = Q(\sqrt{abc})$ for some $t \in D \setminus \{a, b, c\}$, for instance, $t = d$.

Case (II). The field $K$ does not include the field $Q(\sqrt{abcs})$ for any $s \in D \setminus \{a, b, c\}$.

In the case (I), we can deduce that the four parameters $a, b, c, d$ with $c \geq d$ must lie on suitable two planes and in the case (II), $a, b, e, g$ with $e > g$ do on four planes, respectively. However, the order of the parameters would be destroyed. Then we can prove that any real octic fields $K$ does not have a power integral basis[PNM].

**Remark 2.** Recently, in [PNM] we proved that all the 2-elementary abelian fields $K$ with degree $[K : Q] \geq 8$ are non-monogenic except for the field $Q(\sqrt{-1}, \sqrt{2}, \sqrt{-3}) = Q(\zeta_{24})$.

**Problem.** For a primitive element $\xi$ in $K$, let $\text{Ind}(\xi), \hat{m}(K)$ and $m(K)$ be the index $\sqrt{\frac{d_K(\xi)}{D_K}}$ of an element $\xi$, the minimum index $\min_{\xi \in K}\text{Ind}(\xi)$ of $K$ and the field index $\min_{\xi \in K}\text{Ind}(\xi)$ of $K$, respectively. Let the fields $K$ run through all the real octic fields whose Galois groups are 2-elementary abelian. Then evaluate the values of

$$\inf_K \hat{m}(K) \text{ and } \inf_K m(K),$$
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