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Kyoto University
On the Hecke eigenvalues of Siegel cusp forms of genus 2

Winfried Kohnen

Denote by $S_k(\Gamma_1)$ be the space of cusp forms of integral weight $k$ on the full modular group $\Gamma_1 := SL_2(\mathbb{Z})$ and let $f \in S_k(\Gamma_1)$ be a normalized Hecke eigenform. Denote by $\lambda(n) (n \in \mathbb{N})$ the Hecke eigenvalues of $f$. Then using a classical theorem of Landau together with the analytic properties of the Hecke $L$-function $L(f, s)$ and the Rankin-Selberg zeta function attached to $f$ it is not difficult to see that the sequence $(\lambda(n))_{n\in\mathbb{N}}$ changes sign infinitely often, i.e. there are infinitely many $n$ such that $\lambda(n) > 0$ and there are infinitely many $n$ such that $\lambda(n) < 0$. Indeed, this is true for the Fourier coefficients of any non-zero cusp form of any level (supposing that these coefficients are real).

A very natural question to ask is to what extent this result generalizes to Siegel modular forms. Here we consider the simplest case, namely the case of genus 2.

Let $S_k(\Gamma_2)$ be the space of Siegel cusp forms of integral weight $k$ on $\Gamma_2 := Sp_2(\mathbb{Z}) \subset GL_4(\mathbb{Z})$ and let $F \in S_k(\Gamma_2)$ be a non-zero Hecke eigenform. Denote by $\lambda(n) (n \in \mathbb{N})$ the eigenvalues of $F$ under the usual Hecke operators $T(n) (n \in \mathbb{N})$.

Note that the $\lambda(n)$ are no longer “proportional” (in any reasonable sense) to the Fourier coefficients of $F$.

One has

\[(1) \quad \sum_{n \geq 1} \lambda(n)n^{-s} = \zeta(2s - 2k + 4)^{-1}Z_F(s) \quad (\Re(s) >> 0)\]

where

\[Z_F(s) = \prod_p Z_{F,p}(p^{-s})^{-1} \quad (\Re(s) >> 0)\]

is the spinor zeta function of $F$. Here

\[Z_{F,p}(X) = (1 - \alpha_{0,p}X)(1 - \alpha_{0,p}\alpha_{1,p}X)(1 - \alpha_{0,p}\alpha_{2,p}X)(1 - \alpha_{0,p}\alpha_{1,p}\alpha_{2,p}X)\]

and $\alpha_{0,p}, \alpha_{1,p}$ and $\alpha_{2,p}$ are “the” Satake $p$-parameters of $F$ (cf. [1]).

If $k$ is even let $S_k^*(\Gamma_2) \subset S_k(\Gamma_2)$ be the Maass subspace, in other words the subspace spanned by the images of the Saito-Kurokawa lifts of Hecke eigenforms in $S_{2k-2}(\Gamma_1)$. Recall that $S_k^*(\Gamma_2)$ is Hecke-invariant and for a non-zero Hecke eigenform $F \in S_k^*(\Gamma_2)$ there exist a unique normalized Hecke eigenform $f \in S_{2k-2}(\Gamma_1)$ such that

\[(2) \quad Z_F(s) = \zeta(s - k + 1)\zeta(s - k + 2)L(f, s).\]
Theorem 1 [2]. Let $k$ be even and let $F \in S_k^*(\Gamma_2)$ be a non-zero Hecke eigenform. Then $\lambda(n) > 0$ for all $n$.

The proof follows from explicitly exploiting the relations given by (2) between the $\lambda(n)$ and the eigenvalues of the form $f$ and using Deligne's theorem (previously the Ramanujan-Petersson conjecture) for the latter.

Theorem 2 [4]. Let $F \in S_k(\Gamma_2)$ be a non-zero Hecke eigenform and suppose that $F$ is in the orthogonal complement of the space $S_k^*(\Gamma_2)$ if $k$ is even. Then the sequence $(\lambda(n))_{n \in \mathbb{N}}$ has infinitely many sign changes.

The proof uses (1) together with the analytic properties of the spinor zeta function $Z_F(s)$ coupled with the fact that the generalized Ramanujan-Petersson conjecture for $F$ as considered is true (as proved by Weissauer), i.e. one has

$$|\alpha_{1,p}| = |\alpha_{2,p}| = 1 \quad (\forall p).$$

For details we refer to [4].

Taking Theorem 2 for granted, a natural question is when the first negative eigenvalue occurs. Extending previous work in the case of elliptic modular forms [3], it seems possible that one can prove that there exists

$$n \ll \epsilon k^{2+\epsilon}$$

such that $\lambda(n) < 0$ for $F$ as in Theorem 2, where the constant implied in $\ll \epsilon$ depends only on $\epsilon$. For details we refer to [5].

References


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