APPROXIMATING SOLUTIONS OF NONLINEAR VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $A$ be an inverse-strongly-monotone operator of $C$ into the dual space $E^*$ of $E$. In this paper, we introduce the following iterative scheme for finding a solution of the variational inequality problem for $A$: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\Pi_C$ is the generalized projection from $E$ onto $C$, $J$ is the duality mapping from $E$ into $E^*$ and $\{\lambda_n\}$ is a sequence of positive real numbers. Then we obtain a weak convergence theorem (Theorem 3.1). Using this result, we consider the problem of finding a minimizer of a convex function, the problem of finding a point $u \in E$ satisfying $0 = Au$ and so on.

1. INTRODUCTION

Let $E$ be a real Banach space with norm $\| \cdot \|$, let $E^*$ denote the dual of $E$ and let $\langle x, f \rangle$ denote the value of $f \in E^*$ at $x \in E$. Let $C$ be a nonempty closed convex subset of $E$ and let $A$ be a monotone operator of $C$ into $E^*$. Then we deal with the problem of finding

(1.1) a point $u \in C$ such that $\langle v - u, Au \rangle \geq 0$ for all $v \in C$.

This problem is called the variational inequality problem; see [14] and [13]. The set of solutions of the variational inequality problem is denoted by $\text{VI}(C, A)$. An operator $A$ of $C$ into $E^*$ is said to be inverse-strongly-monotone if there exists a positive real number $\alpha$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$; see [6], [15] and [9]. For such a case, $A$ is said to be $\alpha$-inverse-strongly-monotone.

For finding a zero point of an inverse-strongly-monotone operator of the Euclidean space $\mathbb{R}^N$ into itself, Gol'shtein and Tret'yakov [8] introduced the following scheme: $x_1 = x \in \mathbb{R}^N$ and

(1.2) $x_{n+1} = x_n - \lambda_n Ax_n$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ generated by (1.2) converges to some element of $A^{-1}0$, where $A^{-1}0 = \{u \in \mathbb{R}^N : Au = 0\}$.

In the case when $A$ is an inverse-strongly-monotone operator of a closed convex subset $C$ of a Hilbert space $H$ into $H$, one method of finding a point $u \in \text{VI}(C, A)$ is the projection algorithm: $x_1 = x \in C$ and

(1.3) $x_{n+1} = P_C(x_n - \lambda_n Ax_n)$

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for every \( n = 1, 2, \ldots \), where \( P_C \) is the metric projection of \( H \) onto \( C \) and \( \{\lambda_n\} \) is a sequence of positive numbers. Iiduka, Takahashi and Toyoda [9] proved that the sequence \( \{x_n\} \) generated by (1.3) converges weakly to some element of \( \text{VI}(C, A) \).

In the case when the space is a Banach space \( E \), Alber [1] proved the following strong convergence theorem by the generalized projection algorithm:

**Theorem 1.1** (Alber [1]). Let \( C \) be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space \( E \). Suppose an operator \( A \) of \( E \) into \( E^* \) satisfies the following conditions:

(i) \( A \) is uniformly monotone, that is, \( (x - y, Ax - Ay) \geq \psi(||x - y||) \) for all \( x, y \in E \),

(ii) \( \text{VI}(C, A) \neq \emptyset \),

(iii) \( A \) has \( \phi \)-arbitrary growth, that is, \( \|Ay\| \leq \phi(||y - z||) \) for all \( y \in E \) and \( \{z\} = \text{VI}(C, A) \), where \( \phi(t) \) is a continuous nondecreasing function with \( \phi(0) \geq 0 \).

Define a sequence \( \{x_n\} \) as follows: \( x_1 = x \in E \) and

\[
x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)
\]

for every \( n = 1, 2, \ldots \), where \( \Pi_C \) is the generalized projection from \( E \) onto \( C \), \( J \) is the duality mapping from \( E \) into \( E^* \) and \( \{\lambda_n\} \) is a positive nonincreasing sequence which satisfies \( \lim_{n \to \infty} \lambda_n = 0 \) and

\[
\sum_{n=1}^{\infty} \lambda_n = \infty.
\]

Then the sequence \( \{x_n\} \) converges strongly to a unique element \( z \) of \( \text{VI}(C, A) \).

On the other hand, for finding a zero point of a maximal monotone operator, by using the proximal point algorithm, Kamimura, Kohsaka and Takahashi [12] proved the following weak convergence theorem:

**Theorem 1.2** (Kamimura, Kohsaka and Takahashi [12]). Let \( E \) be a uniformly convex and uniformly smooth Banach space whose duality mapping \( J \) is weakly sequentially continuous. Let \( A \subset E \times E^* \) be a maximal monotone operator, let \( J_r = (J + rA)^{-1}J \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in E \) and

\[
x_{n+1} = J_{r_n} x_n
\]

for every \( n = 1, 2, \ldots \), where \( \{r_n\} \subset (0, \infty) \) satisfies \( \limsup_{n \to \infty} r_n > 0 \). If \( A^{-1}0 \neq \emptyset \), then the sequence \( \{x_n\} \) converges weakly to an element \( z \) of \( A^{-1}0 \). Further \( z = \lim_{n \to \infty} \Pi_{A^{-1}0} (x_n) \), where \( \Pi_{A^{-1}0} \) is the generalized projection from \( E \) onto \( A^{-1}0 \).

In this paper, motivated by Alber [1], we introduce an iterative scheme for finding a solution of the variational inequality problem for an operator \( A \) which satisfies the following conditions in a 2-uniformly convex and uniformly smooth Banach space \( E \):

1. \( A \) is inverse-strongly-monotone,
2. \( \text{VI}(C, A) \neq \emptyset \),
3. \( \|Ay\| \leq \|Ay - Au\| \) for all \( y \in C \) and \( u \in \text{VI}(C, A) \).

Then we obtain a weak convergence theorem (Theorem 3.1). Further, using this result, we consider the minimization problem (Theorem 3.3 and Corollary 3.5), the complementarity problem (Theorem 3.7), the problem of finding a point \( u \in E \) satisfying \( 0 = Au \) (Theorem 3.4) and so on.

### 2. Preliminaries

Let \( E \) be a real Banach space. When \( \{x_n\} \) is a sequence in \( E \), we denote strong convergence of \( \{x_n\} \) to \( x \in E \) by \( x_n \to x \) and weak convergence by \( x_n \rightharpoonup x \). A multivalued operator \( T : E \to 2^{E^*} \) with domain \( \text{D}(T) = \{z \in E : Tz \neq \emptyset\} \) and range \( \text{R}(T) = \bigcup \{Tz : z \in \text{D}(T)\} \) is said to be monotone if \( \langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \) for
each \( x_i \in D(T) \) and \( y_i \in Tx_i, i = 1, 2 \). A monotone operator \( T \) is said to be maximal if its graph \( G(T) = \{(x, y) : y \in Tx\} \) is not properly contained in the graph of any other monotone operator.

Let \( U = \{x \in E : \|x\| = 1\} \). A Banach space \( E \) is said to be strictly convex if for any \( x, y \in U \),

\[
x \neq y \text{ implies } \left\| \frac{x+y}{2} \right\| < 1.
\]

It is also said to be uniformly convex if for each \( \varepsilon \in (0, 2] \), there exists \( \delta > 0 \) such that for any \( x, y \in U \),

\[
\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.
\]

It is known that a uniformly convex Banach space is reflexive and strictly convex. And we define a function \( \delta : [0, 2] \rightarrow [0, 1] \) called the modulus of convexity of \( E \) as follows:

\[
\delta(\varepsilon) = \inf \left\{1 - \left\| \frac{x+y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.
\]

It is known that \( E \) is uniformly convex if and only if \( \delta(\varepsilon) > 0 \) for all \( \varepsilon \in (0, 2] \). Let \( p \) be a fixed real number with \( p \geq 2 \). Then \( E \) is said to be \( p \)-uniformly convex if there exists a constant \( c > 0 \) such that \( \delta(\varepsilon) \geq c\varepsilon^p \) for all \( \varepsilon \in [0, 2] \). For example, see [4] and [23] for more details. We know the following fundamental characterization [4, 5] of \( p \)-uniformly convex Banach spaces:

Lemma 2.1 ([4, 5]). Let \( p \) be a real number with \( p \geq 2 \) and let \( E \) be a Banach space. Then \( E \) is \( p \)-uniformly convex if and only if there exists a constant \( c \) with \( 0 < c \leq 1 \) such that

\[
\frac{1}{2}(\|x + y\|^p + \|x - y\|^p) \geq \|x\|^p + c^p\|y\|^p
\]

for all \( x, y \in E \).

The best constant \( 1/c \) in Lemma 2.1 is called the \( p \)-uniformly convexity constant of \( E \); see [4]. Putting \( x = (u + v)/2 \) and \( y = (u - v)/2 \) in (2.1), we readily conclude that, for all \( u, v \in E \),

\[
\frac{1}{2}(\|u\|^p + \|v\|^p) \geq \left\| \frac{u+v}{2} \right\|^p + c^p\left\| \frac{u-v}{2} \right\|^p.
\]

A Banach space \( E \) is said to be smooth if the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for all \( x, y \in U \). It is also said to be uniformly smooth if the limit (2.3) is attained uniformly for \( x, y \in U \). One should note that no Banach space is \( p \)-uniformly convex for \( 1 < p < 2 \); see [23] for more details. It is well known that Hilbert and the Lebesgue \( L^q \) \((1 < q \leq 2)\) spaces are \( 2 \)-uniformly convex and uniformly smooth. Let \( X \) be a Banach space and let \( L^q(X) = L^q(\Omega, \Sigma, \mu; X), 1 \leq q \leq \infty \), be the Lebesgue-Bochner space on an arbitrary measure space \((\Omega, \Sigma, \mu)\). Let \( 2 \leq p < \infty \) and let \( 1 < q \leq p \). Then \( L^q(X) \) is \( p \)-uniformly convex if and only if \( X \) is \( p \)-uniformly convex; see [23]. For the weak convergence in the Lebesgue spaces \( L^p(p \geq 2) \), see Asayama, Iiduka and Takahashi [10].

On the other hand, with each \( p > 1 \), the (generalized) duality mapping \( J_p \) from \( E \) into \( 2E^* \) is defined by

\[
J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}
\]

for all \( x \in E \). In particular, \( J = J_2 \) is called the normalized duality mapping. The duality mapping \( J \) has the following properties:
• If $E$ is smooth, then $J$ is single-valued;
• if $E$ is strictly convex, then $J$ is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;
• if $E$ is reflexive, then $J$ is surjective;
• if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$.

See [22] for more details. The duality mapping $J$ from a smooth Banach space $E$ into $E^*$ is said to be \textit{weakly sequentially continuous} if $x_n \rightharpoonup x$ implies that $Jx_n \rightharpoonup Jx$, where $\rightharpoonup$ implies the weak* convergence; see [7]. It is also known that

$$p\langle y - x, j_x \rangle \leq \|y\|^p - \|x\|^p$$

(2.4)

for all $x, y \in E$ and $j_x \in J_p(x)$. We know the following result [24], which characterizes a $p$-uniformly convex Banach space.

\textbf{Lemma 2.2 ([24])}. Let $p$ be a given real number with $p \geq 2$ and let $E$ be a $p$-uniformly convex Banach space. Then

$$\|x + y\|^p \geq \|x\|^p + p\langle y, j_x \rangle + \frac{c^p}{2^{p-1}}\|y\|^p$$

for all $x, y \in E$ and $j_x \in J_p(x)$, where $J_p$ is the generalized duality mapping of $E$ and $1/c$ is the $p$-uniformly convexity constant of $E$.

Further we know the following result [5, 25], which characterizes a $p$-uniformly convex Banach space.

\textbf{Lemma 2.3 ([5, 25])}. Let $p$ be a given real number with $p \geq 2$ and let $E$ be a $p$-uniformly convex Banach space. Then, for all $x, y \in E$, $j_x \in J_p(x)$ and $j_y \in J_p(y)$,

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p}||x - y||^p,$$

where $J_p$ is the generalized duality mapping of $E$ and $1/c$ is the $p$-uniformly convexity constant of $E$.

Let $E$ be a smooth Banach space. We know the following function studied in Alber [1], Kamimura and Takahashi [11] and Reich [16]:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. It is obvious from the definition of $\phi$ that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$. The following lemma which was proved by Kamimura and Takahashi [11] is important:

\textbf{Lemma 2.4 ([11])}. Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n \to \infty} \phi(x_n, y_n) = 0$, then

$$\lim_{n \to \infty} \|x_n - y_n\| = 0.$$

Let $E$ be a reflexive, strictly convex and smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. For each $x \in E$, there corresponds a unique element $x_0 \in C$ (denoted by $\Pi_C(x)$) such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C$ is called the \textit{generalized projection} from $E$ onto $C$; see Alber [1]. If $E$ is a Hilbert space, then $\Pi_C$ is coincident with the metric projection from $E$ onto $C$. We also know the following lemmas [1]; see also [11]:
Lemma 2.5 ([1]; see also [11]). Let $E$ be a smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, let $x \in E$ and let $x_0 \in C$. Then

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x)$$

if and only if

$$\langle y - x_0, Jx_0 - Jx \rangle \geq 0 \text{ for all } y \in C.$$ 

Lemma 2.6 ([1]; see also [11]). Let $E$ be a reflexive, strictly convex and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then

$$\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x)$$

for all $y \in C$.

Using Lemmas 2.4 and 2.6, we have the following lemma:

Lemma 2.7 ([10]). Let $S$ be a nonempty closed convex subset of a uniformly convex and smooth Banach space $E$. Let $\{x_n\}$ be a sequence in $E$. Suppose that, for all $u \in S$,

$$(2.5) \quad \phi(u, x_{n+1}) \leq \phi(u, x_n)$$

for every $n = 1, 2, \ldots$. Then $\{\Pi_S(x_n)\}$ is a Cauchy sequence.

Let $E$ be a reflexive, strictly convex and smooth Banach space and let $J$ be the duality mapping from $E$ into $E^*$. Then $J^{-1}$ is also single-valued, one-to-one and surjective, and it is the duality mapping from $E^*$ into $E$. We make use of the following mapping $V$ studied in Alber [1]:

$$(2.6) \quad V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping $g$ defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous and convex function from $E^*$ into $(-\infty, \infty)$. We know the following lemma [1]:

Lemma 2.8 ([1]). Let $E$ be a reflexive, strictly convex and smooth Banach space and let $V$ be as in (2.6). Then

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

An operator $A$ of $C$ into $E^*$ is said to be hemicontinuous if for all $x, y \in C$, the mapping $f$ of $[0,1]$ into $E^*$ defined by $f(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of $E^*$. We denote by $N_C(v)$ the normal cone for $C$ at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^*: \langle v - y, x^* \rangle \geq 0 \text{ for all } y \in C\}.$$ 

We know the following theorem [17]:

Theorem 2.9 (Rockafellar [17]). Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $A$ be a monotone and hemicontinuous operator of $C$ into $E^*$. Let $T \subset E \times E^*$ be an operator defined as follows:

$$T_v = \begin{cases} A v + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then $T$ is maximal monotone and $T^{-1}0 = VI(C, A)$.

We also know the following lemma (Lemma 7.1.7 of [22]):
Lemma 2.10 ([22]). Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $A$ be a monotone and hemicontinuous operator of $C$ into $E^*$. Then

\[ \text{VI}(C, A) = \{ u \in C : \langle v - u, Av \rangle \geq 0 \text{ for all } v \in C \}. \]

It is obvious from Lemma 2.10 that the set $\text{VI}(C, A)$ is a closed convex subset of $C$. Further, we know the following lemma (Theorem 7.1.8 of [22]):

Lemma 2.11 ([22]). Let $C$ be a nonempty compact convex subset of a Banach space $E$ and let $A$ be a monotone and hemicontinuous operator of $C$ into $E^*$. Then the set $\text{VI}(C, A)$ is nonempty.

3. Weak convergence theorems

Let $C$ be a nonempty closed convex subset of a Banach space $E$. If an operator $A$ of $C$ into $E^*$ is $\alpha$-inverse-strongly-monotone, then $A$ is Lipschitz continuous, that is, $\|Ax - Ay\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$.

Now we can state the following weak convergence theorem for finding a solution of the variational inequality for an inverse-strongly-monotone operator in a 2-uniformly convex and uniformly smooth Banach space:

Theorem 3.1. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous and let $C$ be a nonempty closed convex subset of $E$. Let $A$ be an operator of $C$ into $E^*$ which satisfies the conditions (1), (2) and (3). Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

\[ x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n) \]

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z$ of $\text{VI}(C, A)$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_{\text{VI}(C, A)}(x_n)$.

Using Theorem 3.1, we consider some weak convergence theorems in a 2-uniformly convex and uniformly smooth Banach space. We first study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space. Before considering this problem, we state the following lemma which was proved by Baillon and Haddad [3]:

Lemma 3.2 ([3]). Let $E$ be a Banach space, let $f$ be a continuously Fréchet differentiable convex functional on $E$ and let $\nabla f$ be the gradient of $f$. If $\nabla f$ is $1/\alpha$-Lipschitz continuous, then $\nabla f$ is $\alpha$-inverse-strongly-monotone.

Now we can consider the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space.

Theorem 3.3. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a functional on $E$ which satisfies the following conditions:

1. $f$ is a continuously Fréchet differentiable convex functional on $E$ and $\nabla f$ is $1/\alpha$-Lipschitz continuous,
2. $S = \arg \min_{y \in C} f(y) = \{ z \in C : f(z) = \min_{y \in C} f(y) \} \neq \emptyset$,
3. $\|\nabla f|_C(y)\| \leq \|\nabla f|_C(y) - \nabla f|_C(u)\|$ for all $y \in C$ and $u \in S$.

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

\[ x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n \nabla f|_C(x_n)) \]
for every \( n = 1, 2, \ldots \), where \( \{\lambda_n\} \) is a sequence of positive numbers. If \( \{\lambda_n\} \) is chosen so that \( \lambda_n \in [a, b] \) for some \( a, b \) with \( 0 < a < b < c^2 \alpha/2 \), then the sequence \( \{x_n\} \) converges weakly to some element \( z \) of \( S \), where \( 1/c \) is the 2-uniformly convexity constant of \( E \). Further \( z = \lim_{n \to \infty} \Pi_S(x_n). \)

We next consider the problem of finding a zero point of an inverse-strongly-monotone operator of \( E \) into \( E^* \). In the case when \( C = E \), the condition (3) of Theorem 3.1 holds.

**Theorem 3.4.** Let \( E \) be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping \( J \) is weakly sequentially continuous. Let \( A \) be an operator of \( E \) into \( E^* \) which satisfies the following conditions:

1. \( A \) is \( \alpha \)-inverse-strongly-monotone,
2. \( A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset. \)

Suppose \( x_1 = x \in E \) and \( \{x_n\} \) is given by

\[
x_{n+1} = J^{-1}(Jx_n - \lambda_n Ax_n)
\]

for every \( n = 1, 2, \ldots \), where \( \{\lambda_n\} \) is a sequence of positive numbers. If \( \{\lambda_n\} \) is chosen so that \( \lambda_n \in [a, b] \) for some \( a, b \) with \( 0 < a < b < c^2 \alpha/2 \), then the sequence \( \{x_n\} \) converges weakly to some element \( z \) of \( A^{-1}0 \), where \( 1/c \) is the 2-uniformly convexity constant of \( E \). Further \( z = \lim_{n \to \infty} \Pi_{A^{-1}0}(x_n). \)

Using Theorem 3.4, we can also consider the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space.

**Corollary 3.5.** Let \( E \) be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping \( J \) is weakly sequentially continuous. Let \( f \) be a functional on \( E \) which satisfies the following conditions:

1. \( f \) is a continuously Fréchet differentiable convex functional on \( E \) and \( \nabla f \) is \( 1/\alpha \)-Lipschitz continuous,
2. \( (\nabla f)^{-1}0 = \{z \in E : f(z) = \min_{y \in E} f(y)\} \neq \emptyset. \)

Suppose \( x_1 = x \in E \) and \( \{x_n\} \) is given by

\[
x_{n+1} = J^{-1}(Jx_n - \lambda_n \nabla f(x_n))
\]

for every \( n = 1, 2, \ldots \), where \( \{\lambda_n\} \) is a sequence of positive numbers. If \( \{\lambda_n\} \) is chosen so that \( \lambda_n \in [a, b] \) for some \( a, b \) with \( 0 < a < b < c^2 \alpha/2 \), then the sequence \( \{x_n\} \) converges weakly to some element \( z \) of \( (\nabla f)^{-1}0 \), where \( 1/c \) is the 2-uniformly convexity constant of \( E \). Further \( z = \lim_{n \to \infty} \Pi_{(\nabla f)^{-1}0}(x_n). \)

Further we consider the problem of finding a unique solution of the variational inequality for a strongly monotone and Lipshitz continuous operator. An operator \( A \) of \( C \) into \( E^* \) is said to be strongly monotone if there exists a positive real number \( \alpha \) such that

\[
\langle x - y, Ax - Ay \rangle \geq \alpha \|x - y\|^2
\]

for all \( x, y \in C \). For such a case, \( A \) is said to be \( \alpha \)-strongly monotone. Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). One method of finding a point \( u \in VI(C, A) \) is the projection algorithm which starts with any \( x_1 = x \in C \) and updates iteratively \( x_{n+1} \) according to the formula (1.3). It is well known that if \( A \) is an \( \alpha \)-strongly monotone and \( \beta \)-Lipschitz continuous operator of \( C \) into \( H \) and \( \{\lambda_n\} \subset (0, 2\alpha/\beta^2) \), then the operator \( P_C(I - \lambda_n A) \) is a contraction of \( C \) into itself. Hence, the Banach contraction principle guarantees that the sequence generated by (1.3) converges strongly to the unique solution of \( VI(C, A) \). Motivated by this result, we obtain the following:
Theorem 3.6. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous and let $C$ be a nonempty closed convex subset of $E$. Let $A$ be an operator of $C$ into $E^*$ which satisfies the following conditions:

1. $A$ is $\alpha$-strongly monotone and $\beta$-Lipschitz continuous,
2. $VI(C, A) \neq \emptyset$,
3. $\|Ay\| \leq \|Ay - Az\|$ for all $y \in C$ and $\{z\} = VI(C, A)$.

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_{C}J^{-1}(Jx_n - \lambda_n A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/(2\beta^2)$, then the sequence $\{x_n\}$ converges weakly to a unique element $z$ of $VI(C, A)$, where $1/c$ is the 2-uniformly convexity constant of $E$.

Finally we consider the complementarity problem. Let $K$ be a nonempty closed convex cone in $E$, let $A$ be an operator of $K$ into $E^*$ and define its polar in $E^*$ to be the set $K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0 \text{ for all } x \in K\}$.

Then an element $u \in K$ is called a solution of the complementarity problem for $A$ if $Au \in K^*$ and $\langle u, Au \rangle = 0$.

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

Theorem 3.7. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping $J$ is weakly sequentially continuous and let $K$ be a nonempty closed convex cone in $E$. Let $A$ be an operator of $K$ into $E^*$ which satisfies the following conditions:

1. $A$ is $\alpha$-inverse-strongly-monotone,
2. $C(K, A) \neq \emptyset$,
3. $\|Ay\| \leq \|Ay - Au\|$ for all $y \in K$ and $u \in C(K, A)$.

Suppose $x_1 = x \in K$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_{K}J^{-1}(Jx_n - \lambda_n A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z$ of $C(K, A)$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_{C(K, A)}(x_n)$.

REFERENCES


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