APPROMATING SOLUTIONS OF NONLINEAR VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $A$ be an inverse-strongly-monotone operator of $C$ into the dual space $E^*$ of $E$. In this paper, we introduce the following iterative scheme for finding a solution of the variational inequality problem for $A$: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\Pi_C$ is the generalized projection from $E$ onto $C$, $J$ is the duality mapping from $E$ into $E^*$ and $\{\lambda_n\}$ is a sequence of positive real numbers. Then we obtain a weak convergence theorem (Theorem 3.1). Using this result, we consider the problem of finding a minimizer of a convex function, the problem of finding a point $u \in E$ satisfying $0 = Au$ and so on.

1. INTRODUCTION

Let $E$ be a real Banach space with norm $\| \cdot \|$, let $E^*$ denote the dual of $E$ and let $\langle x, f \rangle$ denote the value of $f \in E^*$ at $x \in E$. Let $C$ be a nonempty closed convex subset of $E$ and let $A$ be a monotone operator of $C$ into $E^*$. Then we deal with the problem of finding a point $u \in C$ such that $\langle v - u, Au \rangle \geq 0$ for all $v \in C$.

This problem is called the variational inequality problem; see [14] and [13]. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. An operator $A$ of $C$ into $E^*$ is said to be inverse-strongly-monotone if there exists a positive real number $\alpha$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$; see [6], [15] and [9]. For such a case, $A$ is said to be $\alpha$-inverse-strongly-monotone.

For finding a zero point of an inverse-strongly-monotone operator of the Euclidean space $\mathbb{R}^N$ into itself, Gol'shtein and Tret'yakov [8] introduced the following scheme: $x_1 = x \in \mathbb{R}^N$ and

$$x_{n+1} = x_n - \lambda_n Ax_n$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ generated by (1.2) converges to some element of $A^{-1}0$, where $A^{-1}0 = \{u \in \mathbb{R}^N : Au = 0\}$.

In the case when $A$ is an inverse-strongly-monotone operator of a closed convex subset $C$ of a Hilbert space $H$ into $H$, one method of finding a point $u \in VI(C, A)$ is the projection algorithm: $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n Ax_n)$$

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for every \( n = 1, 2, \ldots \), where \( P_{C} \) is the metric projection of \( H \) onto \( C \) and \( \{ \lambda_{n} \} \) is a sequence of positive numbers. Iiduka, Takahashi and Toyoda [9] proved that the sequence \( \{ x_{n} \} \) generated by (1.3) converges weakly to some element of \( \text{VI}(C, A) \).

In the case when the space is a Banach space \( E \), Alber [1] proved the following strong convergence theorem by the generalized projection algorithm:

**Theorem 1.1** (Alber [1]). Let \( C \) be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space \( E \). Suppose an operator \( A \) of \( E \) into \( E^{*} \) satisfies the following conditions:

(i) \( A \) is uniformly monotone, that is, \( \langle x - y, Ax - Ay \rangle \geq \psi(||x - y||) \) for all \( x, y \in E \), where \( \psi(t) \) is a continuous strictly increasing function for all \( t \geq 0 \) with \( \psi(0) = 0 \),

(ii) \( \text{VI}(C, A) \neq \emptyset \),

(iii) \( A \) has \( \phi \)-arbitrary growth, that is, \( ||Ay|| \leq \phi(||y - z||) \) for all \( y \in E \) and \( \{ z \} = \text{VI}(C, A) \), where \( \phi(t) \) is a continuous nondecreasing function with \( \phi(0) \geq 0 \).

Define a sequence \( \{ x_{n} \} \) as follows: \( x_{1} = x \in E \) and

\[
x_{n+1} = \Pi_{C} J^{-1}(Jx_{n} - \lambda_{n} Ax_{n})
\]

for every \( n = 1, 2, \ldots \), where \( \Pi_{C} \) is the generalized projection from \( E \) onto \( C \), \( J \) is the duality mapping from \( E \) into \( E^{*} \) and \( \{ \lambda_{n} \} \) is a positive nonincreasing sequence which satisfies \( \lim_{n \to \infty} \lambda_{n} = 0 \) and \( \sum_{n=1}^{\infty} \lambda_{n} = \infty \). Then the sequence \( \{ x_{n} \} \) converges strongly to a unique element \( z \) of \( \text{VI}(C, A) \).

On the other hand, for finding a zero point of a maximal monotone operator, by using the proximal point algorithm, Kamimura, Kohsaka and Takahashi [12] proved the following weak convergence theorem:

**Theorem 1.2** (Kamimura, Kohsaka and Takahashi [12]). Let \( E \) be a uniformly convex and uniformly smooth Banach space whose duality mapping \( J \) is weakly sequentially continuous. Let \( A \subset E \times E^{*} \) be a maximal monotone operator, let \( J_{r} = (J + rA)^{-1}J \) for all \( r > 0 \) and let \( \{ x_{n} \} \) be a sequence defined as follows: \( x_{1} = x \in E \) and

\[
x_{n+1} = J_{r_{n}} x_{n}
\]

for every \( n = 1, 2, \ldots \), where \( \{ r_{n} \} \subset (0, \infty) \) satisfies \( \limsup_{n \to \infty} r_{n} > 0 \). If \( A^{-1} 0 \neq \emptyset \), then the sequence \( \{ x_{n} \} \) converges weakly to an element \( z \) of \( A^{-1} 0 \). Further \( z = \lim_{n \to \infty} \Pi_{A^{-1} 0}(x_{n}) \), where \( \Pi_{A^{-1} 0} \) is the generalized projection from \( E \) onto \( A^{-1} 0 \).

In this paper, motivated by Alber [1], we introduce an iterative scheme for finding a solution of the variational inequality problem for an operator \( A \) which satisfies the following conditions in a 2-uniformly convex and uniformly smooth Banach space \( E \):

1. \( A \) is inverse-strongly-monotone,
2. \( \text{VI}(C, A) \neq \emptyset \),
3. \( ||Ay|| \leq ||Ay - Au|| \) for all \( y \in C \) and \( u \in \text{VI}(C, A) \).

Then we obtain a weak convergence theorem (Theorem 3.1). Further, using this result, we consider the minimization problem (Theorem 3.3 and Corollary 3.5), the complementarity problem (Theorem 3.7), the problem of finding a point \( u \in E \) satisfying \( 0 = Au \) (Theorem 3.4) and so on.

2. Preliminaries

Let \( E \) be a real Banach space. When \( \{ x_{n} \} \) is a sequence in \( E \), we denote strong convergence of \( \{ x_{n} \} \) to \( x \in E \) by \( x_{n} \to x \) and weak convergence by \( x_{n} \rightharpoonup x \). A multi-valued operator \( T : E \to 2^{E^{*}} \) with domain \( D(T) = \{ z \in E : Tz \neq \emptyset \} \) and range \( R(T) = \bigcup \{ Tz \in E : z \in D(T) \} \) is said to be monotone if \( \langle x_{1} - x_{2}, y_{1} - y_{2} \rangle \geq 0 \) for
each \( x_i \in D(T) \) and \( y_i \in Tx_i, \ i = 1, 2 \). A monotone operator \( T \) is said to be maximal if its graph \( G(T) = \{ (x, y) : y \in Tx \} \) is not properly contained in the graph of any other monotone operator.

Let \( U = \{ x \in E : \|x\| = 1 \} \). A Banach space \( E \) is said to be strictly convex if for any \( x, y \in U \),

\[
x \neq y \implies \left\| \frac{x + y}{2} \right\| < 1.
\]

It is also said to be uniformly convex if for each \( \varepsilon \in (0, 2] \), there exists \( \delta > 0 \) such that for any \( x, y \in U \),

\[
\|x - y\| \geq \varepsilon \implies \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.
\]

It is known that a uniformly convex Banach space is reflexive and strictly convex. And we define a function \( \delta : [0, 2] \to [0, 1] \) called the modulus of convexity of \( E \) as follows:

\[
\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.
\]

It is known that \( E \) is uniformly convex if and only if \( \delta(\varepsilon) > 0 \) for all \( \varepsilon \in (0, 2] \). Let \( p \) be a fixed real number with \( p \geq 2 \). Then \( E \) is said to be \( p \)-uniformly convex if there exists a constant \( c > 0 \) such that \( \delta(\varepsilon) \geq c \varepsilon^p \) for all \( \varepsilon \in (0, 2] \). For example, see [4] and [23] for more details. We know the following fundamental characterization [4, 5] of \( p \)-uniformly convex Banach spaces:

Lemma 2.1 ([4, 5]). Let \( p \) be a real number with \( p \geq 2 \) and let \( E \) be a Banach space. Then \( E \) is \( p \)-uniformly convex if and only if there exists a constant \( c \) with \( 0 < c \leq 1 \) such that

\[
\frac{1}{2}(\|x + y\|^p + \|x - y\|^p) \geq \|x\|^p + c^p \|y\|^p
\]

for all \( x, y \in E \).

The best constant \( 1/c \) in Lemma 2.1 is called the \( p \)-uniformly convexity constant of \( E \); see [4]. Putting \( x = (u + v)/2 \) and \( y = (u - v)/2 \) in (2.1), we readily conclude that, for all \( u, v \in E \),

\[
\frac{1}{2}(\|u\|^p + \|v\|^p) \geq \left\| \frac{u + v}{2} \right\|^p + c^p \left\| \frac{u - v}{2} \right\|^p.
\]

A Banach space \( E \) is said to be smooth if the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for all \( x, y \in U \). It is also said to be uniformly smooth if the limit (2.3) is attained uniformly for \( x, y \in U \). One should note that no Banach space is \( p \)-uniformly convex for \( 1 < p < 2 \); see [23] for more details. It is well known that Hilbert and the Lebesgue \( L^q \) (\( 1 < q \leq 2 \)) spaces are \( 2 \)-uniformly convex and uniformly smooth. Let \( X \) be a Banach space and let \( L^q(X) = L^q(\Omega, \Sigma, \mu; X), 1 \leq q \leq \infty \), be the Lebesgue-Bochner space on an arbitrary measure space \((\Omega, \Sigma, \mu)\). Let \( 2 \leq p < \infty \) and let \( 1 < q < p \). Then \( L^q(X) \) is \( p \)-uniformly convex if and only if \( X \) is \( p \)-uniformly convex; see [23]. For the weak convergence in the Lebesgue spaces \( L^p \) (\( p \geq 2 \)), see Aoyama, Iiduka and Takahashi [10].

On the other hand, with each \( p > 1 \), the (generalized) duality mapping \( J_p \) from \( E \) into \( 2E^* \) is defined by

\[
J_p(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1} \}
\]

for all \( x \in E \). In particular, \( J = J_2 \) is called the normalized duality mapping. The duality mapping \( J \) has the following properties:
If $E$ is smooth, then $J$ is single-valued;
if $E$ is strictly convex, then $J$ is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;
if $E$ is reflexive, then $J$ is surjective;
if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$.

See [22] for more details. The duality mapping $J$ from a smooth Banach space $E$ into $E^*$ is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $Jx_n \rightharpoonup Jx$, where $\rightharpoonup$ implies the weak* convergence; see [7]. It is also known that

$$p\langle y - x, j_x \rangle \leq \|y\|^p - \|x\|^p$$

for all $x, y \in E$ and $j_x \in J_p(x)$. We know the following result [24], which characterizes a $p$-uniformly convex Banach space.

**Lemma 2.2 ([24]).** Let $p$ be a given real number with $p \geq 2$ and let $E$ be a $p$-uniformly convex Banach space. Then

$$\|x + y\|^p \geq \|x\|^p + p\langle y, j_x \rangle + \frac{c^p}{2^{p-2}p} \|y\|^p$$

for all $x, y \in E$ and $j_x \in J_p(x)$, where $J_p$ is the generalized duality mapping of $E$ and $1/c$ is the $p$-uniform convexity constant of $E$.

Further we know the following result [5, 25], which characterizes a $p$-uniformly convex Banach space.

**Lemma 2.3 ([5, 25]).** Let $p$ be a given real number with $p \geq 2$ and let $E$ be a $p$-uniformly convex Banach space. Then, for all $x, y \in E$, $j_x \in J_p(x)$ and $j_y \in J_p(y)$,

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p} \|x - y\|^p,$$

where $J_p$ is the generalized duality mapping of $E$ and $1/c$ is the $p$-uniform convexity constant of $E$.

Let $E$ be a smooth Banach space. We know the following function studied in Alber [1], Kamimura and Takahashi [11] and Reich [16]:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. It is obvious from the definition of $\phi$ that $(\|x\|^2 - \|y\|^2)^2 \leq \phi(x, y)$ for all $x, y \in E$. The following lemma which was proved by Kamimura and Takahashi [11] is important:

**Lemma 2.4 ([11]).** Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n \to \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Let $E$ be a reflexive, strictly convex and smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. For each $x \in E$, there corresponds a unique element $x_0 \in C$ (denoted by $\Pi_C(x)$) such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C$ is called the generalized projection from $E$ onto $C$; see Alber [1]. If $E$ is a Hilbert space, then $\Pi_C$ is coincident with the metric projection from $E$ onto $C$. We also know the following lemmas [1]; see also [11]:
Lemma 2.5 ([1]; see also [11]). Let $E$ be a smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, let $x \in E$ and let $x_0 \in C$. Then
\[ \phi(x_0, x) = \min_{y \in C} \phi(y, x) \]
if and only if
\[ \langle y - x_0, Jx_0 - Jx \rangle \geq 0 \text{ for all } y \in C. \]

Lemma 2.6 ([1]; see also [11]). Let $E$ be a reflexive, strictly convex and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then
\[ \phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x) \]
for all $y \in C$.

Using Lemmas 2.4 and 2.6, we have the following lemma:

Lemma 2.7 ([10]). Let $S$ be a nonempty closed convex subset of a uniformly convex and smooth Banach space $E$. Let $\{x_n\}$ be a sequence in $E$. Suppose that, for all $u \in S$,
\[ \phi(u, x_{n+1}) \leq \phi(u, x_n) \]
for every $n = 1, 2, \ldots$. Then $\{\Pi_S(x_n)\}$ is a Cauchy sequence.

Let $E$ be a reflexive, strictly convex and smooth Banach space and let $J$ be the duality mapping from $E$ into $E^*$. Then $J^{-1}$ is also single-valued, one-to-one and surjective, and it is the duality mapping from $E^*$ into $E$. We make use of the following mapping $V$ studied in Alber [1]:
\[ V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \]
for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping $g$ defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous and convex function from $E^*$ into $(\infty, \infty)$. We know the following lemma [1]:

Lemma 2.8 ([1]). Let $E$ be a reflexive, strictly convex and smooth Banach space and let $V$ be as in (2.6). Then
\[ V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*) \]
for all $x \in E$ and $x^*, y^* \in E^*$.

An operator $A$ of $C$ into $E^*$ is said to be hemicontinuous if for all $x, y \in C$, the mapping $f$ of $[0,1]$ into $E^*$ defined by $f(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of $E^*$. We denote by $N_C(v)$ the normal cone for $C$ at a point $v \in C$, that is,
\[ N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0 \text{ for all } y \in C\}. \]

We know the following theorem [17]:

Theorem 2.9 (Rockafellar [17]). Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $A$ be a monotone and hemicontinuous operator of $C$ into $E^*$. Let $T \subset E \times E^*$ be an operator defined as follows:
\[ T_v = \begin{cases} A v + N_C(v), & v \in C, \\ \emptyset, & v \notin C \end{cases} \]

Then $T$ is maximal monotone and $T^{-1}0 = \text{VI}(C, A)$.

We also know the following lemma (Lemma 7.1.7 of [22]):
Lemma 2.10 ([22]). Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $A$ be a monotone and hemicontinuous operator of $C$ into $E^*$. Then

$$\text{VI}(C, A) = \{u \in C : \langle v - u, Av \rangle \geq 0 \text{ for all } v \in C\}.$$ 

It is obvious from Lemma 2.10 that the set $\text{VI}(C, A)$ is a closed convex subset of $C$. Further, we know the following lemma (Theorem 7.1.8 of [22]):

Lemma 2.11 ([22]). Let $C$ be a nonempty compact convex subset of a Banach space $E$ and let $A$ be a monotone and hemicontinuous operator of $C$ into $E^*$. Then the set $\text{VI}(C, A)$ is nonempty.

3. WEAK CONVERGENCE THEOREMS

Let $C$ be a nonempty closed convex subset of a Banach space $E$. If an operator $A$ of $C$ into $E^*$ is $\alpha$-inverse-strongly-monotone, then $A$ is Lipschitz continuous, that is, $\|Ax - Ay\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$.

Now we can state the following weak convergence theorem for finding a solution of the variational inequality for an inverse-strongly-monotone operator in a 2-uniformly convex and uniformly smooth Banach space:

Theorem 3.1. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous and let $C$ be a nonempty closed convex subset of $E$. Let $A$ be an operator of $C$ into $E^*$ which satisfies the conditions (1), (2) and (3). Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z$ of $\text{VI}(C, A)$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_{\text{VI}(C, A)} (x_n)$.

Using Theorem 3.1, we consider some weak convergence theorems in a 2-uniformly convex and uniformly smooth Banach space. We first study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space. Before considering this problem, we state the following lemma which was proved by Baillon and Haddad [3]:

Lemma 3.2 ([3]). Let $E$ be a Banach space, let $f$ be a continuously Fréchet differentiable convex functional on $E$ and let $\nabla f$ be the gradient of $f$. If $\nabla f$ is $1/\alpha$-Lipschitz continuous, then $\nabla f$ is $\alpha$-inverse-strongly-monotone.

Now we can consider the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space.

Theorem 3.3. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a functional on $E$ which satisfies the following conditions:

1. $f$ is a continuously Fréchet differentiable convex functional on $E$ and $\nabla f$ is $1/\alpha$-Lipschitz continuous,
2. $S = \arg \min_{y \in C} f(y) = \{z \in C : f(z) = \min_{y \in C} f(y) \} \neq \emptyset$,
3. $\|\nabla f|_C(y)\| \leq \|\nabla f|_C(y) - \nabla f|_C(u)\|$ for all $y \in C$ and $u \in S$.

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n \nabla f|_C(x_n))$$
for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z$ of $S$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_S(x_n)$.

We next consider the problem of finding a zero point of an inverse-strongly-monotone operator of $E$ into $E^*$. In the case when $C = E$, the condition (3) of Theorem 3.1 holds.

**Theorem 3.4.** Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous. Let $A$ be an operator of $E$ into $E^*$ which satisfies the following conditions:

1. $A$ is $\alpha$-inverse-strongly-monotone,
2. $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$.

Suppose $x_1 = x \in E$ and $\{x_n\}$ is given by

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z$ of $A^{-1}0$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_{A^{-1}0}(x_n)$.

Using Theorem 3.4, we can also consider the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space.

**Corollary 3.5.** Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous. Let $f$ be a functional on $E$ which satisfies the following conditions:

1. $f$ is a continuously Fréchet differentiable convex functional on $E$ and $\nabla f$ is $1/\alpha$-Lipschitz continuous,
2. $(\nabla f)^{-1}0 = \{z \in E : f(z) = \min_{y \in E} f(y)\} \neq \emptyset$.

Suppose $x_1 = x \in E$ and $\{x_n\}$ is given by

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n \nabla f(x_n))$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z$ of $(\nabla f)^{-1}0$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_{(\nabla f)^{-1}0}(x_n)$.

Further we consider the problem of finding a unique solution of the variational inequality for a strongly monotone and Lipschitz continuous operator. An operator $A$ of $C$ into $E^*$ is said to be strongly monotone if there exists a positive real number $\alpha$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|x - y\|^2$$

for all $x, y \in C$. For such a case, $A$ is said to be $\alpha$-strongly monotone. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. One method of finding a point $u \in \text{VI}(C, A)$ is the projection algorithm which starts with any $x_1 = x \in C$ and updates iteratively $x_{n+1}$ according to the formula (1.3). It is well known that if $A$ is an $\alpha$-strongly monotone and $\beta$-Lipschitz continuous operator of $C$ into $H$ and $\{\lambda_n\} \subset (0, 2\alpha/\beta^2)$, then the operator $P_C(I - \lambda_n A)$ is a contraction of $C$ into itself. Hence, the Banach contraction principle guarantees that the sequence generated by (1.3) converges strongly to the unique solution of $\text{VI}(C, A)$. Motivated by this result, we obtain the following:
Theorem 3.6. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous and let $C$ be a nonempty closed convex subset of $E$. Let $A$ be an operator of $C$ into $E^*$ which satisfies the following conditions:

1. $A$ is $\alpha$-strongly monotone and $\beta$-Lipschitz continuous,
2. $\mathrm{VI}(C, A) \neq \emptyset$,
3. $\|Ay\| \leq \|Ay - Az\|$ for all $y \in C$ and $\{z\} = \mathrm{VI}(C, A)$.

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_{C} J^{-1} (Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/(2\beta^2)$, then the sequence $\{x_n\}$ converges weakly to a unique element $z$ of $\mathrm{VI}(C, A)$, where $1/c$ is the 2-uniformly convexity constant of $E$.

Finally we consider the complementarity problem. Let $K$ be a nonempty closed convex cone in $E$, let $A$ be an operator of $K$ into $E^*$ and define its polar in $E^*$ to be the set

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0 \text{ for all } x \in K\}.$$ 

Then an element $u \in K$ is called a solution of the complementarity problem for $A$ if

$$Au \in K^* \text{ and } \langle u, Au \rangle = 0.$$ 

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

Theorem 3.7. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous and let $K$ be a nonempty closed convex cone in $E$. Let $A$ be an operator of $K$ into $E^*$ which satisfies the following conditions:

1. $A$ is $\alpha$-inverse-strongly-monotone,
2. $C(K, A) \neq \emptyset$,
3. $\|Ay\| \leq \|Ay - Au\|$ for all $y \in K$ and $u \in C(K, A)$.

Suppose $x_1 = x \in K$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_{K} J^{-1} (Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z$ of $C(K, A)$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_{C(K,A)}(x_n)$.

References


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