数理解析研究所講究録
1523 巻 2006 年 1-11

On the basis problem for Siegel modular forms of squarefree level

by SiegfriedBöcherer

Abstract

This is a report on joint work with H.Katsurada and R.Schulze-Pillot ; we also use some recent results by Y.Hironaka and F.Sato [12] generalizing parts of [14]; this allows us to treat general squarefree levels (not only prime levels).

The most familiar examples of (holomorphic) Siegel modular forms are theta series attached to lattices in a positive definite quadratic space. The basis problem for Siegel (cuspidal) modular forms asks whether one can get all cusp forms of a given congruence subgroup of type $\Gamma_0(N)$ as linear combinations of such theta series attached to lattices of level dividing N. For level N=1 this was solved affirmatively provided that the weight is divisible by 4 and large enough [2]. For higher levels only for degree 1 there are definite results due to the deep work of Waldspurger [17]. If one wants to follow the lines of thought of these works in the case of arbitrary degree n and level N one has to consider for a given cusp form f of degree n and weight k an integral of type

$$\Lambda_E^{2n,k}(f)(z) := \int_{\Gamma_0(N)\backslash\mathbb{H}_n} f(w)\overline{E^{2n}(\begin{pmatrix} w & 0 \\ 0 & -\bar{z} \end{pmatrix})}det(w)^k d^*w$$

where E is an appropriate Eisenstein series of degree $2n$ and weight k. In the case of nontrivial levels one has to make a choice. The most ambitious choice is to take E to be the genus theta series of a fixed genus of quadratic forms. There will in general be delicate problems related with the bad primes and there seems at the moment not much hope to do this in general (see however [4] where some cases are done from this point of view with some pain). The problem of the contribution of bad primes to such integrals is also addressed in other works (mostly with aims different from ours, see e.g. [16]).

In this work we have a more modest aim. There is one type of Eisenstein series, for which we can compute $\Lambda_E(f)$ in a particularly simple fashion, namely

$$F_k^n(Z,\chi) := \sum_{C,D} \chi(det(C))det(CZ+D)^{-k}$$

where χ is any Dirichlet character modulo N, $N > 1$. The integral was computed explicitely in [5, 7]. It can be zero, if the level is not squarefree; we will be interested only in the squarefree case; as essential new tools (not available when [5, 7] were written) we use

- the injectivity of the Hecke operators $U(p)$ if $p \mid N$ [3]
- the Eisenstein series F is a linear combination of genus theta series for genera of positive quadratic forms of levels dividing N (at least if the weight k is large enough and of course the character has to be quadratic) ([14] and more generally [12]).

The first of these statements allows us (when combined with the information from [5, 7] about the contribution of the bad primes, where $U(p)$ appears in the "numerator") to deduce the *injectivity* of the map $\Lambda_F^{2n,k}$ on the space of cusp forms in question, provided that the weight is large enough. It was somewhat surprising for us that we do not need any kind of newform argument here. The second statement implies that the *image* of $\Lambda_F^{2n,k}$ consists of linear combinations of theta series.
Our results (see §3 for precise statements) say that Siegel cusp forms of squarefree level and quadratic nebentypus are linear combinations of appropriate theta series, provided that the weight is large enough. Here "appropriate" means that the theta series in question have level (dividing) N and they also have the correct nebentypus. We have to allow all such theta series; in particular, we cannot fix the genus of the quadratic forms, even the quadratic space cannot be specified! From the point of view of theta liftings, this is not satisfying, because we have possibly to deal with automorphic forms on several orthogonal groups simultaneously! In this way we can avoid more delicate questions about the bad primes.
It should be clear (at least to experts) that our methods immediately (but with much more burden concerning terminology) carry over to theta series with harmonic coefficients and also to the case of vector valued modular forms. Also non-cusp forms can be included. We will briefly describe the main modifications in section 4.

There are a few cases, where the simple Eisenstein series F is indeed a genus theta series (possibly after "Hecke summation"). One such example is the case of degree $n \geq 3$, weight 2, trivial character and the genus of quaternary quadratic forms of level p arising from a definite quaternion algebra; this case

is (at least implicitly) considered in [5].
In the last section we consider another such case where our method works also outside the range of convergence, namely for modular forms of prime level $p \equiv 3 \bmod 4$ and primitive nebentypus. Here we can give a characterization of those modular forms which can be represented by (positive definite) binary quadratic forms of level p in terms of L -functions; we rediscover in explicit form some of the consequences of the Gelbart-Jacquet lift [10].
Finally we point out that our overall strategy is not new at all. E.g. we used it before [5, §4] to consider the basis problem for small weights $\frac{n}{2} \leq k \leq n$ (again in the squarefree case). The property of the Eisenstein series F (after Hecke summation!) to be a linear combination of (genus-) theta series was in that case a consequence of the theory of singular modular forms. The injectivity of $U(p)$ allows to reformulate some of the statements there in a more elegant way (in particular [5, Theorem 4.1]).

§1 Preliminaries

For basic facts about Siegel modular forms we refer to [1, 9, 11]. The group $GSp(n,\mathbb{R})$ acts on the upper half space $\mathbb{H}_n$ in the usual way. Let $\rho : GL(n,\mathbb{C}) \longrightarrow V_\rho$ be a finite dimensional (irreducible) polynomial representation. Then we define the slash-operator for functions $f : \mathbb{H}_n \longrightarrow V_\rho$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ by

$$(f \mid_\rho M)(Z) := (\sqrt{\mu(M)})^{\sum \lambda_i} \rho(CZ+D)^{-1} f((AZ+B)(CZ+D)^{-1})$$

Here $\mu(M)$ denotes the similitude factor of M and $(\lambda_1, \ldots, \lambda_n)$ denote the weight of the representation ρ. In the case of one-dimensional representations, we write just k instead of det^k. For a natural number N we put $\Gamma_0(N) := \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{Z}) \mid C \equiv 0 \bmod N\}$; we view a Dirichlet character mod N as a character of $\Gamma_0^n(N)$ by $\chi(M) := \chi(det(D))$. Then the space of Siegel modular forms of degree n, weight ρ and character χ for the congruence subgroup $\Gamma_0(N)$ is the space of all holomorphic functions $f : \mathbb{H} \longrightarrow V_\rho$ which satisfy

$$f \mid_\rho M = \chi(M) f$$

for all $M \in \Gamma_0^n(N)$. We denote this space by $[\Gamma_0^n(N), \rho, \chi]$ and the subspace of cuspforms by $[\Gamma_0^n(N), \rho, \chi]_0$. In the case of scalar-valued weights we just write k instead of det^k.

§2. The pullback formula for F

Here we just recall the result of a computation done in [5, 7]. We can use a more general framework for the moment: Let $N > 1$ be arbitrary, χ an arbitrary Dirichlet character mod N and k a positive number with $\chi(-1) = (-1)^k$. Then for a complex number s with $k + 2\Re(s) > n+1$ the degree n Eisenstein series

$$F_k^n(Z,\chi,s) := \sum_{C,D} \chi(det(C))det(CZ+D)^{-k}\frac{\det(Y)^s}{|det(CZ+D)|^{2s}}$$

converges absolutely and uniformly in domains of type $\Im(Z) \geq \lambda \cdot 1_n$ For a cusp form $f \in [\Gamma_0^n(N), k, \chi]_0$ we consider

$$\Lambda_F^{2n,k}(f)(z,s) := \int_{\Gamma_0(N)\backslash\mathbb{H}_n} f(w)\overline{F_k^{2n}(\begin{pmatrix} w & 0 \\ 0 & -\bar{z} \end{pmatrix}, \chi, \bar{s})}det(w)^k d^*w$$

The result can be formulated completely linearly, but we prefer to assume that f is an eigenform for all Hecke operators coming from Hecke pairs $(Sp(n,\mathbb{Q}_p), Sp(n,\mathbb{Z}_p))$ for all "good primes" p (i.e.comprime to N). Then we can associate to such an eigenform f the standard L-function

$$L^N(f,s) := \prod_{(p,N)=1} \frac{1}{(1-\chi(p)p^{-s})} \prod_{i=1}^{n} \frac{1}{(1-\chi(p)\alpha_i(p)p^{-s})(1-\chi(p)\alpha_i(p)^{-1}p^{-s})}$$

Here the α_i denote the Satake parameters attached to the eigenform f. It is well known that this (partial) Euler product converges absolutely for $\Re(s) >> 0$ and has a meromorphic continuation to the whole complex plane. We also need Hecke operators for the bad primes; we describe them in greater detail: Let

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \qquad (d_i \mid d_{i+1})$$

be an (integral) elementary divisor matrix with $det(D) \mid N^\infty$. Then

$$GL(n,\mathbb{Z}) \cdot D \cdot GL(n,\mathbb{Z}) \longmapsto \Gamma_0(N) \begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix} \cdot \Gamma_0(N)$$

induces an embedding of a GL(n)-Hecke algebra into the Hecke algebra of the pair $(\Gamma_0(N), Sp(n,\mathbb{Z}[\frac{1}{N}])$. For D as above, we define the Hecke operator $T(D)$ on $[\Gamma_0(N),k,\chi]_0$ by

$$f \mid T(D) := \sum_i \chi(\det(D)\det(\alpha_i)) f \mid_k \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

where

$$\Gamma_0(N)\begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix} \cdot \Gamma_0(N) = \dot{\bigcup_i} \Gamma_0(N) \cdot \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

(one may choose representatives with $\gamma_i = 0$).
Then the formulas (2.37) and (3.23) from [7] give for an eigenfunction f for all good Hecke operators the formula ($\Re(s) >> 0$)

$$\Lambda_F^{2n,k}(f)(z,s) = \Omega(s)\frac{N^{\frac{n(n+1)-nk}{2}}}{\mathcal{L}^N(k+2s,\chi)} \cdot L^N(f \mid_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, k+2s-n) \times$$

$$\sum_D f \mid_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}(z) \mid U(N) \cdot T(D) det(D)^{-k-2s}$$

Here D runs over all elementary divisor matrices with $\det(D) \mid N^\infty$, Ω is essentially a Γ-factor

$$\Omega(s) = (-1)^{\frac{nk}{2}} 2^{\frac{n(n+1)}{2}+1-2ns} \pi^{\frac{n(n+1)}{2}} \frac{\Gamma_n(1+s-\frac{n}{2})\Gamma_n(1+s-\frac{n(n+1)}{2})}{\Gamma_n(k+s)\Gamma_n(k+s-\frac{n}{2})}$$

and $\mathcal{L}^N(s)$ comes from a normalizing factor of the Eisenstein series:

$$\mathcal{L}^N(s,\chi) = L^n(s,\chi)\prod_{i=1}^{n} L^N(2s-2i,\chi^2).$$

To analyse this formula, we should mention the following well known facts

- $U(N)$ commutes with all the $T(D)$
- The $T(D)$ are *weakly* multiplicative, i.e.

  $$T(D_1 \cdot D_2) = T(D_1) \circ T(D_2)$$

  if $\det(D_1)$ and $\det(D_2)$ are coprime.

- There is Tamagawa's rationality theorem:

  For $p \mid N$ we write $T_p(i_1, \ldots, i_n)$ instead of $T\left(\begin{pmatrix} p^{i_1} & & \\ & \ddots & \\ & & p^{i_n} \end{pmatrix}\right)$ and

  $\pi(p)_{n,i} := T_p(\underbrace{1, \ldots 1}_{i}; \underbrace{0, \ldots, 0}_{n-i})$; then

$$\sum_{0 \le i_1 \le \cdots \le i_n} T_p(i_1, \ldots, i_n)\, X^{i_1 + \cdots + i_n} = \frac{1}{\sum_{i=0}^{n} (-1)^i p^{\frac{i(i-1}{2}} \pi(p)_{n,i} X^i}$$

- The operator $U(N)$ is injective on $[\Gamma_0(N)^n, k, \chi]_0$; we may therefore consider its inverse $U(N)^{-1}$ on this space.

Finally let us denote by $T_N(s)$ the endomorphism of $[\Gamma_0^n(N), k, \chi]_0$ defined by

$$f \longmapsto f \mid T_N(s) := f \mid \left(\prod_{p|N} \sum_{i=0}^{n} (-1)^i p^{\frac{i(i-1)}{2}} \pi(p)_{n,i} p^{-is} \right)$$

and by W_N the isomorphism $[\Gamma_0^n(N), k, \chi]_0 \simeq [\Gamma_0^n(N), k, \overline{\chi}]_0$ defined by the "Fricke involution"

$$f \longmapsto f \mid W_N := f \mid_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

Then the integral formula from above can be rewritten as

$$\Lambda_F^{2n,k}(f \mid T_N(k+2s)^{-1} \mid U(N)^{-1} \mid W_N^{-1}, s) = \Omega(s) \cdot L^n(f, k+2s) \cdot f(z)$$

Only a simple very special consequence of the formula above will be needed later on:

Proposition: *Let χ be a quadratic character. Assume that k is large enough such that*

- *The Γ-factor $\Omega(s)$ has neither pole nor zero in $s = 0$*

- *For all eigenforms $f \in [\Gamma_0(N), k, \chi]_0$ of the Hecke operators at the good places, the Euler product $L^N(f, s)$ converges absolutely in $s = k$.*

Then

$$f \longmapsto \Lambda_F^{2n,k}(F \mid T(N)(k)^{-1} \mid U(N)^{-1} \mid W_N^{-1}, 0)$$

defines an automorphism Λ *of the space* $[\Gamma_0^n(N), k, \chi]_0$

Remark: The conditions on k are certainly satisfied for $k > 2n+1$ (standard elementary estimate [1]); more sophisticated estimates show that $k > \frac{5n}{2} + 1$ is sufficient [8]. Any progress towards Ramanujan-Petersson will improve this bound.

§3. Theta series

We start with the following elementary
Observation: Let N be squarefree and let that χ be a quadratic character mod N and assume that $F_k^{2n}(Z, \chi, 0)$ is a linear combination of theta series attached to appropriate lattices L_i in positive definite quadratic spaces; here "appropriate " means that the rank of L_i is 2k, the levels of L_i divides N and the nebentypus characters fit, i.e. $\chi = \left(\frac{(-1)^k det(L)}{\cdot}\right)$:

$$F_k^{2n}(Z, \chi, 0) = \sum_i a_i \vartheta^{2n}(L_i, Z)$$

Under these assumptions, for all $f \in [\Gamma_0^n(N), k, \chi]_0$

$$\Lambda_F^{2n,k}(f) = \sum_i \overline{a_i} < f, \vartheta^n(L_i) > \cdot \vartheta^n(L_i)$$

At the moment we do not have to care about the nature of the coefficients a_i (they will only depend on the genus of the L_i). The only interesting point here is that the image of $\Lambda_F^{2n,k}$ consists of theta series!

Theorem (Katsurada/Schulze-Pillot and Hironaka/Sato):
The assumptions above are true for $k > 2n + 1$*; more precisely, all Siegel Eisenstein series (of weight k, level* N *with* N *squarefree and quadratic nebentypus* χ*) can be written as linear combinations of genus theta series.*

Combining the proposition of §2, the observation and the Theorem from above we obtain
Theorem: *Assume that* N *is squarefree and* $k > 2n+1$*; then all cusp forms in* $[\Gamma_0^n(N), k, \chi]$ *are linear combination of theta series.*

§4 Variants

§4.1 Holomorphic differential operators

The calculus of holomorphic differential operators as described in [13] and already used in [6] allows us to extend our results to the case of vector-valued modular forms and theta series with harmonic coefficients.
We need these differential operators here only for the "convergent case" (i.e. we apply it to an Eisenstein series of degree 2n, weight k with $s = 0$ and $k > 2n + 1$. We give a very short summary of the main facts needed here. For details we refer to [11, 13, 6]. For a polynomial represention $\rho = det^k \otimes \rho_0$ there is a holomorphic differential operator $\mathcal{D}_{k,\rho}$ acting on $\mathbb{H}_{2n}$, which is a polynomial in the partial derivatives, evaluated for $z_2 = 0$. This operator maps C^∞-functions F on $\mathbb{H}_{2n}$ to $V_\rho \otimes V_\rho$ valued functions on $\mathbb{H}_n \times \mathbb{H}_n$ and satisfies for all $M \in Sp(n, \mathbb{R})$

$$\begin{aligned} \mathcal{D}_{k,\rho}(F \mid_k M^{\uparrow}) &= (\mathcal{D}_{k,\rho}F) \mid_\rho^{(1)} M \\ \mathcal{D}_{k,\rho}(F \mid_k M^{\downarrow}) &= (\mathcal{D}_{k,\rho}F) \mid_\rho^{(2)} M \end{aligned}$$

Here the arrows $\uparrow$ and $\downarrow$ denote the standard embeddings of $Sp(n)$ into $Sp(2n)$ and on the right side the upper indices (1) and (2) indicate to which component of $\mathbb{H}_n \times \mathbb{H}_n$ the element M has to be applied.
We apply such a differential operator to our Eisenstein series F and integrate then against a vector-valued cusp form of weight ρ.
Now we denote by $\theta^n(m, \rho, N, \chi)$ the vector space generated by (V_ρ- valued) theta series for positive definite quadratic forms of rank $m = 2k$ and level dividing N with character χ. Then by the same kind of argument as before, we obtain

Theorem:

$$[\Gamma_0^n(N), \rho, \chi]_0 \subset \theta^n(m, \rho, \chi) \quad \text{if} \quad \frac{m}{2} \geq 2n + 2.$$

§4.2 : The case of non-cusp forms

So far, we have only considered cusp forms (and Siegel Eisenstein series). To include non-cusp forms, we have to consider pullbacks involving Eisenstein series (and differential operators) for

$$\mathbb{H}_r \times \mathbb{H}_n \hookrightarrow \mathbb{H}_{n+r} \qquad (0 \leq r \leq n)$$

Also, there are several Siegel ϕ-operators to be considered, therefore the Eisenstein series of type F will not be sufficient, we will need some variants of them (still with the same kind of Hecke operators). We arrive at
Theorem:

$$[\Gamma_0^n(N), \rho, \chi] = \theta^n(m, \rho, \chi) \quad \text{if} \quad \frac{m}{2} \geq 2n + 2.$$

§5 The case of binary quadratic forms
We include this section for two reasons; first of all (as already mentioned in the introduction) it is one of the cases where our method really uses the genus theta series (and we are out of the range of convergence of both the Eisenstein series and the Euler product); the second reason is that the result (about binary theta series of degree one) seems not to be known to many number theorists.

Here p should be a prime congruent to 3 modulo 4 and $\chi_p := \left(\frac{\div}{p}\right)$ the quadratic character mod p. Then the Eisenstein series $F_1^2(Z, \chi_p, s)$ has a pole of first order in $s_0 = \frac{1}{2}$ and

$$\mathcal{F}_p(Z) := Res_{s=s_0} F_1^2(Z, \chi_p, s)$$

defines a holomorphic modular form of weight 1; moreover there is a constant $c_p \neq 0$ such that

$$\mathcal{F}_p = c_p \sum_{i=1}^{h(-p)} \vartheta^2(S_i, Z)$$

where S_i runs over representatives of the integral equaivalence classes of binary quadratic forms of discriminant $-p$; most of these statements are part of the "folklore", for an explicit statement we refer to [15].

Theorem: *Assume that $f \in [\Gamma_0(p), 1, \chi_p]_0$ is an eigenform of all Hecke operators. Then*

$$\begin{aligned} &f \text{ is a linear combination of binary theta series } \iff \\ &L^p(f, s) \text{ has a simple pole in } s = 1. \end{aligned}$$

As before, using differential operators, we obtain
Theorem' *Assume that $f \in [\Gamma_0(p), k, \chi_p]_0$ is a Hecke eigenform for all Hecke*

operators. Then

$$\begin{aligned}&f \text{ is a linear combination of binary theta series}\\&\text{(possibly with harmonic polynomials)}\\&\quad\Longleftrightarrow\quad L^p(f,s) \text{ has a simple pole in } s=1.\end{aligned}$$

In principle, such statements are known from Gelbart-Jacquet [10]. One can easily reformulate this in terms of the complete L-function.

References

[1] Andrianov, A.: Quadratic Forms and Hecke Operators. Grundlehren 286, Springer 1987

[2] Böcherer, S.: Siegel modular forms and theta series. In: THETA FUNCTIONS. Proc.Symp.Pure Math. 49 (1989)

[3] Böcherer, S. On the Hecke operator U(p). J.Math.Kyoto Univ. 45,807-829 (2005)

[4] Böcherer, S.: The genus version of the basis problem for elliptic modular forms. In: Automorphic Forms and Zeta Funktions. Proceedings of the conference in memory of T.Arakawa. World Scientific 2005

[5] Böcherer, S., Schulze-Pillot, R.: Siegel modular forms and theta series attached to definite quaternion algebras. Nagoya Math.J. 121, 35-96 (1991)

[6] Böcherer, S., Schulze-Pillot, R.: Siegel modular forms and theta series attached to definite quaternion algebras II. Nagoya Math.J.147, 71-106(1997)

[7] Böcherer, S., Schmidt, C.G.: p-adic measures attached to Siegel modular forms. Ann.Inst.Fourier 50,1375-1443(2000)

[8] Duke, W., Howe, R., Li, J.-S.: Estimating Hecke eigenvalues of Siegel modular forms. Duke Math.J. 67, 219-240(1992)

[9] Freitag, E.: Siegelsche Modulfunktionen. Grundlehren 254, Springer 1983

[10] Gelbart, S., Jacquet, H.: A relation between automorphic forms on $GL(2)$ and $GL(3)$. Ann.Scient.Ec.Norm.Sup.11, 471-552(1978)

[11] Godement, R.: Exp.4-9 Seminaire Cartan 10 (1957/58)

[12] Hironaka, Y., Sato, F.: Linear independence of local densities. Notes 2005

[13] Ibukiyama, T.:On differential operators on automorphic forms and invariant pluriharmonic polynomials. Comm.Math.Univ.St.Pauli 48, 103-118(1999)

[14] Katsurada, H., Schulze-Pillot, R.: Genus theta series, Hecke operators and the basis problem. In:Automorphic Forms and Zeta Funktions. Proceedings of the conference in memory of T.Arakawa. World Scientific 2005

[15] Nagaoka, S.: A remark on Serre's example of p-adic Eisenstein series. Math.Z.235, 227-250(2000)

[16] Shimura, G.: Euler Products and Eisenstein Series. Regional Conference Series in Math. 93. AMS 1997

[17] Waldspurger, J.-L.: L'engendrement par des series de theta de certains espaces de formes modulaires. Invent.Math.50, 135-168(1978)

Siegfried Böcherer
Kunzenhof 4B
79117 Freiburg
Germany
boecherer@t-online.de