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Geometry of polysymbols

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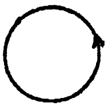
Introduction

We discuss a multiple generalization of the classical tame symbol on a Riemann surface. Let us start to explain a motivation of our work which is coming from the analogies between knots and primes.

knot $K: S^1 = K(\mathbb{Z},1) \hookrightarrow \mathbb{R}^3$	$\longleftrightarrow$	prime $\mathrm{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}},1) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$
	$\longleftrightarrow$	Frobenius
double cover $Y \to X = \mathbb{R}^3 \setminus K$	$\longleftrightarrow$	double étale cover $Y \to X = \mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$
For a 2-component link $K \cup L$, monodromy around L in $\mathrm{Gal}(Y/X)$ $=$ linking number $\mathrm{lk}(K,L)$ mod 2	$\longleftrightarrow$	For 2 odd primes p, q, Frobenius over q in $\mathrm{Gal}(Y/X)$ $=$ Legendre symbol $\left(\frac{p^*}{q}\right)$

where we set $p^* := (-1)^{\frac{p-1}{2}}p$.

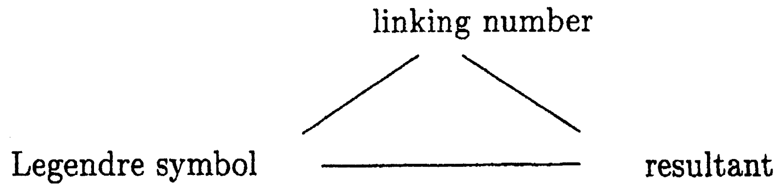
Now, let us consider the polynomial ring $\mathbb{F}_l[X]$ (l being an odd prime) in place of $\mathbb{Z}$. For monic irreducible polynomials $f, g \in \mathbb{F}_l[X]$ s.t. $(f,g) = 1$, we have:

$$\begin{aligned}\left(\frac{f}{g}\right) = 1 \;&\Leftrightarrow\; \exists h \in \mathbb{F}_l[X] \text{ s.t. } h^2 \equiv f \bmod g \\ &\Leftrightarrow\; f \bmod g \in (\mathbb{F}_l[X]/(g))^2 \\ &\Leftrightarrow\; \prod_{g(\beta)=0} f(\beta) \in (\mathbb{F}_l^{\times})^2.\end{aligned}$$

where $R(f,g) := \prod_{g(\beta)=0} f(\beta) = \prod_{f(\alpha)=g(\beta)=0} (\beta - \alpha)$ is the resultant of f and g. Hence we have

$$\left(\frac{f}{g}\right) = R(f,g)^{\frac{l-1}{2}}.$$

Note that the resultant $R(f, g)$ can be defined for any ground field k and $f, g \in k[X]$. Summing up, we have the following analogies:



The symbol in the title means the tame symbol over $\mathbb{C}$ defined by

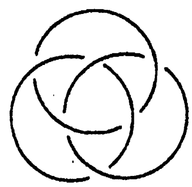
$$\{f, g\}_x := (-1)^{\mathrm{ord}_x(f)\mathrm{ord}_x(g)} \frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}}$$

for $f, g \in \mathbb{C}(X)$ and $x \in \mathbb{C} \cup \{\infty\}$. The connection with the resultant is given by

$$R(f, g) = \prod_{g(\beta)=0} \{f, g\}_\beta$$

for $f, g \in \mathbb{C}[X]$ and hence the tame symbol plays a local and more basic role like the Hilbert symbol.

Now, it is known in knot theory that there is a higher order generalization of the linking number for a link. For example, the Borromean ring $K_1 \cup K_2 \cup K_3$ has the linking numbers $\mathrm{lk}(K_i, K_j) = 0$ for all $i \neq j$ and the triple linking number $\mathrm{lk}(K_1, K_2, K_3) = \pm 1$:



There are known two ways to construct the higher linking numbers; 1) Massey's higher order cup products ([Ma]) 2) Milnor's invariants defined as unipotent monodromies ([Mi]). For $\mathbb{Z}$, both constructions are known ([Mo]). The following question was asked by M. Kapranov:

Question. Is there Massey-Milnor type construction for a multiple $\{f_1, \ldots, f_n\}_x$ (or $R(f_1, \ldots, f_n)$) under a certain condition ?

Our result is to give the Massey type construction of a line bundle with holomorphic flat connection $\langle f_1, \dots, f_n\rangle$, called a polysymbol, on a Riemann surface so that $\{f_1, \dots, f_n\}_x$ is obtained as its holonomy along a loop encircling x, and also to give a global geometric construction of $\langle f_1, \dots, f_n\rangle$.

In the following, we shall discuss only triple symbols, i.e, the case $n = 3$, though we have similar results for any n ([MT]). Let us fix the notations:
$\overline{X} :=$ a closed Riemann surface,
$f_1, f_2, f_3 \in \mathbb{C}(\overline{X})$,
$X := \overline{X} \setminus \cup_{i=1}^{3}\mathrm{Supp}(f_i)$.

1. Massey product construction

For $p \in \mathbb{N}$, the Deligne complex is defined by

$$\mathbb{Z}(p)_D := ((2\pi\sqrt{-1})^p\mathbb{Z} \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_X)$$

which is quasi-isomorphic to

$$(\mathcal{O}^\times_X \xrightarrow{d\log} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_X)[-1].$$

The Deligne cohomology is then defined by the hypercohomology groups

$$\mathbb{H}^q(X, \mathbb{Z}(p)_D) = \mathbb{H}^{q-1}(X, \mathcal{O}^\times_X \xrightarrow{d\log} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_X), \quad q \geq 1.$$

We compute Deligne cohomology groups in terms of Čech cohomology and so we take an open cover $\mathcal{U} = \cup_a U_a$ of X. (For the properties of the Deligne cohomology, we refer to [Br], [EV]).
(1) For $p = q = 1$, we have

$$\begin{array}{ccc} H^0(X, \mathcal{O}^\times_X) & \simeq & \mathbb{H}^1(X, \mathbb{Z}(1)_D) \\ f & \leftrightarrow & [(2\pi\sqrt{-1}n_{ab}, (\log f)_a)] \end{array}$$

where $(\log f)_a$ means a branch of $\log f$ on U_a with $(\log f)_b - (\log f)_a = 2\pi\sqrt{-1}n_{ab}$ on $U_{ab} = U_a \cap U_b$.
Let $\mathrm{Pic}^\nabla(X)$ denote the group of isomorphism classes of line bundles with holomorphic (flat) connection on X.

(2) For $p = q = 2$, we have

$$\begin{array}{ccccc} \mathbb{H}^2(X,\mathbb{Z}(2)_D) & \simeq & \mathbb{H}^1(X,\mathcal{O}_X^\times \overset{\mathrm{dlog}}{\to} \Omega_X{}^1) & \simeq & \mathrm{Pic}^\nabla(X) \\ [((2\pi\sqrt{-1})^2 n_{abc}, (\log f)_{ab}, \Omega_a)] & \leftrightarrow & [(\exp \frac{1}{2\pi\sqrt{-1}}(\log f)_{ab}, \frac{1}{2\pi\sqrt{-1}}\Omega_a)] & & \\ & & [(\xi_{ab}, \omega_a)] & \leftrightarrow & [(L,\nabla)] \end{array}$$

where ξ_{ab} gives the transition function of a $\mathbb{C}$-line bundle L on U_{ab} and local 1-forms ω_a's define a connection ∇ on L.
The Deligne complexes have the product $\mathbb{Z}(p)_D \times \mathbb{Z}(q)_D \to \mathbb{Z}(p+q)_D$ which induces the cup product

$$\mathbb{H}^q(X,\mathbb{Z}(p)_D) \times \mathbb{H}^{q'}(X,\mathbb{Z}(q)) \overset{\cup}{\to} \mathbb{H}^{q+q'}(X,\mathbb{Z}(p+p')_D).$$

By (1) and (2), $f_1 \cup f_2$ determines the isomorphism class of line bundles with holomorphic connection on X, which we denote by $\langle f_1, f_2\rangle$. Deligne ([D]) interpreted the tame symbol $\{f_1, f_2\}_x$ as the holonomy of $\langle f_1, f_2\rangle$ along a loop l based at x_0 and encircling x:

$$\{f_1, f_2\}_x = \exp \frac{1}{2\pi\sqrt{-1}} \left(\int_l \log f_1 \frac{df_2}{f_2} - \log f_2(x_0) \int_l \frac{df_1}{f_1} \right).$$

Now, assume that $f_1 \cup f_2 = f_2 \cup f_3 = 0$ in the following so that

$$\begin{array}{llll} & \alpha_1 \cup \alpha_2 = \partial\alpha_{12}, & \alpha_2 \cup \alpha_3 = \partial\alpha_{23}, & \exists \alpha_{ij} \in C^1(\mathcal{U}, \mathbb{Z}(2)_D) \\ \Leftrightarrow & (\log f_1)_a \frac{df_2}{f_2} = \frac{df_{12}}{f_{12}}, & (\log f_2)_a \frac{df_3}{f_3} = \frac{df_{23}}{f_{23}}, & \exists f_{ij} \in H^0(X, \mathcal{O}_X^\times) \\ & +\text{some equations} & & \end{array}$$

We fix branches $(\log f_{ij})_a$'s on U_a's so that we have a unique cohomology class of the Massey product $[\alpha_1 \cup \alpha_{23} + \alpha_{12} \cup \alpha_3] \in \mathbb{H}^2(X, \mathbb{Z}(3)_D)$. We then define $\langle f_1, f_2, f_3\rangle$ as the corresponding isomorphism class of line bundles with holomorphic connection under the isomorphisms:

$$\mathbb{H}^2(X,\mathbb{Z}(3)_D) \simeq \mathbb{H}^2(X,\mathbb{Z}(2)_D) \simeq \mathrm{Pic}^\nabla(X).$$

Let $x_0 \in X$ be a base point and put $a_i := \frac{1}{2\pi\sqrt{-1}} \log f_i(x_0)$, $a_{ij} := \frac{1}{(2\pi\sqrt{-1})^2} \log f_{ij}(x_0)$ and $\omega_i := \frac{1}{2\pi\sqrt{-1}} \frac{df_i}{f_i}$.

Theorem 1. For $[l] \in \pi_1(X, x_0)$, the holonomy of $\langle f_1, f_2, f_3\rangle$ along l is

$$\exp(2\pi\sqrt{-1} m_{123}(l)),$$

where $m_{123}(l)$ is given by

$$m_{123}(l) = \int_l \omega_1\omega_2\omega_3 + a_1 \int_l \omega_2\omega_3 + a_{12}\int_l \omega_3 - \int_l \omega_1 a_{23} - \int_l \omega_1\omega_2 a_3 - a_1\int_l \omega_2 a_3 + \int_l \omega_1 a_2 a_3.$$

Here $\int_l \omega_{i_1}\cdots\omega_{i_k}$ denotes the iterated integral ([C])

$$\int_{0\le t_1<\cdots<t_k\le 1} F_1(t_1)\cdots F_k(t_k)dt_1\cdots dt_k,\ l^*(\omega_{i_j}) = F_j(t_j)dt_j.$$

For $x \in \overline{X}$, we define $\{f_1, f_2, f_3\}_x$ by the holonomy of $\langle f_1, f_2, f_3\rangle$ along a loop encircling x.

Theorem 2. (Reciprocity) $\prod_{x\in\overline{X}}\{f_1, f_2, f_3\}_x = 1$.

2. Global geometric construction

We give a global geometric construction of $\langle f_1, f_2, f_3\rangle$, which generalizes those by S. Bloch ([Bl]) and R. Hain ([H]) for $\langle f_1, f_2\rangle$.
We set:

$$N_4(R) := \{\begin{pmatrix}1 & x_1 & x_{12} & x_{123}\\ 0 & 1 & x_2 & x_{23}\\ 0 & 0 & 1 & x_3\\ 0 & 0 & 0 & 1\end{pmatrix} \mid x_* \in R\}\ (R: \text{a ring}),$$

$$C_4 := \{\begin{pmatrix}1 & 0 & 0 & z\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \mid z \in \mathbb{C}\},$$

$$P := N_4(\mathbb{Z})\backslash N_4(\mathbb{C}),\ \ B := N_4(\mathbb{Z})\backslash N_4(\mathbb{C})/C_4.$$

The projection $P \to B$ is a principal $\mathbb{C}^\times$-bundle. The 1-form

$$\begin{aligned}\theta\ &:= dx_{123} - x_{12}dx_3 - x_1 d_{23} + x_1 x_2 dx_3\\ &= (1,4)\text{-entry of } x^{-1}dx\end{aligned}$$

is left $N_4(\mathbb{Z})$-invariant, right $\mathbb{C}^\times$-invariant and a Maurer-Cartan form along fibers so that it boils down to a connection form on P. Fixing a base point $x_0 \in X$ and $N_4(\mathbb{Z})AC_4 \in B$ where

$$A := \begin{pmatrix} 1 & a_1 & a_{12} & a_{123} \\ 0 & 1 & a_2 & a_{23} \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we define a holomorphic map

$$T(f_1, f_2, f_3) : X \to B$$

by

$$T(f_1, f_2, f_3) := N_4(\mathbb{Z})A \begin{pmatrix} 1 & \int_{\gamma_x} \omega_1 & \int_{\gamma_x} \omega_1\omega_2 & \int_{\gamma_x} \omega_1\omega_2\omega_3 \\ 0 & 1 & \int_{\gamma_x} \omega_2 & \int_{\gamma_x} \omega_2\omega_3 \\ 0 & 0 & 1 & \int_{\gamma_x} \omega_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} C_4$$

where γ_x is a path from x_0 to x. The map $T(f_1, f_2, f_3)$ is shown to be independent of the choice of γ_x. By computing the holonomy of $T(f_1, f_2, f_3)^*(P, \theta)$, we have

Theorem 3. $\langle f_1, f_2, f_3 \rangle =$ the isomorphism class of $T(f_1, f_2, f_3)^*(P, \theta)$.

Remark. We expect that $\langle f_1, f_2, f_3 \rangle$ would be an obstruction to a variation of the mixed Hodge structure V on X so that the weight filtration $V \supset W_3 \supset W_2 \supset W_1 \supset 0$ satisfies the properties:
(i) $V/W_3 = \mathbb{Z}, W_3/W_2 = \mathbb{Z}(1), W_2/W_1 = \mathbb{Z}(2), W_1 = \mathbb{Z}(3)$,
(ii) V/W_1 and W_3 are classified respectively by

$$T(f_1, f_2) : X \longrightarrow N_3(\mathbb{Z})\backslash N_3(\mathbb{C})/C_3$$
$$T(f_2, f_3) : X \longrightarrow N_3(\mathbb{Z})\backslash N_3(\mathbb{C})/C_3,$$

where $T(f_i, f_{i+1})$ $(i = 1, 2)$ is defined in a similar manner to $T(f_1, f_2, f_3)$.

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