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Commutation relation of Hecke operators for Arakawa lifting

Atsushi Murase and Hiro-aki Narita

Abstract

The aim of this note is to make an announcement of our recent results [M-N] on Arakawa lifting, i.e. a theta lifting from elliptic cusp forms to automorphic forms on $Sp(1, q)$ (cf. [Ar-1], [N-1]). More precisely, restricting ourselves to the case of $q = 1$, we reformulate Arakawa's lifting as a theta lifting from automorphic forms $(f, f')$ on $GL_2 \times B^\times$ to forms $\mathcal{L}(f, f')$ on $GSp(1, 1)$, where $B^\times$ denotes the multiplicative group of a definite quaternion algebra over $\mathbb{Q}$. We show that this modified lifting satisfies a good commutation relation of Hecke operators. As an application we give all non-Archimedean local factors of spinor $L$-functions attached to the lifting in terms of Hecke eigenvalues for $(f, f')$.

1

1.1 Notation

For an algebraic group $G$ over $\mathbb{Q}$, $G_v$ stands for the group of $\mathbb{Q}_v$-points of $G$, where $\mathbb{Q}_v$ denotes the $p$-adic field (resp. the field of real numbers) when $v = p$ is a finite prime (resp. $v = \infty$). By $G_{\mathbb{A}}$ (resp. $G_{\mathbb{A}, f}$), we denote the adelization of $G$ (resp. the group of finite adeles in $G_{\mathbb{A}}$). Let $\psi$ be the additive character of $\mathbb{Q}_{\mathbb{A}}/\mathbb{Q}$ such that $\psi(x_{\infty}) = e(x_{\infty})$ for $x_{\infty} \in \mathbb{R}$, where we put $e(z) = \exp(2\pi iz)$ for $z \in \mathbb{C}$. We denote by $\psi_v$ the restriction of $\psi$ to $\mathbb{Q}_v$ for a prime $v$ of $\mathbb{Q}$.

1.2

Let $B$ be a definite quaternion algebra over $\mathbb{Q}$. In what follows, we fix an identification between $B_{\infty} := B \otimes_{\mathbb{Q}} \mathbb{R}$ and the Hamilton quaternion algebra $\mathbb{H}$, and an embedding $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$. Let $B \ni b \mapsto \overline{b} \in B$ be the main involution of $B$, and put $\text{tr}(b) := b + \overline{b}$ and $n(b) := bb$ for $b \in B$. Let $B^\times := B \setminus \{0\}$ be the multiplicative group of $B$. The center $Z(B^\times)$ of $B^\times$ is $\mathbb{Q}^\times \cdot 1$. Let $d_B$ be the discriminant of $B$. By definition, $d_B$ is the product of finite primes $p$ such that $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra.
We let $G = GSp(1, 1)$ be an algebraic group over $\mathbb{Q}$ defined by
\[ G_{\mathbb{Q}} = \{ g \in M_2(B) \mid {}^t \overline{g} Q g = \nu(g) Q, \nu(g) \in \mathbb{Q}^* \}, \]
where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Denote by $Z_G$ the center of $G$.

The Lie group $G_{\infty}^1 := \{ g \in G_{\infty} \mid \nu(g) = 1 \}$ acts on the hyperbolic 4-space $X := \{ z \in \mathbb{H} \mid \text{tr}(z) > 0 \}$ by linear fractional transformations $g \cdot z := (az + b)(cz + d)^{-1}$, $(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\infty}^1, z \in X)$.

Let $\mu : G_{\infty}^1 \times X \to \mathbb{H}^\times$ be the automorphy factor given by $\mu((a b \begin{pmatrix} c \\ d \end{pmatrix}), z) := cz + d$. The stabilizer subgroup $K_{\infty}$ of $z_0 := 1 \in X$ in $G_{\infty}^1$ is a maximal compact subgroup of $G_{\infty}^1$, which is isomorphic to $Sp^{*}(1) \times Sp^{*}(1)$, where $Sp^{*}(1) := \{ z \in \mathbb{H} \mid n(z) = 1 \}$.

Let $\kappa$ be a positive integer. Denote by $(\sigma_\kappa, V_\kappa)$ the representation of $\mathbb{H}$ given as $\mathbb{H}^\epsilon \to M_2(\mathbb{C}) \to \text{End}(V_\kappa)$, where the second arrow indicates the $\kappa$-th symmetric power representation of $M_2(\mathbb{C})$.

Then $\tau_{\kappa}(k_{\infty}) := \sigma_\kappa(\mu(k_{\infty}, z_0))$, $(k_{\infty} \in K_{\infty})$ gives rise to an irreducible representation of $K_{\infty}$ of dimension $\kappa + 1$.

Define $\omega_\kappa : G_{\infty}^1 \to \text{End}(V_\kappa)$ by
\[ \omega_\kappa(g) := \sigma_\kappa(D(g))^{-1} n(D(g))^{-1}, \quad (g \in G_{\infty}^1), \]
where $D(g) := \frac{1}{2} (g \cdot z_0 + 1) \mu(g, z_0)$. It is known that $\omega_\kappa$ is a matrix coefficient of the discrete series representation with minimal $K_{\infty}$-type $(\tau_{\kappa}, V_\kappa)$ (cf. [Ar-2, §2.6]). This discrete series is a quaternionic discrete series in the sense of B. Gross and N. Wallach [G-W]. We note that $\omega_\kappa$ is integrable if $\kappa > 4$.

Throughout the paper, we fix a maximal order $\mathcal{O}$ of $B$. We also fix a two-sided ideal $\frak{A}$ of $\mathcal{O}$ satisfying the following conditions:

(i) If $p \not\parallel d_B$, then $\frak{A}_p = \mathcal{O}_p$.

(ii) If $p|d_B$, then $\frak{A}_p = \mathfrak{P}_p^{e_p}$ with $e_p \in \{0, 1\}$, where $\mathfrak{P}_p$ is the maximal ideal of $\mathcal{O}_p$.

We set
\[ D = \prod_{p|d_B, e_p=0} p. \]

Note that $D = 1$ if and only if $e_p = 1$ for any $p|d_B$. Let $L := (\mathcal{O} \oplus \mathfrak{A}^{-1})$, which is a maximal lattice of $B^{\oplus 2}$. For a finite prime $p$, $K_p = \{ k \in G_p \mid kL_p = L_p \}$ is a maximal compact subgroup of $G_p$, where $L_p := L \otimes \mathbb{Z}_p$. We set $K_f := \prod_{p<\infty} K_p$. 

Definition 1.1. For an even integer $\kappa > 4$, let $S_\kappa$ be the space of smooth functions $F : \mathbb{G} \to V_\kappa$ satisfying the following conditions:

1. $F(z \gamma g k_\infty) = \tau_\kappa(k_\infty)^{-1} F(g)$ \(\forall (z, \gamma, g, k_f, k_\infty) \in \mathbb{Z}_{\mathbb{G,Q}} \times \mathbb{G}_\mathbb{Q} \times \mathbb{G}_\mathbb{A} \times K_f \times K_\infty\),
2. $F$ is bounded,
3. $c_\kappa \int_{H_{\infty}} \omega_\kappa(h_{\infty}^{-1} g_{\infty}) F(g_{f} h_{\infty}) dh_{\infty} = F(g_{f} g_{\infty})$ for any fixed $(g_f, g_\infty) \in \mathbb{G}_{\mathbb{Q},f} \times \mathbb{G}_{\mathbb{Q}}$,

where $c_\kappa := 2^{-4 \pi^{-2}} \kappa(\kappa-1)$.

Here we remark that this automorphic form has been verified to be cuspidal (cf. [Arakawa, Proposition 3.1]) and to generate a quaternionic discrete series at the infinite place (cf. [Narasimhan, Theorem 8.7]).

Next let $H$ and $H'$ be algebraic groups over $\mathbb{Q}$ defined by $H_\mathbb{Q} = GL_2(\mathbb{Q})$ and $H'_\mathbb{Q} = B^x$ respectively, and denote by $Z_H$ and $Z_{H'}$ the center of $H$ and $H'$ respectively. We define an action of $SL_2(\mathbb{R})$ on the complex upper half plane $\mathfrak{h} := \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}$ as usual. Let $U_\infty := \{h \in SL_2(\mathbb{R}) | h \cdot i = i\} = SO(2)$ and $U'_\infty := \{h' \in \mathbb{H} | n(h') = 1\} = Sp^*(1)$. Moreover, we put $U_f = \prod_{p<\infty} U_p$ and $U'_f = \prod_{p<\infty} U'_p$, where $U_p := \{u = (a_{cd}) \in GL_2(\mathbb{Z}_p) \}$.\[c \in D\mathbb{Z}_p\}$ and $U'_p := O_p^x$.

Definition 1.2. (1) Let $S_\kappa(D)$ be the space of smooth functions $f$ on $H_\mathbb{A}$ satisfying the following conditions:

1. $f(z \gamma h u_f u_\infty) = j(u_\infty, i)^{-\kappa} f(h)$ \(\forall (z, \gamma, h, u_f, u_\infty) \in \mathbb{Z}_{H,\mathbb{Q}} \times H_\mathbb{Q} \times H_\mathbb{A} \times U_f \times U_\infty\),
2. For any fixed $h_f \in H_{\mathbb{Q},f}$, $h \in H_\mathbb{Q}$, $\delta \cdot h_\infty \cdot i \mapsto j(h_\infty, i)^{\kappa} f(h_f h_\infty)$ is holomorphic for $h_\infty \in \mathbb{H}$,
3. $f$ is bounded,

where $j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) := ct + d$ denotes the standard $\mathbb{C}$-valued automorphy factor of $SL_2(\mathbb{R}) \times \mathfrak{h}$.

(2) Furthermore, $A_\kappa$ stands for the space of smooth $V_\kappa$-valued functions $f'$ on $H'_\mathbb{A}$ such that $f'(z' \gamma' h' u'_f u'_\infty) = \sigma_\kappa(u'_\infty)^{-1} f(h')$ \(\forall (z', \gamma', h', u'_f, u'_\infty) \in \mathbb{Z}_{H',\mathbb{Q}} \times H'_\mathbb{Q} \times H'_\mathbb{A} \times U'_f \times U'_\infty\).

2 Arakawa lift

2.1 Metaplectic representation

We fix a prime $v$ of $\mathbb{Q}$. When $v = p$ is a finite prime (resp. $v = \infty$), $| \cdot |_v$ denotes the $p$-adic valuation (resp. the usual absolute value for $\mathbb{R}$). For $X = \begin{pmatrix} x \\ y \end{pmatrix} \in B^{\mathbb{Q}}_v$, we put $X^*(\bar{x}, \bar{y})$. For a finite prime $p$, let $\mathbb{V}_p$ be the space of functions on $B^{\mathbb{Q}}_p \times Q^*_p$ generated by $\varphi_1(x)\varphi_2(t)$, \[\varphi_2(t) = \prod_{v \neq \infty} \varphi_2(t_v)\]
where $\varphi_1$ (resp. $\varphi_2$) is a locally constant and compactly supported function on $B_p^{\oplus 2}$ (resp. $\mathbb{Q}_p^\times$). We also let $V_\infty$ be the space of smooth functions $\varphi$ on $B_\infty^{\oplus 2} \times \mathbb{Q}_\infty^\times = \mathbb{H}_\infty^{\oplus 2} \times \mathbb{R}^\times$ such that, for any fixed $t \in \mathbb{R}^\times$, $X \mapsto \varphi(X, t)$ is rapidly decreasing on $\mathbb{H}_\infty^{\oplus 2}$.

**Lemma 2.1.** There exists a smooth representation $r = r_v$ of $G_v \times H_v \times H_v'$ on $V_v$ given as follows:

For $\varphi \in V_v$, $X \in B_v^{\oplus 2}$ and $t \in \mathbb{Q}_v^\times$,

\[
\begin{align*}
    r(1, (\frac{b}{1}), 1)\varphi(X, t) &= \psi_v(\frac{bt}{2} \text{tr}(X^*QX))\varphi(X, t), \\
    r(1, (\frac{a}{a'}), 1)\varphi(X, t) &= |a|^{\frac{7}{v^2}} |a'|^{-\frac{1}{2}} \varphi(aX, (aa')^{-1}t), \\
    r(1, (1), 1)\varphi(X, t) &= \int_{B_v^{\oplus 2}} \psi_v(t \text{tr}((Y^*QX)))\varphi(Y, t)d_QY, \\
    r(1, 1, z)\varphi(X, t) &= |n(z)|^\frac{\theta}{v^2} \varphi(Xz, n(z)^{-1}t).
\end{align*}
\]

Here $d_QY$ is the Haar measure on $B_v^{\oplus 2}$ self-dual with respect to the pairing

$B_v^{\oplus 2} \times B_v^{\oplus 2} \ni (Y, Y') \mapsto \psi_v(\text{tr}(Y^*QY'))$.

2.2

When $v = p < \infty$, we put

$\varphi_{0,p}(X, t) := \text{char}_{L_p}(X) \text{char}_{\mathbb{Z}_p^\times}(t)$,

where $\text{char}_{L_p}$ (resp. $\text{char}_{\mathbb{Z}_p^\times}$) is the characteristic function of $L_p = t(O_p \oplus \mathfrak{U}_p^{-1})$ (resp. $\mathbb{Z}_p^\times$). When $v = \infty$, we put

$\varphi_{0,\infty}(X, t) := \begin{cases} 
    t^{\frac{3+\kappa}{2}} \sigma_\kappa((1, 1)X)e(\frac{it}{2} \text{tr}(X^*X)) & (t > 0) \\
    0 & (t < 0) 
\end{cases}$.

Let $V_\Lambda$ be the restricted tensor product of $V_v$ with respect to $\{\varphi_{0,p}\}_{p < \infty}$. By $r_\Lambda$ we denote a smooth representation of $G_\Lambda \times H_\Lambda \times H_\Lambda'$ on $V_\Lambda$ given as

$r_\Lambda(g, h, h')\varphi := \otimes_{v} r_v(g_v, h_v, h'_v)\varphi_v$

for $\varphi = \otimes \varphi_v \in V_\Lambda$ and $(g = (g_v), h = (h_v), h' = (h'_v)) \in G_\Lambda \times H_{\Lambda} \times H_{\Lambda}'$.

We define a function $\varphi_0^\kappa \in V_\Lambda$ by

$\varphi_0^\kappa(X, t) := \varphi_{0,\infty}(X_\infty, t_\infty) \prod_{p < \infty} \varphi_{0,p}(X_p, t_p)$.
for $X = (X_v) \in B_{\mathbb{A}}^\oplus 2$ and $t = (t_v) \in \mathbb{Q}_\mathbb{A}^x$, and set
\[
\theta^\kappa(g, h, h') := \sum_{(X,t) \in B_{\mathbb{Q}^\times}^2} r_{\mathbb{A}}(g, h, h') \varphi_0^\kappa(X, t), \quad ((g, h, h') \in G_{\mathbb{A}} \times H_{\mathbb{A}} \times H'_{\mathbb{A}}). \tag{2.6}
\]
This series is uniformly convergent on any compact subset of $G_{\mathbb{A}} \times H_{\mathbb{A}} \times H'_{\mathbb{A}}$, and satisfies
\[
\theta^\kappa(\gamma g k_f, \gamma_1 h u_f u_\infty, \gamma_2 h' u'_f u'_\infty) = \tau_\kappa(k_\infty)^{-1} j(u_\infty, i)^{-\kappa} \theta(g, h, h') \sigma_\kappa(u'_\infty)
\]
for $(\gamma, g, k_f, k_\infty) \in G_{\mathbb{Q}} \times G_{\mathbb{A}} \times K_f \times K_\infty$, $(\gamma_1, h, u_f, u_\infty) \in H_{\mathbb{Q}} \times H_{\mathbb{A}} \times U_f \times U_\infty$ and $(\gamma_2, h', u'_f, u'_\infty) \in H'_{\mathbb{Q}} \times H'_{\mathbb{A}} \times U'_f \times U'_\infty$. It is also verify that $\theta^\kappa$ is $Z_{G,\mathbb{A}} \times Z_{H,\mathbb{A}} \times Z_{H',\mathbb{A}^{-}}$-invariant.

For $f \in S_\kappa(D)$ and $f' \in A_\kappa$, we set
\[
\mathcal{L}(f, f')(g) := \int_{Z_{H,\mathbb{A}}H_{\mathbb{Q}} \backslash H_{\mathbb{A}}} dh \int_{Z_{H',\mathbb{A}}H'_{\mathbb{Q}} \backslash H'_{\mathbb{A}}} dh' \theta^\kappa(g, h, h') \overline{f(h)} f'(h') \quad (g \in G_{\mathbb{A}}). \tag{2.7}
\]

**Theorem 2.2 (Arakawa, Narita).** Suppose $\kappa > 6$.

(i) The integral (2.7) is absolutely convergent.

(ii) $\mathcal{L}(f, f')(g) \in S_{\kappa}$.

**Proof.** Since $G_{\mathbb{A}} = Z_{G,\mathbb{A}} G_{\mathbb{Q}} G_{\infty}^1 K_f$ (cf. [Shim-2, Theorem 6.14]), it is sufficient to consider the restriction of $\mathcal{L}(f, f')$ to $G_{\infty}^1$. By a standard argument, we see that $\mathcal{L}(f, f')_{| G_{\infty}^1}$ is a finite linear combination of original Arakawa lift (cf. [Ar-1], [N-1, §4] and [N-3, Theorem 4.1]), from which the theorem follows.

**Remark 2.3.** At the Archimedean place our lifting reads a correspondence between the quaternionic discrete series of $G_{\infty}^1$ with minimal $K_\infty$-type $\tau_\kappa$ and the discrete series representation of $O^*(4) \simeq SL_2(\mathbb{R}) \times SU(2)$ given by the direct product of the holomorphic discrete series of $SL_2(\mathbb{R})$ with Blattner parameter $\kappa$ and the representation $\sigma_\kappa$ of $Sp^*(1) \simeq SU(2)$. This is compatible with the result [J] of J. S. Li on theta correspondences for unitary representations with non-zero cohomology (cf. [J, §6, (I_1)]).

For the case of $GSp(1, q)$ we would be able to give an adelic reformulation of the lifting similarly. In view of [N-3, Theorem 4.1] and [J, §6, (I_1)], the weight of elliptic cusp forms or the Blatter parameter of the holomorphic discrete series of $SL_2(\mathbb{R})$ should be $\kappa - 2q + 2$ for this general case.

**3 Main results**

**3.1**

To state our results, we need to review several facts on Hecke operators.
3.2

First we consider the case where $p \nmid d_B$. We fix an isomorphism of $B_p$ onto $M_2(\mathbb{Q}_p)$ such that $\mathcal{O}_p$ maps onto $M_2(\mathbb{Z}_p)$ and that the main involution of $B_p$ corresponds to an involution of $M_2(\mathbb{Q}_p)$ given by

$$M_2(\mathbb{Q}_p) \ni X \mapsto w^{-1}Xw, \quad (w = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})).$$

The reduced trace $\text{tr}$ corresponds to the trace $\text{Tr}$ of $M_2(\mathbb{Q}_p)$. We henceforth identify $B_p$ with $M_2(\mathbb{Q}_p)$ using the above isomorphism. Then $G_p, K_p, H_p'$ and $U_p'$ are identified with $GSp(J, \mathbb{Q}_p), GSp(J, \mathbb{Z}_p), GL_2(\mathbb{Q}_p)$ and $GL_2(\mathbb{Z}_p)$ respectively, where $GSp(J)$ is the group of similitudes of $J = (\begin{smallmatrix} 0 & \nu \\ \nu & 0 \end{smallmatrix})$. Note that we can identify $U_p$ with $U_p'$ by the isomorphism $B_p \simeq M_2(\mathbb{Q}_p)$ fixed above.

Define Hecke operators $T_p^i$ ($i = 0, 1, 2$) on $S_\kappa$ by

$$T_p^i F(g) = \int_{G_p} F(gx) \Phi_p^i(x) dx,$$

where $\Phi_p^0$, $\Phi_p^1$ and $\Phi_p^2$ are the characteristic function of $K_p \cdot \text{diag}(p, p, p) K_p$, $K_p \cdot \text{diag}(p, p, 1, 1) K_p$ and $K_p \cdot \text{diag}(\Pi, \Pi) K_p$ respectively. Note that $(T_p^0)^2 F = F$ for any $F \in S_\kappa$.

We also define Hecke operators $T_p$ and $T_p'$ on $S_\kappa(D)$ and $A_\kappa$ by

$$T_p f(h) = \int_{H_p} f(hx) \phi_p(x) dx,$$

$$T_p' f'(h') = \int_{H_p'} f'(h'x') \phi'_p(x') dx',$$

where $\phi_p = \phi'_p$ is the characteristic function of $GL_2(\mathbb{Z}_p) \cdot \text{diag}(p, 1) GL_2(\mathbb{Z}_p)$.

3.3

We next consider the case where $p|d_B$, i.e., $B_p$ is a division algebra. In this case, we fix a prime element $\Pi$ of $B_p$ and put $\pi := n(\Pi)$. Then $\pi$ is a prime element of $\mathbb{Q}_p$.

Define Hecke operators $T_p^i$ ($i = 0, 1$) on $S_\kappa$ by

$$T_p^i F(g) = \int_{G_p} F(gx) \Phi_p^i(x) dx,$$

where $\Phi_p^0$ and $\Phi_p^1$ are the characteristic functions of $K_p \cdot \text{diag}(\Pi, \Pi) \cdot K_p$ and $K_p \cdot \text{diag}(1, \pi) K_p$ respectively. Note that $(T_p^0)^2 F = F$ for any $F \in S_\kappa$. We also define Hecke operators $T_p$ and $T_p'$ on $S_\kappa(D)$ and $A_\kappa$ by

$$T_p f(h) = \int_{H_p} f(hx) \phi_p(x) dx,$$

$$T_p' f'(h') = \int_{H_p'} f'(h'x') \phi'_p(x') dx'.$$
Here $\phi_p'$ is the characteristic function of $U_p'\Pi U' = \Pi U_p'$ and $\phi_p$ is defined as follows: If $p|D$, $\phi_p$ is the sum of the characteristic functions of $U_p(0\ 0\ 1)U_p$ and $U_p(0\ 0\ p)U_p$. If $p \nmid D$, $\phi_p$ is the characteristic function of $U_p(0\ 0\ 1)U_p$.

3.4

We say that $F \in S_\kappa$ is a Hecke eigenform if $F$ is a common eigenfunction of the Hecke operators $T_p^i$ for any $p < \infty$. Let $F \in S_\kappa$ be a Hecke eigenform with $T_p^i F = \Lambda_p^i F$ ($\Lambda_p^i \in \mathbb{C}$). We define the spinor $L$-function of $F$ by

$$L(F, \text{spin}, s) = \prod_{p<\infty} L_p(F, \text{spin}, s),$$

where $L_p(F, \text{spin}, s) = Q_p(F, p^{-s})^{-1}$,

$$Q_p(F, t) = \begin{cases} 1 - p^{\kappa-3}\Lambda_p^1 t + p^{2\kappa-5}(\Lambda_p^2 + p^2 + 1)t^2 - p^{3\kappa-6}\Lambda_p^1 t^3 + p^{4\kappa-6}t^4 & \text{if } p \nmid D_B, \\ 1 - (p^{\kappa-3}\Lambda_p^1 - p^{\kappa-3}(p^A - 1)\Lambda_p^0)t + p^{2\kappa-3}(\Lambda_p^0)^2t^2 & \text{if } p|D_B, \\ \end{cases}$$

and

$$A_p = \begin{cases} 1 & \text{if } p \nmid D, \\ 2 & \text{if } p|D. \\ \end{cases}$$

The Euler factor for $p \nmid D_B$ (resp. $p|D_B$) is given by the formula for the denominator of the Hecke series in [Shim-1, Theorem 2] (resp. [H-S, §4] and [Su, (1-34)]), under the normalization of the Hecke eigenvalues

$$\left\{(\Lambda_p^0, \Lambda_p^1, \Lambda_p^2) \to (p^{2(\kappa-3)}\Lambda_p^0, p^{\kappa-3}\Lambda_p^1, p^{2(\kappa-3)}\Lambda_p^2) \ (p \nmid D_B) \right\}$$

$$\left\{(\Lambda_p^0, \Lambda_p^1) \to (p^{\kappa-3}\Lambda_p^0, p^{\kappa-3}\Lambda_p^1) \ (p|D_B) \right\}.$$

We say that $f \in S_\kappa(D)$ (resp. $f' \in A_\kappa$) is a Hecke eigenform if $f$ (resp. $f'$) is a common eigenfunction of $T_p$ (resp. $T_p'$) for any $p < \infty$. For Hecke eigenforms $f \in S_\kappa(D)$ and $f' \in A_\kappa$ with $T_p f = \lambda_p f$ and $T_p f' = \lambda'_p f'$ ($\lambda_p, \lambda'_p \in \mathbb{C}$), we define $L$-functions

$$L^D(f, s) = \prod_{p \nmid D} \left(1 - \lambda_p p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1},$$

$$L^{D_B}(f', s) = \prod_{p \mid D_B} \left(1 - \lambda'_p p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1}.$$

When $D = 1$, we write $L(f, s)$ for $L^D(f, s)$, which is the usual Hecke $L$-function of $f$. 
We are now able to state the main result.

**Theorem 3.1.** Let \( f \in S_\kappa(D) \) and \( f' \in A_\kappa \), and suppose that
\[
T_p f = \lambda_p f, \\
T'_p f' = \lambda'_p f',
\]
for each \( p < \infty \). Then \( F(g) := \mathcal{L}(f, f')(g) \) is a Hecke eigenform and the Hecke eigenvalues are given as follows:

(i) If \( p \nmid d_B \), we have
\[
T^0_p F = F, \\
T^1_p F = (p\lambda_p + p\lambda'_p) F, \\
T^2_p F = (p\lambda_p\lambda'_p + p^2 - 1) F.
\]

(ii) If \( p \mid d_B \), we have
\[
T^0_p F = \lambda'_p F, \\
T^1_p F = (p\lambda_p + (p - 1)\lambda'_p) F.
\]

**Remark 3.2.** Noting that the elliptic cusp forms are assumed to have the trivial central character, we see that all the Hecke operators above for the cusp forms are self-adjoint with respect to the Petersson inner product. We can thus remove the complex conjugates of their Hecke eigenvalues in the formula above.

**Corollary 3.3.** Let \( f \) and \( f' \) be as in Theorem 3.1. Then we have
\[
\mathcal{L}(f, f')(\text{spin}, s) = \mathcal{L}^D(f, s) \mathcal{L}^{d_B}(f', s) \prod_{p\mid D} (1 - \{\lambda_p + (1 - p)\lambda'_p\}p^{\kappa-2-s} + p^{2\kappa-3-2\epsilon})^{-1}.
\]

In particular, if \( D = 1 \), we have
\[
\mathcal{L}(f, f')(\text{spin}, s) = \mathcal{L}(f, s) \mathcal{L}^{d_B}(f', s).
\]

The results above are deduced from the commutation relation of Hecke operators for the metaplectic representation \( r \) as follows:

**Proposition 3.4.** For a function \( \phi \) on \( H_p \), we put \( \hat{\phi}(h) = \phi(h^{-1}) \) \( (h \in H_p) \). We define \( \hat{\phi}' \) for \( \phi' : H'_p \to \mathbb{C} \) in a similar manner.

(1) Suppose that \( p \nmid d_B \). Then we have
(i) $r(\Phi^1_p, 1, 1) \varphi_{0,p} = p \cdot r(1, \Phi_p, 1) \varphi_{0,p} + p \cdot r(1, 1, \Phi'_p) \varphi_{0,p}$,
(ii) $r(\Phi^2_p, 1, 1) \varphi_{0,p} + (1 - p^2) r(\Phi^0_p, 1, 1) \varphi_{0,p} = p \cdot r(1, \Phi_p, \Phi'_p) \varphi_{0,p}$

(2) Suppose that $p \mid d_B$. Then we have

$r(\Phi^0_p, 1, 1) \varphi_{0,p} = r(1, 1, \Phi'_p) \varphi_{0,p}$,
$r(\Phi^1_p, 1, 1) \varphi_{0,p} = p \cdot r(1, \Phi_p, 1) \varphi_{0,p} + (p - 1) r(1, 1, \Phi'_p) \varphi_{0,p}$

Remark 3.5. When $p \nmid d_B$ the formula for the Hecke eigenvalues is essentially the same as the corresponding result of Yoshida lifting (cf. [Y, Theorem 6.1]). For such $p$ this leads to the following decomposition

$L_p(\mathcal{L}(f, f'), \text{spin}, s) = (1 - \lambda_p p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1} (1 - \lambda'_p p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}$

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