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Geometry of polysymbols

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Introduction

We discuss a multiple generalization of the classical tame symbol on a Riemann surface. Let us start to explain a motivation of our work which is coming from the analogies between knots and primes.

\[
\text{knot } K : S^1 = K(\mathbb{Z}, 1) \hookrightarrow \mathbb{R}^3 \quad \longleftrightarrow \quad \text{prime } \text{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \text{Spec}(\mathbb{Z})
\]

\[
\text{double cover } Y \rightarrow X = \mathbb{R}^3 \setminus K \quad \longleftrightarrow \quad \text{double \'{e}tale cover } Y \rightarrow X = \text{Spec}(\mathbb{Z}[\frac{1}{p}])
\]

For a 2-component link \(K \cup L\), monodromy around \(L\) in \(\text{Gal}(Y/X)\) \(\longleftrightarrow\) Frobenius over \(q\) in \(\text{Gal}(Y/X)\)

\[= \text{linking number } \text{lk}(K, L) \mod 2 \quad \text{=} \quad \text{Legendre symbol } \left(\frac{p}{q}\right)\]

where we set \(p^* := (-1)^{\frac{p-1}{2}}p\).

Now, let us consider the polynomial ring \(\mathbb{F}_l[X]\) (\(l\) being an odd prime) in place of \(\mathbb{Z}\). For monic irreducible polynomials \(f, g \in \mathbb{F}_l[X]\) s.t. \((f, g) = 1\), we have:

\[
\left(\frac{f}{g}\right) = 1 \iff \exists h \in \mathbb{F}_l[X] \text{ s.t. } h^2 \equiv f \mod g
\]

\[\iff f \mod g \equiv (\mathbb{F}_l[X]/(g))^2 \]

\[\iff \prod_{g(\beta) = 0} f(\beta) = (\mathbb{F}_l^x)^2.\]

where \(R(f, g) := \prod_{g(\beta) = 0} f(\beta) = \prod_{f(\alpha) = g(\beta) = 0} (\beta - \alpha)\) is the resultant of \(f\) and \(g\). Hence we have

\[
\left(\frac{f}{g}\right) = R(f, g)^{\frac{l-1}{2}}.
\]
Note that the resultant $R(f, g)$ can be defined for any ground field $k$ and $f, g \in k[X]$.
Summing up, we have the following analogies:

\[
\begin{array}{c}
\text{linking number} \\
\text{Legendre symbol} \\
\text{resultant}
\end{array}
\]

The symbol in the title means the tame symbol over $\mathbb{C}$ defined by

\[
\{f, g\}_x := (-1)^{\text{ord}_x(f)\text{ord}_x(g)} \frac{f^{\text{ord}_x(g)}}{g^{\text{ord}_x(f)}}
\]

for $f, g \in \mathbb{C}(X)$ and $x \in \mathbb{C} \cup \{\infty\}$. The connection with the resultant is given by

\[
R(f, g) = \prod_{g(\beta) = 0} \{f, g\}_\beta
\]

for $f, g \in \mathbb{C}[X]$ and hence the tame symbol plays a local and more basic role like the Hilbert symbol.

Now, it is known in knot theory that there is a higher order generalization of the linking number for a link. For example, the Borromean ring $K_1 \cup K_2 \cup K_3$ has the linking numbers $\text{lk}(K_i, K_j) = 0$ for all $i \neq j$ and the triple linking number $\text{lk}(K_1, K_2, K_3) = \pm 1$:

There are known two ways to construct the higher linking numbers; 1) Massey’s higher order cup products ([Ma]) 2) Milnor’s invariants defined as unipotent monodromies ([Mi]). For $\mathbb{Z}$, both constructions are known ([Mo]). The following question was asked by M. Kapranov:

Question. Is there Massey-Milnor type construction for a multiple $\{f_1, \ldots, f_n\}_x$ (or $R(f_1, \ldots, f_n)$) under a certain condition?
Our result is to give the Massey type construction of a line bundle with holomorphic flat connection \( \langle f_1, \ldots, f_n \rangle \), called a polysymbol, on a Riemann surface so that \( \{f_1, \ldots, f_n\}_x \) is obtained as its holonomy along a loop encircling \( x \), and also to give a global geometric construction of \( \langle f_1, \ldots, f_n \rangle \).

In the following, we shall discuss only triple symbols, i.e., the case \( n = 3 \), though we have similar results for any \( n \) ([MT]). Let us fix the notations:
\( \overline{X} := \) a closed Riemann surface,
\( f_1, f_2, f_3 \in \mathbb{C}(\overline{X}) \),
\( X := \overline{X} \setminus \bigcup_{i=1}^{3} \text{Supp}(f_i) \).

1. Massey product construction

For \( p \in \mathbb{N} \), the Deligne complex is defined by
\[
Z(p)_D := ((2\pi\sqrt{-1})^p \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow^{d} \Omega_X^1 \rightarrow^{d} \cdots \rightarrow^{d} \Omega_X^{p-1})
\]
which is quasi-isomorphic to
\[
(\mathcal{O}_X^\times \rightarrow^{d\log} \Omega_X^1 \rightarrow^{d} \cdots \rightarrow^{d} \Omega_X^{p-1})[-1].
\]
The Deligne cohomology is then defined by the hypercohomology groups
\[
\mathbb{H}^q(X, Z(p)_D) = \mathbb{H}^{q-1}(X, \mathcal{O}_X^\times \rightarrow^{d\log} \Omega_X^1 \rightarrow^{d} \cdots \rightarrow^{d} \Omega_X^{p-1}), \quad q \geq 1.
\]
We compute Deligne cohomology groups in terms of Čech cohomology and so we take an open cover \( U = \cup_a U_a \) of \( X \). (For the properties of the Deligne cohomology, we refer to [Br], [EV]).
(1) For \( p = q = 1 \), we have
\[
H^0(X, \mathcal{O}_X^\times) \cong \mathbb{H}^1(X, Z(1)_D) \quad \leftrightarrow \quad [(2\pi\sqrt{-1}n_{ab}, (\log f)_a)]
\]
where \( (\log f)_a \) means a branch of \( \log f \) on \( U_a \) with \( (\log f)_b - (\log f)_a = 2\pi\sqrt{-1}n_{ab} \) on \( U_{ab} = U_a \cap U_b \).
Let \( \text{Pic}^\nabla(X) \) denote the group of isomorphism classes of line bundles with holomorphic (flat) connection on \( X \).
(2) For $p = q = 2$, we have
\[
\mathbb{H}^2(X, \mathbb{Z}(2)_D) \cong \mathbb{H}^1(X, \mathcal{O}_X^\times \log) \cong \text{Pic}^\nabla(X)
\]
\[
[(2\pi\sqrt{-1})^2 n_{abc}, (\log f)_{ab}, \Omega_a) \leftrightarrow [(\exp \frac{1}{2\pi\sqrt{-1}} (\log f)_{ab}, \frac{1}{2\pi\sqrt{-1}} \Omega_a)] \leftrightarrow [(L, \nabla)]
\]
where $\xi_{ab}$ gives the transition function of a $\mathbb{C}$-line bundle $L$ on $U_{ab}$ and local 1-forms $\omega_a$'s define a connection $\nabla$ on $L$.

The Deligne complexes have the product $\mathbb{H}^q(X, \mathbb{Z}(p)_D) \otimes \mathbb{H}^q(X, \mathbb{Z}(q)_D) \rightarrow \mathbb{H}^{q+q'}(X, \mathbb{Z}(p+q')_D)$.

By (1) and (2), $f_1 \cup f_2$ determines the isomorphism class of line bundles with holomorphic connection on $X$, which we denote by $\langle f_1, f_2 \rangle$. Deligne ([D]) interpreted the tame symbol $\{f_1, f_2\}_x$ as the holonomy of $\langle f_1, f_2 \rangle$ along a loop $l$ based at $x_0$ and encircling $x$:
\[
\{f_1, f_2\}_x = \exp \frac{1}{2\pi\sqrt{-1}} \left( \int_{l} (\log f_1)_{a_1} f_2 - \log f_2(x_0) \int_{l} \frac{df_1}{f_1} \right).
\]

Now, assume that $f_1 \cup f_2 = f_2 \cup f_3 = 0$ in the following so that
\[
\alpha_1 \cup \alpha_2 = \partial \alpha_{12}, \quad \alpha_2 \cup \alpha_3 = \partial \alpha_{23}, \quad \exists \alpha_{ij} \in C^1(U, \mathbb{Z}(2)_D)
\]
\[
\leftrightarrow (\log f_1)_a f_2 = f_2, \quad (\log f_2)_a f_3 = f_3, \quad \exists f_{ij} \in H^0(X, \mathcal{O}_X^\times)
\]
\[
+ \text{some equations}
\]

We fix branches $(\log f_{ij})_a$'s on $U_a$'s so that we have a unique cohomology class of the Massey product $[\alpha_1 \cup \alpha_{23} + \alpha_{12} \cup \alpha_3] \in \mathbb{H}^2(X, \mathbb{Z}(3)_D)$. We then define $\langle f_1, f_2, f_3 \rangle$ as the corresponding isomorphism class of line bundles with holomorphic connection under the isomorphisms:
\[
\mathbb{H}^2(X, \mathbb{Z}(3)_D) \cong \mathbb{H}^2(X, \mathbb{Z}(2)_D) \cong \text{Pic}^\nabla(X).
\]

Let $x_0 \in X$ be a base point and put $a_i := \frac{1}{2\pi\sqrt{-1}} \log f_i(x_0)$, $a_{ij} := \frac{1}{(2\pi\sqrt{-1})^2} \log f_{ij}(x_0)$ and $\omega_i := \frac{1}{2\pi\sqrt{-1}} \frac{df_i}{f_i}$.

**Theorem 1.** For $[l] \in \pi_1(X, x_0)$, the holonomy of $\langle f_1, f_2, f_3 \rangle$ along $l$ is
\[
\exp(2\pi\sqrt{-1} m_{123}(l)),
\]
where $m_{123}(l)$ is given by

$$m_{123}(l) = \int_{l} \omega_1 \omega_2 \omega_3 + a_1 \int \omega_2 \omega_3 - \int \omega_1 a_{23} - a_1 \int \omega_2 a_3 + \int \omega_1 a_{23}.$$ 

Here $\int \omega_i \cdots \omega_i$ denotes the iterated integral ([C])

$$\int_{0 \leq t_1 < \cdots < t_k \leq 1} F_1(t_1) \cdots F_k(t_k) dt_1 \cdots dt_k,$$

For $x \in \overline{X}$, we define $\{f_1, f_2, f_3\}_x$ by the holonomy of $\langle f_1, f_2, f_3 \rangle$ along a loop encircling $x$.

**Theorem 2.** (Reciprocity) $\prod_{x \in \overline{X}} \{f_1, f_2, f_3\}_x = 1$.

### 2. Global geometric construction

We give a global geometric construction of $\langle f_1, f_2, f_3 \rangle$, which generalizes those by S. Bloch ([Bl]) and R. Hain ([H]) for $\langle f_1, f_2 \rangle$.

We set:

$$N_4(R) := \left\{ \begin{pmatrix} 1 & x_1 & x_{12} & x_{123} \\ 0 & 1 & x_2 & x_{23} \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_* \in R \right\} \quad (R: \text{a ring}),$$

$$C_4 := \left\{ \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\},$$

$$P := \frac{N_4(\mathbb{Z}) \setminus N_4(\mathbb{C})}{C_4}, \quad B := \frac{N_4(\mathbb{Z}) \setminus N_4(\mathbb{C})}{C_4}.$$

The projection $P \rightarrow B$ is a principal $\mathbb{C}^x$-bundle. The 1-form

$$\theta := dx_{123} - x_{12}dx_3 - x_1d_{23} + x_1x_2dx_3$$

is the $(1,4)$-entry of $x^{-1}dx$. 

is left $N_4(\mathbb{Z})$-invariant, right $\mathbb{C}^\times$-invariant and a Maurer-Cartan form along fibers so that it boils down to a connection form on $P$. Fixing a base point $x_0 \in X$ and $N_4(\mathbb{Z})A C_4 \in B$ where

$$A := \begin{pmatrix} 1 & a_1 & a_{12} \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

we define a holomorphic map

$$T(f_1, f_2, f_3) : X \to B$$

by

$$T(f_1, f_2, f_3) := N_4(\mathbb{Z})A \begin{pmatrix} 1 & \int_{\gamma_x} \omega_1 & \int_{\gamma_x} \omega_1 \omega_2 \\ \int_{\gamma_x} \omega_2 & 1 & \int_{\gamma_x} \omega_2 \omega_3 \\ \int_{\gamma_x} \omega_3 & \int_{\gamma_x} \omega_3 \omega_2 & 1 \end{pmatrix} C_4$$

where $\gamma_x$ is a path from $x_0$ to $x$. The map $T(f_1, f_2, f_3)$ is shown to be independent of the choice of $\gamma_x$. By computing the holonomy of $T(f_1, f_2, f_3)^\ast(P, \theta)$, we have

**Theorem 3.** $\langle f_1, f_2, f_3 \rangle = \text{the isomorphism class of } T(f_1, f_2, f_3)^\ast(P, \theta)$.

**Remark.** We expect that $\langle f_1, f_2, f_3 \rangle$ would be an obstruction to a variation of the mixed Hodge structure $V$ on $X$ so that the weight filtration $V \supset W_3 \supset W_2 \supset W_1 \supset 0$ satisfies the properties:

(i) $V/W_3 = \mathbb{Z}, W_3/W_2 = \mathbb{Z}(1), W_2/W_1 = \mathbb{Z}(2), W_1 = \mathbb{Z}(3),$

(ii) $V/W_1$ and $W_3$ are classified respectively by

$$T(f_1, f_2) : X \to N_3(\mathbb{Z}) \backslash N_3(\mathbb{C})/C_3$$

$$T(f_2, f_3) : X \to N_3(\mathbb{Z}) \backslash N_3(\mathbb{C})/C_3,$$

where $T(f_i, f_{i+1}) (i = 1, 2)$ is defined in a similar manner to $T(f_1, f_2, f_3)$. 


References


