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APPROXIMATING SOLUTIONS OF NONLINEAR VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. Let C be a nonempty closed convex subset of a Banach space E and let A be an inverse-strongly-monotone operator of C into the dual space E^* of E. In this paper, we introduce the following iterative scheme for finding a solution of the variational inequality problem for A: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where Π_C is the generalized projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. Then we obtain a weak convergence theorem (Theorem 3.1). Using this result, we consider the problem of finding a minimizer of a convex function, the problem of finding a point $u \in E$ satisfying $0 = Au$ and so on.

1. INTRODUCTION

Let E be a real Banach space with norm $\|\cdot\|$, let E^* denote the dual of E and let $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. Let C be a nonempty closed convex subset of E and let A be a monotone operator of C into E^*. Then we deal with the problem of finding

(1.1) a point $u \in C$ such that $\langle v - u, Au\rangle \geq 0$ for all $v \in C$.

This problem is called the *variational inequality* problem; see [14] and [13]. The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$. An operator A of C into E^* is said to be *inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$; see [6], [15] and [9]. For such a case, A is said to be α-inverse-strongly-monotone.

For finding a zero point of an inverse-strongly-monotone operator of the Euclidean space $\mathbb{R}^N$ into itself, Gol'shteĭn and Tret'yakov [8] introduced the following scheme: $x_1 = x \in \mathbb{R}^N$ and

$$x_{n+1} = x_n - \lambda_n Ax_n \tag{1.2}$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ generated by (1.2) converges to some element of $A^{-1}0$, where $A^{-1}0 = \{u \in \mathbb{R}^N : Au = 0\}$.

In the case when A is an inverse-strongly-monotone operator of a closed convex subset C of a Hilbert space H into H, one method of finding a point $u \in \mathrm{VI}(C, A)$ is the projection algorithm: $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n Ax_n) \tag{1.3}$$

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for every $n = 1, 2, \ldots$, where P_C is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive numbers. Iiduka, Takahashi and Toyoda [9] proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to some element of $\mathrm{VI}(C, A)$.

In the case when the space is a Banach space E, Alber [1] proved the following strong convergence theorem by the *genelarized projection* algorithm:

Theorem 1.1 (Alber [1]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Suppose an operator A of E into E^* satisfies the following conditions:*

(i) *A is uniformly monotone, that is, $\langle x - y, Ax - Ay\rangle \geq \psi(\|x - y\|)$ for all $x, y \in E$, where $\psi(t)$ is a continuous strictly increasing function for all $t \geq 0$ with $\psi(0) = 0$,*
(ii) *$\mathrm{VI}(C, A) \neq \emptyset$,*
(iii) *A has ϕ-arbitrary growth, that is, $\|Ay\| \leq \phi(\|y - z\|)$ for all $y \in E$ and $\{z\} = \mathrm{VI}(C, A)$, where $\phi(t)$ is a continuous nondecreasing function with $\phi(0) \geq 0$.*

Define a sequence $\{x_n\}$ as follows: $x_1 = x \in E$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where Π_C is the generalized projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a positive nonincreasing sequence which satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a unique element z of $\mathrm{VI}(C, A)$.

On the other hand, for finding a zero point of a maximal monotone operator, by using the *proximal point algorithm*, Kamimura, Kohsaka and Takahashi [12] proved the following weak convergence theorem:

Theorem 1.2 (Kamimura, Kohsaka and Takahashi [12]). *Let E be a uniformly convex and uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let $A \subset E \times E^*$ be a maximal monotone operator, let $J_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$x_{n+1} = J_{r_n} x_n$$

for every $n = 1, 2, \ldots$, where $\{r_n\} \subset (0, \infty)$ satisfies $\limsup_{n\to\infty} r_n > 0$. If $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element z of $A^{-1}0$. Further $z = \lim_{n\to\infty} \Pi_{A^{-1}0}(x_n)$, where $\Pi_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$.

In this paper, motivated by Alber [1], we introduce an iterative scheme for finding a solution of the variational inequality problem for an operator A which satisfies the following conditions in a 2-uniformly convex and uniformly smooth Banach space E:

(1) A is inverse-strongly-monotone,
(2) $\mathrm{VI}(C, A) \neq \emptyset$,
(3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in \mathrm{VI}(C, A)$.

Then we obtain a weak convergence theorem (Theorem 3.1). Further, using this result, we consider the minimization problem (Theorem 3.3 and Corollary 3.5), the complementarity problem (Theorem 3.7), the problem of finding a point $u \in E$ satisfying $0 = Au$ (Theorem 3.4) and so on.

2. Preliminaries

Let E be a real Banach space. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. A multi-valued operator $T : E \to 2^{E^*}$ with domain $\mathrm{D}(T) = \{z \in E : Tz \neq \emptyset\}$ and range $\mathrm{R}(T) = \bigcup\{Tz \in E : z \in \mathrm{D}(T)\}$ is said to be *monotone* if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$. A monotone operator T is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *strictly convex* if for any $x, y \in U$,

$$x \neq y \text{ implies } \left\|\frac{x+y}{2}\right\| < 1.$$

It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. And we define a function $\delta : [0, 2] \to [0, 1]$ called the *modulus of convexity* of E as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}.$$

It is known that E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. Then E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$. For example, see [4] and [23] for more details. We know the following fundamental characterization [4, 5] of p-uniformly convex Banach spaces:

Lemma 2.1 ([4, 5]). *Let p be a real number with $p \geq 2$ and let E be a Banach space. Then E is p-uniformly convex if and only if there exists a constant c with $0 < c \leq 1$ such that*

$$\frac{1}{2}(\|x + y\|^p + \|x - y\|^p) \geq \|x\|^p + c^p\|y\|^p \tag{2.1}$$

for all $x, y \in E$.

The best constant $1/c$ in Lemma 2.1 is called the *p-uniformly convexity constant* of E; see [4]. Putting $x = (u + v)/2$ and $y = (u - v)/2$ in (2.1), we readily conclude that, for all $u, v \in E$,

$$\frac{1}{2}(\|u\|^p + \|v\|^p) \geq \left\|\frac{u+v}{2}\right\|^p + c^p\left\|\frac{u-v}{2}\right\|^p. \tag{2.2}$$

A Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.3) is attained uniformly for $x, y \in U$. One should note that no Banach space is p-uniformly convex for $1 < p < 2$; see [23] for more details. It is well known that Hilbert and the Lebesgue L^q $(1 < q \leq 2)$ spaces are 2-uniformly convex and uniformly smooth. Let X be a Banach space and let $L^q(X) = L^q(\Omega, \Sigma, \mu; X)$, $1 \leq q \leq \infty$, be the Lebesgue-Bochner space on an arbitrary measure space (Ω, Σ, μ). Let $2 \leq p < \infty$ and let $1 < q \leq p$. Then $L^q(X)$ is p-uniformly convex if and only if X is p-uniformly convex; see [23]. For the weak convergence in the Lebesgue spaces L^p $(p \geq 2)$, see Aoyama, Iiduka and Takahashi [10].

On the other hand, with each $p > 1$, the (generalized) *duality mapping* J_p from E into 2^{E^*} is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. The duality mapping J has the following properties:

- If E is smooth, then J is single-valued;
- if E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;
- if E is reflexive, then J is surjective;
- if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

See [22] for more details. The duality mapping J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* if $x_n \rightharpoonup x$ implies that $Jx_n \overset{*}{\rightharpoonup} Jx$, where $\overset{*}{\rightharpoonup}$ implies the weak* convergence; see [7]. It is also known that

$$p\langle y - x, j_x \rangle \leq \|y\|^p - \|x\|^p \tag{2.4}$$

for all $x, y \in E$ and $j_x \in J_p(x)$. We know the following result [24], which characterizes a p-uniformly convex Banach space.

Lemma 2.2 ([24]). *Let p be a given real number with $p \geq 2$ and let E be a p-uniformly convex Banach space. Then*

$$\|x + y\|^p \geq \|x\|^p + p\langle y, j_x \rangle + \frac{c^p}{2^{p-1}} \|y\|^p$$

for all $x, y \in E$ and $j_x \in J_p(x)$, where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Further we know the following result [5, 25], which characterizes a p-uniformly convex Banach space.

Lemma 2.3 ([5, 25]). *Let p be a given real number with $p \geq 2$ and let E be a p-uniformly convex Banach space. Then, for all $x, y \in E$, $j_x \in J_p(x)$ and $j_y \in J_p(y)$,*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p} \|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Let E be a smooth Banach space. We know the following function studied in Alber [1], Kamimura and Takahashi [11] and Reich [16]:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. It is obvious from the definition of ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$. The following lemma which was proved by Kamimura and Takahashi [11] is important:

Lemma 2.4 ([11]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed convex subset of E. For each $x \in E$, there corresponds a unique element $x_0 \in C$ (denoted by $\Pi_C(x)$) such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The mapping Π_C is called the *generalized projection* from E onto C; see Alber [1]. If E is a Hilbert space, then Π_C is coincident with the metric projection from E onto C. We also know the following lemmas [1]; see also [11]:

Lemma 2.5 ([1]; see also [11]). *Let E be a smooth Banach space, let C be a nonempty closed convex subset of E, let $x \in E$ and let $x_0 \in C$. Then*

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x)$$

if and only if

$$\langle y - x_0, Jx_0 - Jx \rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.6 ([1]; see also [11]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x)$$

for all $y \in C$.

Using Lemmas 2.4 and 2.6, we have the following lemma:

Lemma 2.7 ([10]). *Let S be a nonempty closed convex subset of a uniformly convex and smooth Banach space E. Let $\{x_n\}$ be a sequence in E. Suppose that, for all $u \in S$,*

$$\phi(u, x_{n+1}) \leq \phi(u, x_n) \tag{2.5}$$

for every $n = 1, 2, \ldots$. Then $\{\Pi_S(x_n)\}$ is a Cauchy sequence.

Let E be a reflexive, strictly convex and smooth Banach space and let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one and surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.6}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping g defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous and convex function from E^* into $(-\infty, \infty)$. We know the following lemma [1]:

Lemma 2.8 ([1]). *Let E be a reflexive, strictly convex and smooth Banach space and let V be as in* (2.6). *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

An operator A of C into E^* is said to be *hemicontinuous* if for all $x, y \in C$, the mapping f of [0,1] into E^* defined by $f(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $\mathrm{N}_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$\mathrm{N}_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0 \text{ for all } y \in C\}.$$

We know the following theorem [17]:

Theorem 2.9 (Rockafellar [17]). *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + \mathrm{N}_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = \mathrm{VI}(C, A)$.

We also know the following lemma (Lemma 7.1.7 of [22]):

Lemma 2.10 ([22]). *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemicontinuous operator of C into E^*. Then*

$$\mathrm{VI}(C, A) = \{u \in C : \langle v - u, Av \rangle \geq 0 \text{ for all } v \in C\}.$$

It is obvious from Lemma 2.10 that the set $\mathrm{VI}(C, A)$ is a closed convex subset of C. Further, we know the following lemma (Theorem 7.1.8 of [22]):

Lemma 2.11 ([22]). *Let C be a nonempty compact convex subset of a Banach space E and let A be a monotone and hemicontinuous operator of C into E^*. Then the set $\mathrm{VI}(C, A)$ is nonempty.*

3. Weak convergence theorems

Let C be a nonempty closed convex subset of a Banach space E. If an operator A of C into E^* is α-inverse-strongly-monotone, then A is *Lipschitz continuous*, that is, $\|Ax - Ay\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$.

Now we can state the following weak convergence theorem for finding a solution of the variational inequality for an inverse-strongly-monotone operator in a 2-uniformly convex and uniformly smooth Banach space:

Theorem 3.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous and let C be a nonempty closed convex subset of E. Let A be an operator of C into E^* which satisfies the conditions* (1), (2) *and* (3). *Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element z of $\mathrm{VI}(C, A)$, where $1/c$ is the 2-uniformly convexity constant of E. Further $z = \lim_{n\to\infty} \Pi_{\mathrm{VI}(C,A)}(x_n)$.

Using Theorem 3.1, we consider some weak convergence theorems in a 2-unifomly convex and uniformly smooth Banach space. We first study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space. Before considering this problem, we state the following lemma which was proved by Baillon and Haddad [3]:

Lemma 3.2 ([3]). *Let E be a Banach space, let f be a continuously Fréchet differentiable convex functional on E and let ∇f be the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse-strongly-monotone.*

Now we can consider the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space.

Theorem 3.3. *Let E be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous and let C be a nonempty closed convex subset of E. Let f be a functional on E which satisfies the following conditions:*

(1) *f is a continuously Fréchet differentiable convex functional on E and ∇f is $1/\alpha$-Lipshitz continuous,*
(2) $S = \arg\min_{y\in C} f(y) = \{z \in C : f(z) = \min_{y\in C} f(y)\} \neq \emptyset$,
(3) $\|\nabla f|_C(y)\| \leq \|\nabla f|_C(y) - \nabla f|_C(u)\|$ *for all $y \in C$ and $u \in S$.*

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n \nabla f|_C(x_n))$$

for every $n = 1, 2, \ldots$, *where* $\{\lambda_n\}$ *is a sequence of positive numbers. If* $\{\lambda_n\}$ *is chosen so that* $\lambda_n \in [a, b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *then the sequence* $\{x_n\}$ *converges weakly to some element* z *of* S, *where* $1/c$ *is the* 2*-uniformly convexity constant of* E. *Further* $z = \lim_{n\to\infty} \Pi_S(x_n)$.

We next consider the problem of finding a zero point of an inverse-strongly-monotone operator of E into E^*. In the case when $C = E$, the condition (3) of Theorem 3.1 holds.

Theorem 3.4. *Let* E *be a* 2*-uniformly convex and uniformly smooth Banach space whose duality mapping* J *is weakly sequentially continuous. Let* A *be an operator of* E *into* E^* *which satisfies the following conditions:*

(1) A *is* α*-inverse-strongly-monotone,*
(2) $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$.

Suppose $x_1 = x \in E$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, *where* $\{\lambda_n\}$ *is a sequence of positive numbers. If* $\{\lambda_n\}$ *is chosen so that* $\lambda_n \in [a, b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *then the sequence* $\{x_n\}$ *converges weakly to some element* z *of* $A^{-1}0$, *where* $1/c$ *is the* 2*-uniformly convexity constant of* E. *Further* $z = \lim_{n\to\infty} \Pi_{A^{-1}0}(x_n)$.

Using Theorem 3.4, we can also consider the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Banach space.

Corollary 3.5. *Let* E *be a* 2*-uniformly convex and uniformly smooth Banach space whose duality mapping* J *is weakly sequentially continuous. Let* f *be a functional on* E *which satisfies the following cinditions:*

(1) f *is a continuously Fréchet differentiable convex functional on* E *and* ∇f *is* $1/\alpha$*-Lipshitz continuous,*
(2) $(\nabla f)^{-1}0 = \{z \in E : f(z) = \min_{y\in E} f(y)\} \neq \emptyset$.

Suppose $x_1 = x \in E$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n \nabla f(x_n))$$

for every $n = 1, 2, \ldots$, *where* $\{\lambda_n\}$ *is a sequence of positive numbers. If* $\{\lambda_n\}$ *is chosen so that* $\lambda_n \in [a, b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *then the sequence* $\{x_n\}$ *converges weakly to some element* z *of* $(\nabla f)^{-1}0$, *where* $1/c$ *is the* 2*-uniformly convexity constant of* E. *Further* $z = \lim_{n\to\infty} \Pi_{(\nabla f)^{-1}0}(x_n)$.

Further we consider the problem of finding a unique solution of the variational inequality for a strongly monotone and Lipshitz continuous operator. An operator A of C into E^* is said to be *strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|x - y\|^2$$

for all $x, y \in C$. For such a case, A is said to be α-strongly monotone. Let C be a nonempty closed convex subset of a Hilbert space H. One method of finding a point $u \in \mathrm{VI}(C, A)$ is the projection algorithm which starts with any $x_1 = x \in C$ and updates iteratively x_{n+1} according to the formula (1.3). It is well known that if A is an α-strongly monotone and β-Lipschitz continuous operator of C into H and $\{\lambda_n\} \subset (0, 2\alpha/\beta^2)$, then the operator $P_C(I - \lambda_n A)$ is a contraction of C into itself. Hence, the Banach contraction principle guarantees that the sequence generated by (1.3) converges strongly to the unique solution of $\mathrm{VI}(C, A)$. Motivated by this result, we obtain the following:

Theorem 3.6. *Let E be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous and let C be a nonempty closed convex subset of E. Let A be an operator of C into E^* which satisfies the following conditions:*

(1) *A is α-strongly monotone and β-Lipschitz continuous,*
(2) $\mathrm{VI}(C, A) \neq \emptyset$,
(3) $\|Ay\| \leq \|Ay - Az\|$ *for all $y \in C$ and* $\{z\} = \mathrm{VI}(C, A)$.

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/(2\beta^2)$, then the sequence $\{x_n\}$ converges weakly to a unique element z of $\mathrm{VI}(C, A)$, where $1/c$ is the 2-uniformly convexity constant of E.

Finally we consider the complementarity problem. Let K be a nonempty closed convex cone in E, let A be an operator of K into E^* and define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0 \text{ for all } x \in K\}.$$

Then an element $u \in K$ is called a solution of the *complementarity problem* for A if

$$Au \in K^* \text{ and } \langle u, Au \rangle = 0.$$

The set of solutions of the complementarity problem is denoted by $\mathrm{C}(K, A)$.

Theorem 3.7. *Let E be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous and let K be a nonempty closed convex cone in E. Let A be an operator of K into E^* which satisfies the following conditions:*

(1) *A is α-inverse-strongly-monotone,*
(2) $\mathrm{C}(K, A) \neq \emptyset$,
(3) $\|Ay\| \leq \|Ay - Au\|$ *for all $y \in K$ and $u \in \mathrm{C}(K, A)$.*

Suppose $x_1 = x \in K$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element z of $\mathrm{C}(K, A)$, where $1/c$ is the 2-uniformly convexity constant of E. Further $z = \lim_{n\to\infty} \Pi_{\mathrm{C}(K,A)}(x_n)$.

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