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Commutation relation of Hecke operators for Arakawa lifting

Atsushi Murase and Hiro-aki Narita

Abstract
The aim of this note is to make an announcement of our recent results [M-N] on Arakawa lifting, i.e. a theta lifting from elliptic cusp forms to automorphic forms on $Sp(1,q)$ (cf. [Ar-1], [N-1]). More precisely, restricting ourselves to the case of $q = 1$, we reformulate Arakawa's lifting as a theta lifting from automorphic forms $(f, f')$ on $GSp(1,1)$, where $B^\times$ denotes the multiplicative group of a definite quaternion algebra over $Q$. We show that this modified lifting satisfies a good commutation relation of Hecke operators. As an application we give all non-Archimedean local factors of spinor $L$-functions attached to the lifting in terms of Hecke eigenvalues for $(f, f')$.

1

1.1 Notation
For an algebraic group $G$ over $Q$, $G_v$ stands for the group of $Q_v$-points of $G$, where $Q_v$ denotes the $p$-adic field (resp. the field of real numbers) when $v = p$ is a finite prime (resp. $v = \infty$). By $G_A$ (resp. $G_{A,f}$), we denote the adelization of $G$ (resp. the group of finite adeles in $G_A$). Let $\psi$ be the additive character of $Q_A/Q$ such that $\psi(x_\infty) = e(x_\infty)$ for $x_\infty \in R$, where we put $e(z) = \exp(2\pi iz)$ for $z \in C$. We denote by $\psi_v$ the restriction of $\psi$ to $Q_v$ for a prime $v$ of $Q$.

1.2
Let $B$ be a definite quaternion algebra over $Q$. In what follows, we fix an identification between $B_\infty := B \otimes Q R$ and the Hamilton quaternion algebra $H$, and an embedding $H \hookrightarrow M_2(C)$. Let $B \ni b \mapsto \bar{b} \in B$ be the main involution of $B$, and put $\text{tr}(b) := b + \bar{b}$ and $n(b) := bb$ for $b \in B$. Let $B^\times := B \setminus \{0\}$ be the multiplicative group of $B$. The center $Z(B^\times)$ of $B^\times$ is $Q^\times \cdot 1$. Let $d_B$ be the discriminant of $B$. By definition, $d_B$ is the product of finite primes $p$ such that $B_p := B \otimes Q_p$ is a division algebra.
We let $G = GSp(1, 1)$ be an algebraic group over $\mathbb{Q}$ defined by

$$G_{\mathbb{Q}} = \{ g \in M_2(B) \mid {}^t\overline{g}Qg = \nu(g)Q, \nu(g) \in \mathbb{Q}^x \},$$

where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Denote by $Z_G$ the center of $G$.

The Lie group $G^1_\infty := \{ g \in G_{\infty} \mid \nu(g) = 1 \}$ acts on the hyperbolic 4-space $\mathcal{X} := \{ z \in \mathbb{H} \mid \text{tr}(z) > 0 \}$ by linear fractional transformations

$$g \cdot z := (az + b)(cz + d)^{-1}, \quad (g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in G^1_\infty, z \in \mathcal{X}).$$

Let $\mu : G^1_\infty \times \mathcal{X} \to \mathbb{H}^x$ be the automorphy factor given by $\mu((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), z) := cz + d$. The stabilizer subgroup $K_\infty$ of $z_0 := 1 \in \mathcal{X}$ in $G^1_\infty$ is a maximal compact subgroup of $G^1_\infty$, which is isomorphic to $Sp^*(1) \times Sp^*(1)$, where $Sp^*(1) := \{ z \in \mathbb{H} \mid n(z) = 1 \}$.

Let $\kappa$ be a positive integer. Denote by $(\sigma_\kappa, V_\kappa)$ the representation of $\mathbb{H}$ given as $\mathbb{H}^\epsilon \to M_2(\mathbb{C}) \to \text{End}(V_\kappa)$, where the second arrow indicates the $\kappa$-th symmetric power representation of $M_2(\mathbb{C})$. Then $\tau_\kappa(k_\infty) := \sigma_\kappa(\mu(k_\infty, z_0))$, $(k_\infty \in K_\infty)$ gives rise to an irreducible representation of $K_\infty$ of dimension $\kappa + 1$.

Define $\omega_\kappa : G^1_\infty \to \text{End}(V_\kappa)$ by

$$\omega_\kappa(g) := \sigma_\kappa(D(g))^{-1}n(D(g))^{-1}, \quad (g \in G^1_\infty),$$

where $D(g) := \frac{1}{2}(g \cdot z_0 + 1)\mu(g, z_0)$. It is known that $\omega_\kappa$ is a matrix coefficient of the discrete series representation with minimal $K_\infty$-type $(\tau_\kappa, V_\kappa)$ (cf. [Ar-2, §2.6]). This discrete series is a quaternionic discrete series in the sense of B. Gross and N. Wallach [G-W]. We note that $\omega_\kappa$ is integrable if $\kappa > 4$.

Throughout the paper, we fix a maximal order $\mathcal{O}$ of $B$. We also fix a two-sided ideal $\mathfrak{A}$ of $\mathcal{O}$ satisfying the following conditions:

(i) If $p \nmid d_B$, then $\mathfrak{A}_p = \mathcal{O}_p$.

(ii) If $p|d_B$, then $\mathfrak{A}_p = \mathfrak{P}_p^{e_p}$ with $e_p \in \{0, 1\}$, where $\mathfrak{P}_p$ is the maximal ideal of $\mathcal{O}_p$.

We set

$$D = \prod_{p|d_B, e_p=0} p.$$ 

Note that $D = 1$ if and only if $e_p = 1$ for any $p|d_B$. Let $L := {}^t(O \oplus \mathfrak{A}^{-1})$, which is a maximal lattice of $B^{\oplus 2}$. For a finite prime $p$, $K_p = \{ k \in G_p \mid kL_p = L_p \}$ is a maximal compact subgroup of $G_p$, where $L_p := L \otimes \mathbb{Z}_p$. We set $K_f := \prod_{p<\infty} K_p.$
Definition 1.1. For an even integer $\kappa > 4$, let $\mathcal{S}_\kappa$ be the space of smooth functions $F: G_\mathbb{A} \to V_\kappa$ satisfying the following conditions:

1. $F(z \gamma g k_f k_\infty) = \tau_\kappa(k_\infty)^{-1}F(g)$ \(\forall (z, \gamma, g, k_f, k_\infty) \in Z_{G, \mathbb{A}} \times G_\mathbb{Q} \times G_\mathbb{A} \times K_f \times K_\infty,\)
2. $F$ is bounded,
3. $c_\kappa \int_{G_\infty} \omega_\kappa(h_\infty^{-1}g_\infty)F(g_f h_\infty)dh_\infty = F(g_f g_\infty)$ for any fixed $(g_f, g_\infty) \in G_{\mathbb{A}, f} \times G_\infty$,

where $c_\kappa := 2^{-4}\pi^{-2}\kappa(\kappa - 1)$.

Here we remark that this automorphic form has been verified to be cuspidal (cf. [Ar-2, Proposition 3.1]) and to generate a quaternionic discrete series at the infinite place (cf. [N-2, Theorem 8.7]).

Next let $H$ and $H'$ be algebraic groups over $\mathbb{Q}$ defined by $H_\mathbb{Q} = GL_2(\mathbb{Q})$ and $H'_\mathbb{Q} = B^x$ respectively, and denote by $Z_H$ and $Z_{H'}$ the center of $H$ and $H'$ respectively.

We define an action of $SL_2(\mathbb{R})$ on the complex upper half plane $\mathfrak{h} := \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}$ as usual.

Let $U_\infty := \{h \in SL_2(\mathbb{R}) | h \cdot i = i\} = SO(2)$ and $U'_\infty := \{h' \in \mathbb{H} | n(h') = 1\} = Sp^*(1)$.

Moreover, we put $U_f = \prod_{p<\infty} U_p$ and $U'_f = \prod_{p<\infty} U'_p$, where $U_p := \{u = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in GL_2(\mathbb{Z}_p) | c \in D\mathbb{Z}_p\}$ and $U'_p := O_p^x$.

Definition 1.2. (1) Let $\mathcal{S}_\kappa(D)$ be the space of smooth functions $f$ on $H_\mathbb{A}$ satisfying the following conditions:

1. $f(z \gamma hu_f u_\infty) = j(u_\infty, i)^{-\kappa}f(h)$ \(\forall (z, \gamma, h, u_f, u_\infty) \in Z_{H, \mathbb{A}} \times H_\mathbb{Q} \times H_\mathbb{A} \times U_f \times U_\infty,\)
2. For any fixed $h_f \in H_{\mathbb{A}, f}$, $h \ni h_\infty \cdot i \mapsto j(h_\infty, i)^{\kappa}f(h_f h_\infty)$ is holomorphic for $h_\infty \in SL_2(\mathbb{R})$,
3. $f$ is bounded,

where $j((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), \tau) := ct + d$ denotes the standard $\mathbb{C}$-valued automorphy factor of $SL_2(\mathbb{R}) \times \mathfrak{h}$.

(2) Furthermore, $\mathcal{A}_\kappa$ stands for the space of smooth $V_\kappa$-valued functions $f'$ on $H_\mathbb{A}'$ such that

$$f'(z' \gamma' h'u_f' u'_\infty) = \sigma_\kappa(u'_\infty)^{-1}f(h') \quad \forall (z', \gamma', h', u'_f, u'_\infty) \in Z_{H', \mathbb{A}} \times H'_\mathbb{Q} \times H'_\mathbb{A} \times U'_f \times U'_\infty.$$

2 Arakawa lift

2.1 Metaplectic representation

We fix a prime $v$ of $\mathbb{Q}$. When $v = p$ is a finite prime (resp. $v = \infty$), $| \ast |_v$ denotes the $p$-adic valuation (resp. the usual absolute value for $\mathbb{R}$). For $X = \begin{pmatrix} x \\ y \end{pmatrix} \in B^p_{\mathbb{Q}}$, we put $X^* := (\bar{x}, \bar{y})$.

For a finite prime $p$, let $V_p$ be the space of functions on $B^p_{\mathbb{Q}} \times \mathbb{Q}_p^*$ generated by $\varphi_1(X)\varphi_2(t)$,
where \( \varphi_1 \) (resp. \( \varphi_2 \)) is a locally constant and compactly supported function on \( B_{p}^{\oplus 2} \) (resp. \( \mathbb{Q}_{p}^{\times} \)). We also let \( V_{\infty} \) be the space of smooth functions \( \varphi \) on \( B_{\infty}^{\oplus 2} \times \mathbb{Q}_{\infty}^{\times} = \mathbb{H}_{\infty}^{\oplus 2} \times \mathbb{R}^{\times} \) such that, for any fixed \( t \in \mathbb{R}^{\times} \), \( X \mapsto \varphi(X, t) \) is rapidly decreasing on \( \mathbb{H}_{\infty}^{\oplus 2} \).

**Lemma 2.1.** There exists a smooth representation \( r = r_{v} \) of \( G_{v} \times H_{v} \times H'_{v} \) on \( V_{v} \) given as follows:

For \( \varphi \in V_{v} \), \( X \in B_{v}^{\oplus 2} \) and \( t \in \mathbb{Q}_{v}^{\times} \),

\[
\begin{align*}
\varphi(g, 1, 1) \varphi(X, t) &= |\nu(g)|^{-\frac{3}{2}} \varphi(g^{-1}X, \nu(g)t), \quad (g \in G_{v}), \\
\varphi(1, (b, 1), 1) \varphi(X, t) &= \psi_{v}(\frac{bt}{2} tr(X^{\ast}QX)) \varphi(X, t), \quad (b \in \mathbb{Q}_{v}), \\
\varphi(1, (a, 1), 1) \varphi(X, t) &= |a|^{-\frac{3}{2}} |a'|^{-\frac{1}{2}} \varphi(aX, (aa')^{-1}t), \quad (a, a' \in \mathbb{Q}_{v}^{\times}), \\
\varphi(1, (b, 1), 1) \varphi(X, t) &= |t|_{v}^{4} \int_{B_{v}^{\oplus 2}} \psi_{v}(t \cdot tr(\cdot X^{\ast}QX)) \varphi(\cdot, t) d_{Q} \cdot \\
\varphi(1, (a, 1), 1) \varphi(X, t) &= |n(z)|^{\frac{3}{v^{2}}} |n(z)|^{-\frac{1}{2}} \varphi(n(z)X, (nn(z))^{-1}t), \quad (z \in B_{v}^{\oplus 2}).
\end{align*}
\]

Here \( d_{Q} \cdot \) is the Haar measure on \( B_{v}^{\oplus 2} \) self-dual with respect to the pairing

\[
B_{v}^{\oplus 2} \times B_{v}^{\oplus 2} \ni (\cdot, \cdot) \mapsto \psi_{v}(tr(\cdot \cdot X^{\ast}QX)) \varphi(\cdot, t)
\]

2.2

When \( v = p < \infty \), we put

\[
\varphi_{0,p}(X, t) := \text{char}_{L_{p}}(X) \text{char}_{\mathbb{Z}_{p}^{\times}}(t),
\]

where char_{L_{p}} (resp. char_{\mathbb{Z}_{p}^{\times}}) is the characteristic function of \( L_{p} = t(O_{p} \oplus \mathfrak{U}_{p}^{-1}) \) (resp. \( \mathbb{Z}_{p}^{\times} \)).

When \( v = \infty \), we put

\[
\varphi_{0,\infty}^{\kappa}(X, t) := \left\{ \begin{array}{ll}
t^{\frac{3}{2} + \frac{3}{2}} \sigma_{\kappa}((1, 1)X)e\left(\frac{it}{2} tr(X^{\ast}X)\right) & (t > 0) \\
0 & (t < 0)
\end{array} \right.
\]

Let \( V_{A} \) be the restricted tensor product of \( V_{v} \) with respect to \( \{ \varphi_{0,p} \}_{p<\infty} \). By \( r_{A} \) we denote a smooth representation of \( G_{A} \times H_{A} \times H'_{A} \) on \( V_{A} \) given as

\[
r_{A}(g, h, h') \varphi := \bigotimes_{v} r_{v}(g_{v}, h_{v}, h'_{v}) \varphi_{v}
\]

for \( \varphi = \otimes \varphi_{v} \in V_{A} \) and \( (g = (g_{v}), h = (h_{v}), h' = (h'_{v})) \in G_{A} \times H_{A} \times H'_{A} \).

We define a function \( \varphi_{0}^{\kappa} \in V_{A} \) by

\[
\varphi_{0}^{\kappa}(X, t) := \varphi_{0,\infty}^{\kappa}(X, t_{\infty}) \prod_{p<\infty} \varphi_{0,p}(X_{p}, t_{p})
\]
for $X = (X_v) \in B_{\mathbb{A}}^\oplus$ and $t = (t_v) \in \mathbb{Q}_\mathbb{A}^\times$, and set

$$\theta^\kappa(g, h, h') := \sum_{(X, t) \in B_{\mathbb{Q}}^\times} r_{\mathbb{A}}(g, h, h') \varphi_0^\kappa(X, t), \quad ((g, h, h') \in G_{\mathbb{A}} \times H_{\mathbb{A}} \times H'_{\mathbb{A}}). \quad (2.6)$$

This series is uniformly convergent on any compact subset of $G_{\mathbb{A}} \times H_{\mathbb{A}} \times H'_{\mathbb{A}}$, and satisfies

$$\theta^\kappa(\gamma g k_f k_{\infty}, \gamma_1 h u_f u_{\infty}, \gamma_2 h' u_f' u_{\infty}') = \tau_\kappa(k_{\infty})^{-1} j(u_{\infty}, i)^{-\kappa} \theta(g, h, h') \sigma_\kappa(u_{\infty}')$$

for $(\gamma, g, k_f, k_{\infty}) \in G_{\mathbb{Q}} \times G_{\mathbb{A}} \times K_f \times K_{\infty}, \ (\gamma_1, h, u_f, u_{\infty}) \in H_{\mathbb{Q}} \times H_{\mathbb{A}} \times U_f \times U_{\infty}$ and $(\gamma_2, h', u_f', u_{\infty}') \in H'_Q \times H'_A \times U'_f \times U'_{\infty}$. It is also verify that $\theta^\kappa$ is $Z_{G, \mathbb{A}} \times Z_{H, \mathbb{A}} \times Z_{H', \mathbb{A}}$-invariant.

For $f \in S_\kappa(D)$ and $f' \in A_\kappa$, we set

$$\mathcal{L}(f, f')(g) := \int_{Z_{H} \mathbb{A} \backslash H_{\mathbb{A}}} dh \int_{Z_{H'} \mathbb{A} \backslash H'_{\mathbb{A}}} dh' \theta^\kappa(g, h, h') \overline{f(h)} f'(h') \quad (g \in G_{\mathbb{A}}). \quad (2.7)$$

**Theorem 2.2 (Arakawa, Narita).** Suppose $\kappa > 6$.

(i) The integral $(2.7)$ is absolutely convergent.

(ii) $\mathcal{L}(f, f')(g) \in S_\kappa$.

**Proof.** Since $G_{\mathbb{A}} = Z_{G, \mathbb{A}} G_{\mathbb{Q}} G_{\infty} \times K_f$ (cf. [Shim-2, Theorem 6.14]), it is sufficient to consider the restriction of $\mathcal{L}(f, f')$ to $G_{\mathbb{Q}}$. By a standard argument, we see that $\mathcal{L}(f, f')|_{G_{\mathbb{Q}}}$ is a finite linear combination of original Arakawa lift (cf. [Ar-l], [N-l, §4] and [N-3, Theorem 4.1]), from which the theorem follows. \(\Box\)

**Remark 2.3.** At the Archimedean place our lifting reads a correspondence between the quaternionic discrete series of $G_{\infty}'$ with minimal $K_{\infty}$-type $\tau_\kappa$ and the discrete series representation of $O^*(4) \simeq SL_2(\mathbb{R}) \times SU(2)$ given by the direct product of the holomorphic discrete series of $SL_2(\mathbb{R})$ with Blattner parameter $\kappa$ and the representation $\sigma_\kappa$ of $Sp^*(1) \simeq SU(2)$. This is compatible with the result [J] of J. S. Li on theta correspondences for unitary representations with non-zero cohomology (cf. [J, §6, (I_1)]).

For the case of $GSp(1, q)$ we would be able to give an adelic reformulation of the lifting similarly. In view of [N-3, Theorem 4.1] and [J, §6, (I_1)], the weight of elliptic cusp forms or the Blatter parameter of the holomorphic discrete series of $SL_2(\mathbb{R})$ should be $\kappa - 2q + 2$ for this general case.

### 3 Main results

#### 3.1

To state our results, we need to review several facts on Hecke operators.
3.2

First we consider the case where \( p \nmid d_B \). We fix an isomorphism of \( B_p \) onto \( M_2(\mathbb{Q}_p) \) such that \( \mathcal{O}_p \) maps onto \( M_2(\mathbb{Z}_p) \) and that the main involution of \( B_p \) corresponds to an involution of \( M_2(\mathbb{Q}_p) \) given by

\[
M_2(\mathbb{Q}_p) \ni X \mapsto w^{-1}Xw, \quad (w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}).
\]

The reduced trace \( \text{tr} \) corresponds to the trace \( \text{Tr} \) of \( M_2(\mathbb{Q}_p) \). We henceforth identify \( B_p \) with \( M_2(\mathbb{Q}_p) \) using the above isomorphism. Then \( G_p, K_p, H'_p \) and \( U'_p \) are identified with \( GSp(J, \mathbb{Q}_p), GSp(J, \mathbb{Z}_p), GL_2(\mathbb{Q}_p) \) and \( GL_2(\mathbb{Z}_p) \) respectively, where \( GSp(J) \) is the group of similitudes of \( J = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \). Note that we can identify \( U_p \) with \( U'_p \) by the isomorphism \( B_p \simeq M_{\mathit{2}}(\mathbb{Q}_p) \) fixed above.

Define Hecke operators \( T_p^i \) \((i = 0, 1, 2)\) on \( S_\kappa \) by

\[
T_p^i F(g) = \int_{G_p} F(gx)\Phi_p^i(x)dx,
\]

where \( \Phi_p^0, \Phi_p^1 \) and \( \Phi_p^2 \) are the characteristic function of \( K_p \cdot \text{diag}(p, p, p)K_p, K_p \cdot \text{diag}(p, p, 1, 1)K_p \) and \( K_p \cdot \text{diag}(p^2, p, p, 1)K_p \) respectively. Note that \( (T_p^0)^2F = F \) for any \( F \in S_\kappa \).

We also define Hecke operators \( T_p \) and \( T_p' \) on \( S_\kappa(D) \) and \( A_\kappa \) by

\[
T_p f(h) = \int_{H_p} f(hx)\phi_p(x)dx,
\]

\[
T_p' f'(h') = \int_{H_p'} f'(h'x')\phi'_p(x')dx',
\]

where \( \phi_p = \phi'_p \) is the characteristic function of \( GL_2(\mathbb{Z}_p) \cdot \text{diag}(p, 1)GL_2(\mathbb{Z}_p) \).

3.3

We next consider the case where \( p|d_B \), i.e., \( B_p \) is a division algebra. In this case, we fix a prime element \( \Pi \) of \( B_p \) and put \( \pi := n(\Pi) \). Then \( \pi \) is a prime element of \( \mathbb{Q}_p \).

Define Hecke operators \( T_p^i \) \((i = 0, 1)\) on \( S_\kappa \) by

\[
T_p^i F(g) = \int_{G_p} F(gx)\Phi_p^i(x)dx,
\]

where \( \Phi_p^0 \) and \( \Phi_p^1 \) are the characteristic functions of \( K_p \cdot \text{diag}(\Pi, \Pi) \cdot K_p \) and \( K_p \cdot \text{diag}(1, \pi)K_p \) respectively. Note that \( (T_p^0)^2F = F \) for any \( F \in S_\kappa \). We also define Hecke operators \( T_p \) and \( T_p' \) on \( S_\kappa(D) \) and \( A_\kappa \) by

\[
T_p f(h) = \int_{H_p} f(hx)\phi_p(x)dx,
\]

\[
T_p' f'(h') = \int_{H_p'} f'(h'x')\phi'_p(x')dx'.
\]
Here \( \phi_p' \) is the characteristic function of \( U'_p \Pi U'_p = \Pi U'_p \) and \( \phi_p \) is defined as follows: If \( p|D \), \( \phi_p \) is the sum of the characteristic functions of \( U_p(\begin{smallmatrix} 0 & 0 \\ \pi & 1 \end{smallmatrix})U_p \) and \( U_p(\begin{smallmatrix} 0 & \pi \\ 0 & 0 \end{smallmatrix})U_p \). If \( p \nmid D \), \( \phi_p \) is the characteristic function of \( U_p(\begin{smallmatrix} 0 & 0 \\ 0 & \pi \end{smallmatrix})U_p \).

3.4

We say that \( F \in S_\kappa \) is a Hecke eigenform if \( F \) is a common eigenfunction of the Hecke operators \( T_p^j \) for any \( p < \infty \). Let \( F \in S_\kappa \) be a Hecke eigenform with \( T_p^j F = \Lambda_p^j F \) (\( \Lambda_p^j \in \mathbb{C} \)). We define the spinor \( L \)-function of \( F \) by

\[
L(F, \text{spin}, s) = \prod_{p<\infty} L_p(F, \text{spin}, s),
\]

where \( L_p(F, \text{spin}, s) = Q_p(F, p^{-s})^{-1} \),

\[
Q_p(F, t) = \begin{cases} 
1 - p^{\kappa-3} \Lambda_p^1 t + p^{2\kappa-5} (\Lambda_p^2 + t^2) + p^{4\kappa-6} t^4 & \text{if } p \nmid d_B, \\
1 - p^{\kappa-3} \Lambda_p^0 - p^{\kappa-3} (p^\lambda - 1) \Lambda_p^0 + p^{2\kappa-3} (\Lambda_p^0)^2 t^2 & \text{if } p|d_B,
\end{cases}
\]

and

\[
A_p = \begin{cases} 
1 & \text{if } p \nmid D, \\
2 & \text{if } p|D.
\end{cases}
\]

The Euler factor for \( p \nmid d_B \) (resp. \( p|d_B \)) is given by the formula for the denominator of the Hecke series in [Shim-1, Theorem 2] (resp. [H-S, §4] and [Su, (1-34)]), under the normalization of the Hecke eigenvalues

\[
\begin{align*}
(\Lambda_p^0, \Lambda_p^1, \Lambda_p^2) &\rightarrow (p^{2(\kappa-3)} \Lambda_p^0, p^{\kappa-3} \Lambda_p^1, p^{2(\kappa-3)} \Lambda_p^2) \quad (p \nmid d_B) \\
(\Lambda_p^0, \Lambda_p^1) &\rightarrow (p^{\kappa-3} \Lambda_p^0, p^{\kappa-3} \Lambda_p^1) \quad (p|d_B).
\end{align*}
\]

We say that \( f \in S_\kappa(D) \) (resp. \( f' \in A_\kappa \)) is a Hecke eigenform if \( f \) (resp. \( f' \)) is a common eigenfunction of \( T_p \) (resp. \( T'_p \)) for any \( p < \infty \). For Hecke eigenforms \( f \in S_\kappa(D) \) and \( f' \in A_\kappa \) with \( T_p f = \lambda_p f \) and \( T_p f' = \lambda_p' f' \) (\( \lambda_p, \lambda_p' \in \mathbb{C} \)), we define \( L \)-functions

\[
L^D(f, s) = \prod_{p|D} (1 - \lambda_pp^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1},
\]

\[
L^{d_B}(f', s) = \prod_{p|d_B} (1 - \lambda'_pp^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}.
\]

When \( D = 1 \), we write \( L(f, s) \) for \( L^D(f, s) \), which is the usual Hecke \( L \)-function of \( f \).
3.5

We are now able to state the main result.

**Theorem 3.1.** Let $f \in S_\kappa(D)$ and $f' \in A_\kappa$, and suppose that

$$
T_p f = \lambda_p f, \\
T'_p f' = \lambda'_p f'
$$

for each $p < \infty$. Then $F(g) := \mathcal{L}(f, f')(g)$ is a Hecke eigenform and the Hecke eigenvalues are given as follows:

(i) If $p \nmid d_B$, we have

$$
T^0_p F = F, \\
T^1_p F = (p\lambda_p + p\lambda'_p) F, \\
T^2_p F = (p\lambda'_p \lambda_p' + p^2 - 1) F.
$$

(ii) If $p | d_B$, we have

$$
T^0_p F = \lambda'_p F, \\
T^1_p F = (p\lambda_p + (p-1)\lambda'_p) F.
$$

**Remark 3.2.** Noting that the elliptic cusp forms are assumed to have the trivial central character, we see that all the Hecke operators above for the cusp forms are self-adjoint with respect to the Petersson inner product. We can thus remove the complex conjugates of their Hecke eigenvalues in the formula above.

**Corollary 3.3.** Let $f$ and $f'$ be as in Theorem 3.1. Then we have

$$
L(\mathcal{L}(f, f'), \text{spin}, s) = L^D(f, s) L^{d_B}(f', s) \prod_{p|D} (1 - \{\lambda_p + (1-p)\lambda'_p\} p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}.
$$

In particular, if $D = 1$, we have

$$
L(\mathcal{L}(f, f'), \text{spin}, s) = L(f, s) L^{d_B}(f', s).
$$

The results above are deduced from the commutation relation of Hecke operators for the metaplectic representation $\tau$ as follows:

**Proposition 3.4.** For a function $\phi$ on $H_p$, we put $\tilde{\phi}(h) = \phi(h^{-1})$ ($h \in H_p$). We define $\tilde{\phi}'$ for $\phi': H'_p \to \mathbb{C}$ in a similar manner.

(1) Suppose that $p \nmid d_B$. Then we have
(i) $r(\Phi_{p}^{1}, 1, 1)\varphi_{0,p} = p \cdot r(1, \widehat{\phi}_{p}, 1)\varphi_{0,p} + p \cdot r(1, 1, \widehat{\phi}_{p}')\varphi_{0,p}$,
(ii) $r(\Phi_{p}^{2}, 1, 1)\varphi_{0,p} + (1 - p^{2})r(\Phi_{p}^{0}, 1, 1)\varphi_{0,p} = p \cdot r(1, \phi_{p}, \phi_{p}')\varphi_{0,p}$.

(2) Suppose that $p \nmid d_{B}$. Then we have

\[ r(\Phi_{p}^{0}, 1, 1)\varphi_{0,p} = r(1, 1, \widehat{\phi}_{p}')\varphi_{0,p}, \]
\[ r(\Phi_{p}^{1}, 1, 1)\varphi_{0,p} = p \cdot r(1, \widehat{\phi}_{p}, 1)\varphi_{0,p} + (p - 1)r(1, 1, \widehat{\phi}_{p}')\varphi_{0,p}. \]

Remark 3.5. When $p \nmid d_{B}$ the formula for the Hecke eigenvalues is essentially the same as the corresponding result of Yoshida lifting (cf. [Y, Theorem 6.1]). For such $p$ this leads to the following decomposition

\[ L_{p}(\mathcal{L}(f, f'), \text{spin}, s) = (1 - \lambda_{p}p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}(1 - \lambda_{p}'p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}. \]

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References


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