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Computational Prospects on Copositive Programming

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Abstract

This short note discusses the so-called copositive program, a related problem to semidefinite programs. Copositive programs are very hard problems. We review some known facts to approximately solve these problems by semidefinite programs or second-order cone programs, and provide hints for further developments.

1 Introduction

It is well-known that some linear conic and convex programs, *e.g.*, Linear Program (LP), Second-Order Cone Program (SOCP), and Semidefinite Program (SDP), can be solved very efficiently by interior-point methods.

In the recent years, there is a momentary increase in interest in copositive program [1, 5, 6, 8, 10], which is related to the above problems, but includes a large class of optimization problems.

The cone of copositive matrices is defined as

$$\mathcal{C}_+^n = \{\boldsymbol{X} \in \mathcal{S}^n : \ \boldsymbol{z}^T \boldsymbol{X} \boldsymbol{z} \geq 0, \ \forall \boldsymbol{z} \geq \boldsymbol{0}\},$$

where $\mathcal{S}^n$ denotes the space of $n \times n$ symmetric matrices. Its dual (employing the inner-product defined in $\mathcal{S}^n$) is the cone of completely positive matrices

$$(\mathcal{C}_+^n)^* = \{\sum_{i=1}^k \boldsymbol{q}_i \boldsymbol{q}_i^T \in \mathcal{S}^n : \ \mathbb{R}^n \ni \boldsymbol{q}_i \geq 0, \ i = 1, 2, \dots, k\}.$$

Essentially, the copositive matrices hide very complex structures in terms of computational solvability since even the problem of checking if a symmetric matrix is copositive or not is NP-complete [7].

Given $\boldsymbol{C}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_m \in \mathcal{S}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$, the copositive program and its dual is defined as follows:

$$\left\{ \begin{array}{ll} \text{maximize} & \langle \boldsymbol{b}, \boldsymbol{y} \rangle \\ \text{subject to} & \boldsymbol{C} - \sum_{p=1}^m \boldsymbol{A}_p y_p \in \mathcal{C}_+^n, \end{array} \right. \tag{1}$$

$$\left\{ \begin{array}{ll} \text{minimize} & \langle \boldsymbol{C}, \boldsymbol{X} \rangle \\ \text{subject to} & \langle \boldsymbol{A}_p, \boldsymbol{X} \rangle = b_p, \quad (p = 1, 2, \dots, m) \\ & \boldsymbol{X} \in (\mathcal{C}_+^n)^*. \end{array} \right.$$

Our primal interest is to develop a new polynomial-time algorithm to solve approximately the copositive program and its dual. In this short note, we start describing known facts and observations in order to obtain a glimpse of further directions.

Section 2 presents some examples of copositive programs mostly related to combinatorial optimization. Sections 3 and 4 give some well-known facts about approximations of the copositive cone, and finally Section 5 gives very preliminary numerical experiments.

2 Examples of Copositive Programs

In this section, we present three examples of optimization problems which can be formulated exactly as copositive programs.

2.1 Standard Quadratic Optimization Problem

It consists in minimizing a quadratic form for the given matrix $\boldsymbol{Q} \in \mathcal{S}^n$ over the simplex:

$$\text{(SQOP)} \quad \begin{cases} \text{minimize} & \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \\ \text{subject to} & \boldsymbol{e}^T \boldsymbol{x} = 1, \ \boldsymbol{x} \in \mathbb{R}^n_+, \end{cases} \tag{2}$$

where $\boldsymbol{e}$ is the vector with all 1's.

Introducing an auxiliary variable λ, (2) is equivalent to

$$\text{maximize } \lambda \text{ subject to } \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \geq \lambda (\boldsymbol{e}^T \boldsymbol{x})^2, \ \forall \boldsymbol{x} \in \mathbb{R}^n_+, \ \boldsymbol{e}^T \boldsymbol{x} = 1,$$

which gives

$$\text{(SQOP)} \quad \begin{cases} \text{maximize} & \lambda \\ \text{subject to} & \boldsymbol{Q} - \lambda \boldsymbol{e}\boldsymbol{e}^T \in \mathcal{C}^n_+, \end{cases}$$

and becomes in the form of (1).

2.2 Maximum Stable Set Problem

Given a graph $G(V, E)$, a subset $V' \subseteq V$ is called a *stable set of G* if the induced subgraph on V' contains no edges. The *maximum stable set problem* consists in finding a stable set of maximal cardinality. This problem is also equivalent to finding the largest clique in the complementary graph.

Let $\alpha(G)$ be the cardinality of the maximum stable set. It is known that the Lovász's theta function $\vartheta(G)$, which can be computed by solving an SDP, gives an upper bound for $\alpha(G)$:

$$\vartheta(G) \equiv \begin{cases} \text{minimize} & \lambda \\ \text{subject to} & \lambda \boldsymbol{I} + \sum_{(i,j) \in E} x_{ij} \boldsymbol{E}_{ij} - \boldsymbol{e}\boldsymbol{e}^T \in \mathcal{S}^n_+, \end{cases}$$

where $\boldsymbol{E}_{ij} \in \mathcal{S}^n$ denotes a matrix which has 1 at positions (i, j) and (j, i), and 0 otherwise.

Recently, de Klerk and Pasechnik showed that the maximum stable set problem can be exactly formulated as a copositive programs [6]:

$$
\text{(MSSP)}\quad \alpha(G) \equiv \begin{cases} \text{minimize} & \lambda \\ \text{subject to} & \lambda \boldsymbol{I} + \sum\limits_{(i,j)\in E} x_{ij}\boldsymbol{E}_{ij} - \boldsymbol{e}\boldsymbol{e}^T \in \mathcal{C}_+^n. \end{cases} \tag{3}
$$

2.3 Quadratic Assignment Problem

The *Quadratic Assignment Problem* (QAP) is a classical location problem. Given $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{S}^n$ and $\boldsymbol{C} \in \mathbb{R}^{n\times n}$, it can be formulated as

$$
\text{(QAP)}\quad \begin{cases} \min & \langle \boldsymbol{X}, \boldsymbol{AXB} + \boldsymbol{C}\rangle \\ \text{subject to} & \boldsymbol{X} \in \{0,1\}^{n\times n}, \\ & \boldsymbol{X} \in \Pi \equiv \{\text{set of all permutation matrices}\}, \end{cases}
$$

which further can be re-written as

$$
\text{(QAP)}\quad \begin{cases} \min & \langle \boldsymbol{X}, \boldsymbol{AXB} + \boldsymbol{C}\rangle \\ \text{subject to} & \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^T = \boldsymbol{I}, \\ & \boldsymbol{X} \in \mathcal{N}^n, \end{cases}
$$

where $\mathcal{N}^n = \{\boldsymbol{X} \in \mathcal{S}^n : \ X_{ij} \geq 0, \ 1 \leq i,j \leq n\}$.

Notice that denoting by $\boldsymbol{x} = vec(\boldsymbol{X})$, the vector formed by stacking the columns of $\boldsymbol{X}$, and using the Kronecker product, we have $\langle \boldsymbol{X}, \boldsymbol{AXB}+\boldsymbol{C}\rangle = \langle \boldsymbol{B}\otimes\boldsymbol{A}+Diag(vec(\boldsymbol{C})), \boldsymbol{x}\boldsymbol{x}^T\rangle = \langle \boldsymbol{B}\otimes\boldsymbol{A}+Diag(vec(\boldsymbol{C})), \boldsymbol{Y}\rangle$ with $\boldsymbol{Y} = \boldsymbol{x}\boldsymbol{x}^T \in \mathcal{S}^{n^2}$.

Very recently, Povh and Rendl showed that the QAP can be also formulated as a copositive program [10]:

$$
\text{(QAP)}\quad \begin{cases} \min & \langle \boldsymbol{B}\otimes\boldsymbol{A}+Diag(vec(\boldsymbol{C})), \boldsymbol{Y}\rangle \\ \text{subject to} & \langle \boldsymbol{e}\boldsymbol{e}^T, \boldsymbol{Y}\rangle = n^2, \\ & \langle \boldsymbol{I}, \boldsymbol{Y}^{ij}\rangle = \delta_{ij}, \quad 1 \leq i,j \leq n \\ & \sum_{i=1}^n \boldsymbol{Y}^{ii} = \boldsymbol{I}, \\ & \boldsymbol{Y} \in (\mathcal{C}_+^{n^2})^*, \end{cases}
$$

where $\boldsymbol{Y}^{ij}$ denotes the (i,j)th sub-block of size $n\times n$ of the matrix $\boldsymbol{Y}$.

3 Parrilo's Hierarchical Approximation of the Copositive Cone

Given $\boldsymbol{M} \in \mathcal{S}^n$, let us define the polynomials

$$
\begin{aligned} P(\boldsymbol{M};\boldsymbol{x}) &= (\boldsymbol{x}\circ\boldsymbol{x})^T\boldsymbol{M}(\boldsymbol{x}\circ\boldsymbol{x}) = \sum_{i,j=1}^n M_{ij}x_i^2x_j^2, \quad \text{and} \\ P_r(\boldsymbol{M};\boldsymbol{x}) &= \left(\sum_{i=1}^n x_i^2\right)^r P(\boldsymbol{M};\boldsymbol{x}), \quad \text{for } r \in \mathbb{N}^*. \end{aligned}
$$

We say that a polynomial is *Sum-Of-Squares* (SOS) *decomposable* if it can be written as a finite sum of squared polynomials.

Let ${}_r\mathcal{K}_+^n = \{\boldsymbol{M} \in \mathcal{S}^n : P_r(\boldsymbol{M};\boldsymbol{x}) \text{ is SOS decomposable}\}$. It can be shown that ${}_0\mathcal{K}_+^n = \mathcal{S}_+^n + \mathcal{N}^n$, and also that

$$
{}_1\mathcal{K}_+^n = \left\{ \boldsymbol{M} \in \mathcal{S}^n : \exists \boldsymbol{P}^i \text{ such that } \begin{array}{rl} \boldsymbol{M} - \boldsymbol{P}^i \in \mathcal{S}_+^n, & 1 \le i \le n, \\ P_{ii}^i = 0, & 1 \le i \le n, \\ P_{jj}^i + 2P_{ij}^j = 0, & 1 \le i \ne j \le n, \\ P_{jk}^i + P_{ik}^j + P_{ij}^k \ge 0, & 1 \le i < j < k \le n \end{array} \right\}
$$

according to Parrilo [8].

Pólya [9] showed that if $\boldsymbol{M} \in \mathcal{S}^n$ is strictly copositive there will be a sufficiently large r such that $P_r(\boldsymbol{M};\boldsymbol{x})$ becomes SOS decomposable. This fact permit us to construct an hierarchical and nested sequence of convex cones

$$
\mathcal{S}_+^n + \mathcal{N}^n = {}_0\mathcal{K}_+^n \subset {}_1\mathcal{K}_+^n \subset {}_2\mathcal{K}_+^n \subset \cdots
$$

which approximate the cone of copositive matrices $\mathcal{C}_+^n$.

In particular, it is known that $\mathcal{S}_+^n + \mathcal{N}^n = \mathcal{C}_+^n$ for $n \le 4$ [3]. However, for $n = 5$, the following matrix is a counter-example:

$$
\begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{pmatrix},
$$

which is copositive but does not belong to $\mathcal{S}_+^n + \mathcal{N}^n$ [3].

4 Polynomial-Time Approximations for the Copositive Programs

For $k = 2, 3, 4$, let us define

$$
{}_k\mathcal{Y}_+^n = \left\{ \mathcal{M} \in \mathcal{S}^n : \begin{array}{l} \boldsymbol{M}_{JJ} \in \mathcal{S}_+^{|J|} + \mathcal{N}^{|J|}, \\ \forall J \subseteq \{1,2,\dots,n\}, \ |J| = k \end{array} \right\}.
$$

Since $\mathcal{S}_+^n + \mathcal{N}^n = \mathcal{C}_+^n$ for $n \le 4$, we have the following relation between the cones which are familiar to us:

$$
\mathcal{S}_+^n + \mathcal{N}^n \subseteq \mathcal{C}_+^n \subseteq {}_4\mathcal{Y}_+^n \subset {}_3\mathcal{Y}_+^n \subset {}_2\mathcal{Y}_+^n, \quad \text{for} \quad n \ge 4. \tag{4}
$$

Therefore, if we consider a linear conic program in the form of (1):

$$
\begin{cases} \text{maximize} & \langle \boldsymbol{b}, \boldsymbol{y} \rangle \\ \text{subject to} & \boldsymbol{C} - \sum_{p=1}^{\bar{m}} \boldsymbol{A}_p y_p \in \mathcal{C}, \end{cases} \tag{5}
$$

where $\mathcal{C}$ is a convex cone, we can obtain a lower bound and upper bounds for the copositive program (1) if we substitute the cones $\mathcal{S}^n_+ + \mathcal{N}^n$ and ${}_k\mathcal{Y}^n_+$ in $\mathcal{C}$ of (5), respectively.

Table 1 shows the relation between the convex cones and the number of constraints $\bar{m}$ and sizes of LP, SOCP, and SDP restrictions of (5) when applying this kind of approximation for (1). In the entries $\binom{n}{2} \times \mathcal{Q}^3_+$ means that there are $\binom{n}{2} = n(n-1)/2$ constraints of SOC type with size 3. $\mathcal{Q}^n_+ = \{x \in \mathbb{R}^n : x_1^2 \geq \|(x_2, \ldots, x_n)^T\|^2\}$.

Table 1: Relation between the convex cones and the number of constraints $\bar{m}$, LP, SOCP and SDP constraints for the linear conic program (5).

cone	$\mathcal{S}^n_+ + \mathcal{N}^n$	$\mathcal{C}^n_+$	${}_2\mathcal{Y}^n_+$	${}_3\mathcal{Y}^n_+$	${}_4\mathcal{Y}^n_+$
# constraints $\bar{m}$	$m + \binom{n}{2}$	m	$m + \binom{n}{2}$	$m + \binom{n}{3}$	$m + \binom{n}{4}$
LP constraints	$\mathbb{R}_+^{n(n-1)/2}$		$\mathbb{R}_+^{n(n-1)/2}$	$\mathbb{R}_+^{n(n-1)(n-2)/6}$	$\mathbb{R}_+^{n(n-1)(n-2)(n-3)/24}$
SOCP constraints			$\binom{n}{2} \times \mathcal{Q}^3_+$		
SDP constraints	$\mathcal{S}^n_+$			$\binom{n}{3} \times \mathcal{S}^3_+$	$\binom{n}{4} \times \mathcal{S}^4_+$
Copositive constraints		$\mathcal{C}^n_+$			

5 Very Preliminary Numerical Experiments

We performed very preliminary numerical experiments to study the relation proposed in the previous section. To compute the lower bound of problem (1), which is of our interest, we solved problem (5) with $\mathcal{S}^n_+ + \mathcal{N}^n$ while for the upper bound with ${}_2\mathcal{Y}^n_+$. The numerical experiments were performed on an Opteron 850 (2.4GHz) with 8GB of main memory. All problems were converted to an SDP and solved by the SDPA 6.2.0 [11].

Four Standard Quadratic Optimization Problems (2) with known optimal values were chosen. Table 2 summarize our results.

We can observe that even such simple relaxations can give very efficient approximation for the copositive programs of small size, excepting ex5.2.

Finally, Table 3 gives the numerical results for the same relaxation for the Maximum Stable Set Problem (3). These problems were extracted from the SDPLIB 1.2 [2]. We also computed the Lovász's theta function $\vartheta(G)$ using SDPA 6.2.0 [11].

Unfortunately in this case, we obtained a huge gap between the lower and upper bounds and since we do not know their optimal values, the results became non-conclusive.

6 Further Directions

We recognize that there are much further work to do.

Table 2: Optimal values of the SQOPs (2) and their lower and upper bounds with their respective computational times.

problem	lower bound time (s)	optimal value	upper bound time (s)
ex5.1 [1] $n = 5$	0.4472 0s	0.5000	0.5000 0s
ex5.2 [1] $n = 12$	0.3090 0s	0.3333	0.5000 0s
ex5.3 [1] $n = 5$	-16.3333 0s	-16.3333	-16.3314 0s
test2.9 [4] $n = 10$	-0.3750 0s	-0.3750	-0.2500 0s

Table 3: Lower bounds, upper bounds and the theta functions $\vartheta(G)$ for the MSSP (3) and their respective computational time for the SDPLIB 1.2 problems [2].

problem	lower bound time (s)	upper bound time (s)	$\vartheta(G)$ time (s)
theta 1 $n = 50$	2.0000 55s	23.0000 60s	23.0000 0s
theta 2 $n = 100$	2.0000 5772s	32.6875 9928s	32.8792 1s

It is still unclear the relation between the Parrilo's hierarchical approximation of the copositive cones and the ones we indicated in Section 4. Also if an explicitly characterization of ${}_2\mathcal{K}_+^n$ is known. Maybe the recent article of de Klerk-Laurent-Parrilo [5] might give a hint.

Also a more exhaustive computational experiments are necessary to evaluate and compare the various approximations of the copositive programs. Specially for larger problems. A drawback is that the computational time might grow rapidly as the number of variables and constraints of the relaxed problem grows combinatorially with the size of the original copositive program.

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