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Some analytic aspects of the mean field equation for arbitrarily-signed vortices[1]

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Abstract

In this note, we review our recent results on the mean field equation for arbitrarily-signed vortices. Our main interest is the blow-up analysis to the equation, that is, the classification of the asymptotic behaviour of the non-compact solution sequences. We also mention some applications of the blow-up analysis including the Trudinger-Moser type inequality relating to the equation.

This is a joint work with Prof. Takashi Suzuki of Osaka University.

1 Introduction

Let (M,g) be a two-dimensional compact orientable Riemannian manifold without boundary. We are concerned with the mean field equation for arbitrarily-signed vortices on (M,g) :

$$-\Delta_g v = \lambda_1 \left(\frac{e^v}{\int_M e^v dv_g} - \frac{1}{|M|} \right) - \lambda_2 \left(\frac{e^{-v}}{\int_M e^{-v} dv_g} - \frac{1}{|M|} \right), \quad \int_M v dv_g = 0, \tag{1}$$

where Δ_g, dv_g, and $|M|$ are the Laplace-Beltrami operator, the volume form, and the volume of M, respectively. λ_1 and λ_2 are non-negative constants.

It describes the mean field of the equilibrium turbulence with arbitrarily signed vortices [24, 12, 22, 26], and is obtained by Joyce and Montgomery [15] and Pointin and Lundgren [34] from different statistical arguments. Here, these vortices are composed of positive and negative intensities with the same absolute value, and v and $\lambda_1 : \lambda_2$ are associated with the stream function of the fluid and the ratio of the numbers of the signed vortices, respectively, see Section 2.

When we assume all the vortices have a definite-signed (constant) intensity, the mean field equation (1) reduces to the case $\lambda_2 = 0$:

$$-\Delta_g v = \lambda \left(\frac{e^v}{\int_M e^v dv_g} - \frac{1}{|M|} \right), \quad \int_M v dv_g = 0, \tag{2}$$

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which is studied by several authors [5, 6, 19] in this context. Moreover this equation and it generalization

$$-\Delta_g v = \lambda\left(\frac{K(x)e^v}{\int_M K(x)e^v dv_g} - \frac{1}{|M|}\right), \quad \int_M v dv_g = 0, \tag{3}$$

with the inhomogeneous coefficient $K(x) \geq 0$ widely appears in several fields such as in the self-dual gauge field theory [45], in the stationary system of chemotaxis or in the self-interacting particles [43], and in the prescribing Gaussian curvature problem [2], see also [25, 42, 4, 21, 20, 41, 13, 37, 29, 30, 3, 9, 10]. Among them, here we recall the following fact:

Fact 1.1 ([4, 21, 20]). *Suppose a sequence of non-negative constants $\lambda_n \longrightarrow \lambda$ as $n \longrightarrow \infty$. Then the sequence of solutions of (2) for $\{\lambda_n\}$ is relatively compact in $E = \{w \in H^1(M); \int_M w = 0\}$ if*

$$\lambda \in [0, \infty)\backslash 8\pi\mathbf{N}.$$

In contrast to Fact 1.1, we establish the following result for the indefinite cases:

Theorem 1.2 (Main Theorem [32]). *Suppose a sequence of pairs of non-negative constants $(\lambda_{1,n}, \lambda_{2,n}) \longrightarrow (\lambda_1, \lambda_2)$ as $n \longrightarrow \infty$. Then the sequence of solutions of (1) for $\{(\lambda_{1,n}, \lambda_{2,n})\}$ is relatively compact in E if*

$$(\lambda_1, \lambda_2) \in ([0, 4(2+\sqrt{5}\pi))\backslash 8\pi\mathbf{N}) \times ([0, 4(2+\sqrt{5}\pi))\backslash 8\pi\mathbf{N}). \tag{4}$$

Especially, every sequence of solution of (1) is relatively compact if

$$(\lambda_1, \lambda_2) \in [0, 8\pi) \times [0, 8\pi).$$

Compared to Fact 1.1, our result Theorem 1.2 seems to be a partial result. Nevertheless it is enough to get the following applications.

The equation (1) is the Euler-Lagrange equation of the functional $J_{\lambda_1,\lambda_2}(u)$ on E:

$$J_{\lambda_1,\lambda_2}(u) = \frac{1}{2}\int_M |\nabla_g u|^2 dv_g - \lambda_1 \log \int_M e^u dv_g - \lambda_2 \log \int_M e^{-u} dv_g,$$

When $\lambda_2 = 0$, the well-known Trudinger-Moser inequality on (M, g) [14] indicates the following inequalities for $J_\lambda(u) := J_{\lambda,0}(u)$:

$$\begin{array}{ll} \inf_{v\in E} J_\lambda(v) > -\infty & \text{if } \lambda \in [0, 8\pi] \\ \inf_{v\in E} J_\lambda(v) = -\infty & \text{if } \lambda > 8\pi. \end{array} \tag{5}$$

Here, we have

$$J_{\lambda_1,\lambda_2}(v) = \frac{1}{2}\left(1 - \frac{\lambda_1}{8\pi} - \frac{\lambda_2}{8\pi}\right)\|v\|_E^2 + \frac{\lambda_1}{8\pi}J_{8\pi}(v) + \frac{\lambda_2}{8\pi}J_{8\pi}(-v),$$

and therefore,

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) > -\infty \quad \text{if } 1 - \frac{\lambda_1}{8\pi} - \frac{\lambda_2}{8\pi} \geq 0.$$

Using Theorem 1.2, however, this trivial inequality is improved as follows :

Corollary 1.3 ([32]). *It hold that*

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) > -\infty \quad \text{if } (\lambda_1, \lambda_2) \in [0, 8\pi] \times [0, 8\pi], \tag{6}$$

and in particular, J_{λ_1,λ_2} *has a global minimizer on* E *if* $0 \leq \lambda_1, \lambda_2 < 8\pi$.

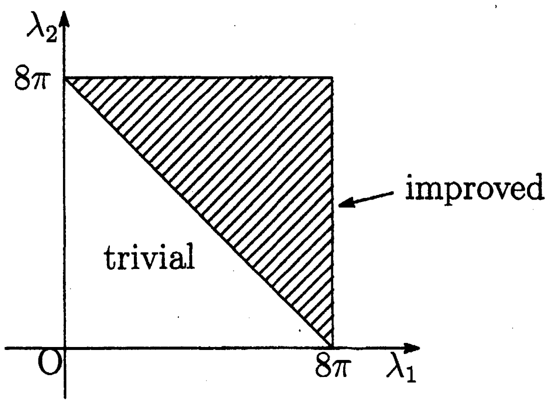

Figure 1: The region of the parameters for the Trudinger-Moser inequality.

Similar inequalities for a bounded domain $\Omega \subset \mathbf{R}^2$ with the Dirichlet or the Neumann boundary conditions are easily derived from Corollary 1.3, see Section 4.

As we see in Corollary 1.3, the functional $J_{\lambda_1,\lambda_2}(v)$ has a global minimizer if $(\lambda_1, \lambda_2) \in [0, 8\pi) \times [0, 8\pi)$ and this indicates the solution of the equation (1). But this functional is not bounded from below if $\lambda_1 > 8\pi$ or $\lambda_2 > 8\pi$, that is, the inequality (6) is optimal

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) = -\infty \quad \text{if } \lambda_1 > 8\pi \text{ or } \lambda_2 > 8\pi. \tag{7}$$

In fact, we have

$$J_{\lambda_1,\lambda_2}(v) = J_{\lambda_1}(v) - \lambda_2 \log \int_M e^{-v} dv_g \tag{8}$$

and from Jensen's inequality

$$\log \int_M e^{-v} dv_g \geq \log |M| . \tag{9}$$

Combining (5), (8), and (9) (and similar argument if $\lambda_2 > 8\pi$), we obtain (7). Therefore we need another device to get solutions if $\lambda_1 > 8\pi$ or $\lambda_2 > 8\pi$. As another application of Theorem 1.2, we are able to prove the existence of solutions of (1) for some of these cases by variational methods.

For $\lambda_2 = 0$ cases (2), we have already two variational methods by Struwe-Tarantello [41] and Ding-Jost-Li-Wang [13]. The former one construct a non-trivial solution by the mountain pass method when M is a flat torus with the fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in (8\pi, 4\pi^2)$. Discussing the general setting of the Riemannian surface, (2) has a non-trivial mountain pass solution (Struwe-Tarantello solution) if $\lambda \in (8\pi, \mu_1 |M|)$, where μ_1 denotes the principal eigenvalue of $-\Delta_g$. On the other hand, Ding-Jost-Li-Wang [13] showed the existence of solution by a higher dimensional min-max method if M has genus $g \geq 1$ and $8\pi < \lambda < 16\pi$. This solution may be a trivial solution $v = 0$ to (2), though we consider that it may be non-trivial if $\lambda \in (8\pi, \min\{\mu_1 |M|, 16\pi\})$ and may be different even from the Struwe-Tarantello mountain pass solution from the structure of each variational scheme and the Chen-Lin's formula [10] to (2) concerning the total degree. See [7] for further discussion on this point.

The important fact in both variational methods is that it is not known whether the Palais-Smale condition for J_λ holds or not if $\lambda > 8\pi$. Therefore it is not obvious that the min-max values determined by each variational schemes are indeed critical values. To overcome this difficulty, they used the so-called *Struwe's Monotonicity Trick*, that is, they showed the min-max values are indeed the critical values for *almost every* parameter values in each range by using the monotonicity of J_λ in λ. Then, using Fact 1.1, they prove the existence of the solutions for parameters in residual set by the approximating argument. For $\lambda_2 \neq 0$ cases, the similar variational structures and the monotonicity in the parameters λ_1 and λ_2 of J_{λ_1,λ_2} also exist and Theorem 1.2 is sufficient to prove the existence of solutions for parameters in residual set. The results are summarized as follows:

Theorem 1.4. *Suppose $\lambda_1 > 8\pi$ or $\lambda_2 > 8\pi$. Then the followings hold:*

1. *(Struwe-Tarantello type solution) The functional J_{λ_1,λ_2} has a mountain-pass non-trivial critical point if*

$$\lambda_1 + \lambda_2 < \mu_1|M|.$$

(When M is the flat torus $[-\frac{1}{2},\frac{1}{2}] \times [-\frac{1}{2},\frac{1}{2}]$, the value $\mu_1|M| = 4\pi^2$.)

2. *(Ding-Jost-Li-Wang type solution) If M has genus ≥ 1, the functional J_{λ_1,λ_2} has a min-max critical point if*

$$\lambda_1 + \lambda_2 < 16\pi.$$

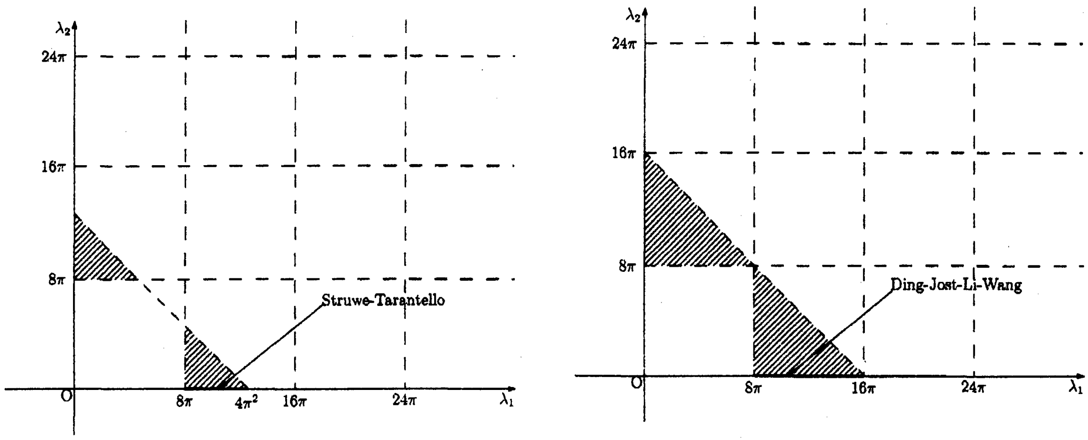

Figure 2: The parameter regions for variational solutions.

See Section 5 for the precise definitions of the min-max schemes. The important question of the non-triviality of Ding-Jost-Li-Wang type solution will be studied in a forthcoming paper.

The key to prove Theorem 1.2 is to assume a solution of (1) to be a solution of *a Liouville system*. Indeed, suppose u_1 and u_2 are solutions of the equations

$$-\Delta_g u_1 = \lambda_1 \left(\frac{e^v}{\int_M e^v dv_g} - \frac{1}{|M|} \right), \quad \int_M u_1 dv_g = 0,$$
$$-\Delta_g u_2 = \lambda_2 \left(\frac{e^{-v}}{\int_M e^{-v} dv_g} - \frac{1}{|M|} \right), \quad \int_M u_2 dv_g = 0.$$

Then it holds that $v = u_1 - u_2$ and u_1 and u_2 are the solutions of the Liouville system

$$
\begin{aligned}
-\Delta_g u_1 &= \lambda_1 \left(\frac{e^{a_{11}u_1 + a_{12}u_2}}{\int_M e^{a_{11}u_1+a_{12}u_2} dv_g} - \frac{1}{|M|} \right) \quad \int_M u_1 dv_g = 0 \\
-\Delta_g u_2 &= \lambda_2 \left(\frac{e^{a_{21}u_1 + a_{22}u_2}}{\int_M e^{a_{21}u_1+a_{22}u_2} dv_g} - \frac{1}{|M|} \right) \quad \int_M u_2 dv_g = 0 \qquad (10)
\end{aligned}
$$

for $A = (a_{ij}) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. When $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, the Liouville system is called the $SU(3)$ Toda system that arises in non-abelian relativistic self-dual gauge theory studied by several authors mathematically [17, 18, 23, 7, 31, 16]. Using the similar argument to [17] for similar results for $SU(3)$ Toda system, we establish Theorem 1.2 and Corollary 1.3. Moreover Theorem 1.4 1 and 2 are similar to the results of [23, 7], respectively. Furthermore, similar to [31] for $SU(3)$ Toda system, we are able to improve Theorem 1.2 as follows by using the rescaling argument à la Li-Shafrir [21]:

Theorem 1.5 ([33]). *The conclusion of Theorem 1.2 also holds if we replace (4) with*

$$(\lambda_1, \lambda_2) \in ([0, 24\pi)\backslash 8\pi\mathbf{N}) \times ([0, 24\pi)\backslash 8\pi\mathbf{N})$$

As we remarked here, the resemblances between our equation (1) and $SU(3)$ Toda system widely exist. Generally speaking, (1) is simpler to handle than $SU(3)$ Toda system. For example, it is not necessary to show Theorem 1.5 to get the Ding-Jost-Li-Wang type solution (Theorem 1.4, 2). On the other hand, it is necessary for $SU(3)$ Toda system to show the fact like Theorem 1.5, which was done in [31], before we get the Ding-Jost-Li-Wang type solution obtained in [7].

We also note here that another result indicates Corollary 1.3 exists [39] but the proof is completely different.

The plan of this note is as follows: in Section 2, we review the derivation of the mean field equation and see the physical meaning of the parameters λ_1 and λ_2. In Section 3, we sketch the proof of our main theorem (Theorem 1.2). We also mention some conjecture relating to Theorem 1.5. In Section 4, we sketch the proof of the Trudinger-Moser type inequality (Corollary 1.3) and discuss other versions of it for $\Omega \subset \mathbf{R}^2$. In Section 5, we see the variational schemes to prove Theorem 1.4.

2 The derivation of the mean field equation

Here we briefly review the derivation of the equation (1) by Pointin and Lundgren [34] and see the physical meaning of the parameters λ_1 and λ_2. To simplify the presentation, we assume the domain is a simply connected $\Omega \subset \mathbf{R}^2$.

Suppose Ω is filled with incompressible non-viscous fluid and the vorticity field has the form $\omega(x,t) = \sum_{i=1}^N \alpha_i \delta_{x_i(t)}$, where δ_x is the Dirac measure supported on x and the intensities $\alpha_1, \ldots, \alpha_N$ satisfy

$$\alpha_i = \begin{cases} \alpha(>0), & (i = 1, \cdots, n^+N) \\ -\alpha, & (i = n^+N+1, \cdots, N), \end{cases}$$

where $0 \leq n^{\pm} \leq 1$ and $n^+ + n^- = 1$. Under the appropriate physical assumptions, the probability density functions of each vortex found in Ω consist of two types :

$$\rho_{1,N}^+ = \int_{\Omega^{N-1}} \mu_N dx_2 \cdots dx_N, \quad \rho_{1,N}^- = \int_{\Omega^{N-1}} \mu_N dx_1 \cdots dx_{N-1}$$

where μ_N is the microcanonical measure at the energy $E = E(x_1, \ldots, x_N)$, that is,

$$\mu_N = \frac{\delta(H-E)}{Q(E)}, \quad Q(E) = \int_{\Omega^N} \delta(H-E) dx_1 \cdots dx_N.$$

Pointin-Lundgren considered here the mean field limit called *the high energy scaling limit* as $n \longrightarrow \infty$ subject to

$$\frac{E}{\alpha^2 N^2} = O(1), \quad \left(\tilde{\beta} :=\right) \alpha^2 N \beta = O(1),$$

where E is the total energy of the vortices and

$$\beta = \frac{1}{k_B} \frac{\partial}{\partial E} S(E)$$

is the inverse temperature of the system. Here k_B is the Boltzmann constant and $S(E)$ is the entropy.

Suppose $\rho_{1,N}^{\pm} \longrightarrow \rho_1^{\pm}$ in this limit in an appropriate sense, where ρ_1^+ and ρ_1^- represent the mean field of the positive and negative vortices, respectively. Then they derived that the function

$$v = -\tilde{\beta}\Phi(x) = -\tilde{\beta} \int_\Omega G_\Omega(x,y) \left[n^+\rho_1^+(y) - n^-\rho_1^-(y)\right] dy$$

proportional to the stream function Φ determined by the mean field of the vortices $\rho_1^{\pm}$ must satisfy

$$-\Delta v = \lambda_1 \frac{e^v}{\int_M e^v dv_g} - \lambda_2 \frac{e^{-v}}{\int_M e^{-v} dv_g} \quad \text{in } \Omega, v = 0 \quad \text{on } \partial\Omega,$$

where $\lambda_1 = -\tilde{\beta} n^+$, $\lambda_2 = -\tilde{\beta} n^-$, and $G_\Omega(x, y)$ is the Green function of $-\Delta$ with the Dirichlet boundary condition. Therefore λ_1 and λ_2 are affected by n^+ and n^-, respectively.

3 On the proof of Theorems 1.2 and 1.5

The procedure of the proof of Theorems 1.2 is as follows:

Step 1 Reduction to the Liouville system.

Step 2 Reduction to the definite signed case with variable coefficient.

Step 3 The analysis of *the collision* of the concentration points.

We have already seen **Step 1** in Section 1, see (10).

Here we recall the results for $\lambda_2 = 0$ cases. Brezis-Merle [4] and subsequent Li-Shafrir [21] and Li [20] established Fact 1.1 by the blow-up analysis, that is, the classification of the limit of the non-compact solution sequences for (2). Below we explain this along [20].

Suppose $\{\lambda_n\}$ be a sequence of non-negative numbers satisfying $\lambda_n \longrightarrow \lambda$ and $\{v_n\}$ be a corresponding non-compact sequence of solutions for (2) in E. Then they proved that

$$\lambda_n \frac{e^{v_n}}{\int_M e^{v_n} dv_g} \longrightarrow \sum_{x_0 \in \mathcal{S}} 8\pi \delta_{x_0} \quad \text{weakly } *$$

up to subsequence by the combination of some elliptic L^1 type estimate and the rescaling argument. Therefore

$$\lambda_n \left(= \int_M \lambda_n \frac{e^{v_n}}{\int_M e^{v_n} dv_g} dv_g \right) \longrightarrow 8\pi \# \mathcal{S} \in 8\pi \mathbf{N}$$

if the solution sequence is non-compact and we get Fact 1.1. There results also holds to some extent to the equation (3) with varying coefficients $\{K_n(x)\}$. Now we discuss $\lambda_2 \neq 0$ cases.

Step 2 Putting $K_{i,n}(x) = e^{-u_{j,n}}$ $(i \neq j)$, we are able to reduce the each component of (10) for $A = (a_{ij}) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ to

$$-\Delta_g u_i = \lambda_{i,n}\left(\frac{K_{i,n}(x)e^{u_{i,n}}}{\int_M K_{i,n}(x)e^{u_{i,n}}dv_g} - \frac{1}{|M|}\right).$$

Similar to the constant coefficient case, we have

$$\lambda_{i,n}\frac{K_{i,n}(x)e^{u_{i,n}}}{\int_M K_{i,n}(x)e^{u_{i,n}}dv_g} \longrightarrow \sum_{x_0\in\mathcal{S}_i} 8\pi\delta_{x_0} \quad \text{weakly} *$$

up to subsequence when $\{u_{i,n}\}$ is not relatively compact in E provided

$$\{K_{i,n}(x)\} \text{ is bounded in } W^{1,\infty}(M), \quad K_{i,n}(x) \geq \exists C > 0,$$

where C is independent of n [20]. But these are heavy for our problem because we have only

$$\{K_{i,n}(x)\} \text{ is bounded in } W^{1,q}(M)\ \forall q \in [1,2), \quad K_{i,n}(x) \geq 0.$$

Indeed, $\nabla K_{i,n}(x) = -e^{-u_{j,n}}\nabla u_{j,n}$ and $\{e^{-u_{j,n}}\}$ is uniformly bounded in $L^\infty(M)$ because there exists $C > 0$ independent of n such that $u_{j,n} \geq -C$ from the existence of the lower bound of the Green function of $-\Delta_g$ in E, see, e.g., [2].

On the other hand, we know $\{u_{j,n}\}$ is bounded only in $W^{1,q}(M)$ $\forall q \in [1,2)$ from the elliptic L^1 theory. Furthermore, though it is possible indeed to assume

$$u_{j,n} \longrightarrow u_j \text{ weakly in } W^{1,q}(M)\ \forall q \in [1,2)$$

up to subsequence and

$$K_{i,n}(x) = e^{-u_{j,n}(x)} \longrightarrow e^{-u_j(x)} \text{ in } L^p(M)\ \forall p \in [1,\infty), \tag{11}$$

the limit function $e^{-u_j(x)}$ take 0 if there exists x_0 satisfying $u_j(x_0) = \infty$. Therefore we are not able to assume the existence of the uniform lower bound C satisfying $K_{i,n}(x) \geq C > 0$.

Although these are difficulties, however, our previous work [30] on the equation (3), the weaker version of [20] in weaker assumptions, is applicable. Under our situation (11), we get the following facts by [30]:

$$\lambda_{i,n}\frac{K_{i,n}(x)e^{u_{i,n}}}{\int_M K_{i,n}(x)e^{u_{i,n}}dv_g} \longrightarrow r_i(x) + \sum_{x_0\in\mathcal{S}_i} m_i(x_0)\delta_{x_0} \quad \text{weakly} *$$

for each i, where $\mathcal{S}_i$ is a finite subset of M satisfying $\mathcal{S}_1 \cup \mathcal{S}_2 \neq \emptyset$, the function $r_i \geq 0$ belongs to $L^1(M) \cap L^\infty_{\text{loc}}(M \backslash \mathcal{S}_i)$, and

$$m_i(x_0) \geq 4\pi \tag{12}$$

for every $x_0 \in \mathcal{S}_i$. Therefore we have

$$\lambda_{i,n} \longrightarrow \lambda_i = \int_M r_i dv_g + \sum_{x_0 \in \mathcal{S}_i} m_i(x_0) \geq 4\pi \# \mathcal{S}_i.$$

Here we note that the set $\mathcal{S}_i$ is exactly the blow-up set of $\{u_{i,n}\}$, that is,

$$\mathcal{S}_i = \{x \in M | \exists x_n \longrightarrow x \text{ s.t. } u_{i,n}(x_n) \longrightarrow +\infty\}.$$

Therefore, if $x_0 \in \mathcal{S}_i \backslash \mathcal{S}_j$, the limit coefficient function e^{-u_j} is positive near x_0 and this enable one to get

$$\begin{array}{ll} \text{if} \quad x_0 \in \mathcal{S}_i \backslash \mathcal{S}_j \quad \text{(not collide)} \quad \Rightarrow \quad m_i(x_0) = 8\pi, \\ \text{if} \quad \exists x_0 \in \mathcal{S}_i \backslash \mathcal{S}_j \quad \Rightarrow \quad r_i \equiv 0 \end{array} .$$

from the careful application of the argument like [20]. Consequently we have

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset \quad \Rightarrow \quad \lambda_{i,n} \longrightarrow \lambda_i = \sum_{x_0 \in \mathcal{S}_i} m_i(x_0) = 8\pi \# \mathcal{S}_i \in 8\pi \mathbf{N}. \tag{13}$$

The rest of the problem is to determine the mass at

$$x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2 \quad \text{(collision)}.$$

Step 3 For this problem, we obtain the following answer:

Proposition 3.1 ([32]). *For every $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$, we have*

$$(m_1(x_0) - m_2(x_0))^2 = 8\pi \left(m_1(x_0) + m_2(x_0)\right) \quad (\cdots \textit{a parabola}).$$

Here we recall (12) for every $x_0 \in \mathcal{S}_i$. Therefore the collision $\mathcal{S}_1 \cap \mathcal{S}_2$ never occur if

$$((m_1(x_0), m_2(x_0)) \leqq)(\lambda_1, \lambda_2) \in [0, 4(2+\sqrt{5}\pi)) \times [0, 4(2+\sqrt{5}\pi))$$

and we get Theorem 1.2 from (13).

We note that the improvement of Theorem 1.2 to Theorem 1.5 is nothing but the improvement of (12) for $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$, see Figure 3.

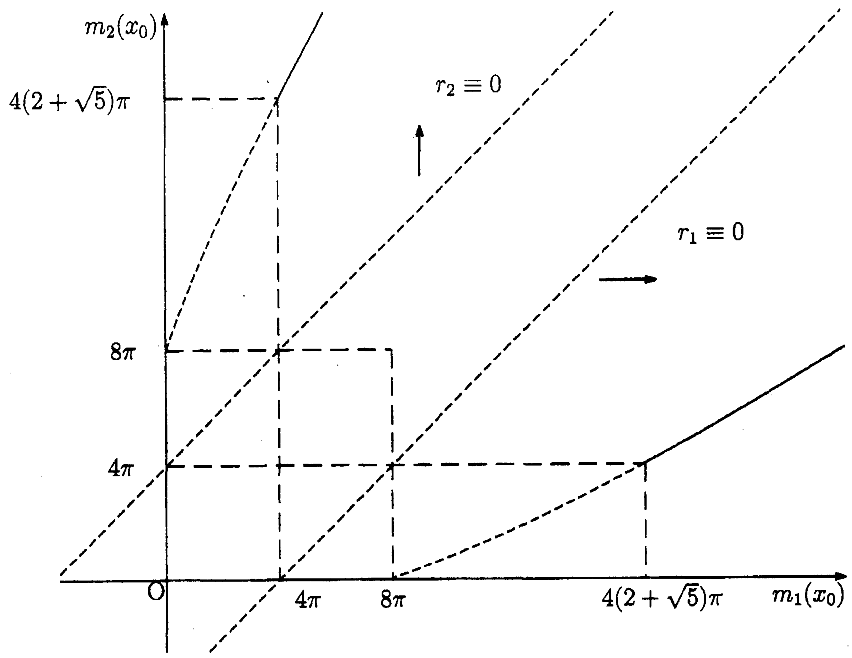

Figure 3: The parabola

Here we remark that the possible values of $(m_1(x_0), m_2(x_0))$ for $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$ will be more restrictive and we expect that

$$(m_1(x_0), m_2(x_0)) = 8\pi\left(\frac{(\ell-1)\ell}{2}, \frac{\ell(\ell+1)}{2}\right), \quad 8\pi\left(\frac{\ell(\ell+1)}{2}, \frac{(\ell-1)\ell}{2}\right)$$

for $\ell = 1, 2, 3, \cdots$.

The idea of the proof of Proposition 3.1 is the *symmetrization* of the Green function established in [38, 29, 30, 43]. Similar methods are also used in the analysis of the two dimensional incompressible Euler equation, see [44, 36, 27] for example. To see the idea of the symmetrization, we consider the following toy model.

$$-\Delta u_n = \lambda_n \frac{e^{u_n}}{\int_\Omega e^{u_n} dx} \quad \text{in } \Omega, \quad u_n = 0 \quad \text{in } \partial\Omega,$$

where $\Omega \subset \mathbf{R}^2$ is the bounded domain with smooth boundary $\partial\Omega$.

Set $\mu_n = \lambda_n \frac{e^{u_n}}{\int_\Omega e^{u_n} dx}$. Then we have

$$\begin{aligned} &\nabla \mu_n = \nabla e^{u_n + \log \lambda_n - \log \int_\Omega e^{u_n} dx} = \mu_n \nabla u_n \\ \Rightarrow \quad &0 = \nabla \cdot (\nabla \mu_n - \mu_n \nabla u_n). \end{aligned}$$

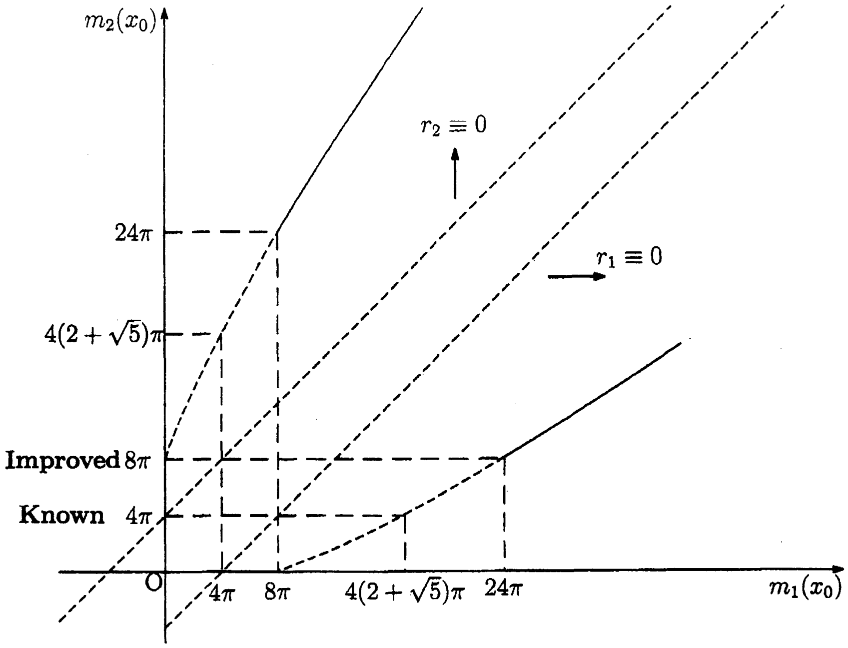

Figure 4: The improvement of Proposition 3.1 in [33].

Here we notice that the last equation is nothing but the stationary equation of chemotaxis. The most interesting feature of this method is to reduce the mean field equation to this chemotaxis equation. In the weak form, we have

$$0 = \int_\Omega \mu_n \Delta\phi + \int_\Omega (\nabla u_n \cdot \nabla\phi)\, \mu_n$$

for every test function $\phi \in C_0^\infty(\Omega)$.

Let $G(x,y)$ be the Green function of $-\Delta$ with the Dirichlet boundary condition. Then

$$\begin{aligned} 0 &= \int_\Omega \mu_n \Delta\phi + \int_\Omega (\nabla u_n \cdot \nabla\phi)\, \mu_n \\ &= \int_\Omega \mu_n \Delta\phi + \int_\Omega\int_\Omega (\nabla_x G(x,y)\mu_n(y) \cdot \nabla\phi(x))\, \mu_n(x) dxdy \\ &= \int_\Omega \mu_n \Delta\phi + \int_\Omega\int_\Omega \rho_\phi(x,y)\mu_n(x)\mu_n(y) dxdy \end{aligned}$$

where

$$\rho_\phi(x,y) = \frac{1}{2}(\nabla_x G(x,y) \cdot \nabla\phi(x) + \nabla_y G(x,y) \cdot \nabla\phi(y)),$$

the *symmetrization* of the Green function.

Indeed we are able to consider $G(x,y) \sim \frac{1}{2\pi}\log|x-y|^{-1}$ without loss of strictness. Then, when $\mu_n \longrightarrow m\delta_0$, we assume ϕ that behaves like $|x|^2$ near the origin and we have

$$\begin{aligned}\rho_\phi(x,y) &= \frac{1}{2}(\nabla_x G(x,y)\cdot\nabla\phi(x) + \nabla_y G(x,y)\cdot\nabla\phi(y))\\ &\sim -\frac{1}{4\pi}\frac{(x-y)\cdot 2x-(x-y)\cdot 2y}{|x-y|^2} \sim -\frac{1}{2\pi}.\end{aligned}$$

On the other hand, $\Delta\phi \sim 4$ if $\phi \sim |x|^2$. Therefore we have

$$\begin{aligned} 0 &= \int_\Omega \mu_n\Delta\phi + \int_\Omega\int_\Omega \rho_\phi(x,y)\mu_n(x)\mu_n(y)dxdy\\ &\longrightarrow 4m - \frac{1}{2\pi}m^2.\\ \Leftrightarrow \quad 0 &= 4m - \frac{1}{2\pi}m^2.\\ \Rightarrow \quad m &= 8\pi\end{aligned}$$

when $m > 0$. This is nothing but the conclusion obtained by Li [20] for $\lambda_2 = 0$ cases, though the method is completely different. Now we have several method to get this value 8π, however, the symmetrization seems to be most easy to handle for complicated case like ours.

To apply similar calculations to our equation

$$-\Delta_g u_i = \lambda_{i,n}\left(\frac{e^{u_{i,n}-u_{j,n}}}{\int_M e^{u_{i,n}-u_{j,n}}dv_g} - \frac{1}{|M|}\right),$$

we set

$$\mu_{i,n} := \lambda_{i,n}\frac{e^{u_{i,n}-u_{j,n}}}{\int_M e^{u_{i,n}-u_{j,n}}dv_g}.$$

Then

$$\begin{aligned}\nabla\mu_{i,n} &= \mu_{i,n}\nabla(u_{i,n}-u_{j,n})\\ \Delta\mu_{i,n} &= \nabla\cdot\mu_{i,n}\nabla(u_{i,n}-u_{j,n})\end{aligned}$$

and we have the weak form

$$0 = \int_\Omega \mu_{i,n}\Delta\phi + \int_\Omega \left(\nabla\left[u_{i,n}-u_{j,n}\right]\cdot\nabla\phi\right)\mu_{i,n}$$

if appropriately localized to flat domain Ω. The second term of the righthand side is transformed as follows:

$$\begin{aligned}
&\int_\Omega \left(\nabla\left[u_{i,n} - u_{j,n}\right] \cdot \nabla\phi\right) \mu_{i,n} \\
&= \int_\Omega \int_\Omega \left(\nabla_x G(x,y)\left[\mu_{i,n}(y) - \mu_{j,n}(y)\right] \cdot \nabla\phi(x)\right) \mu_{i,n}(x) dxdy \\
&= \int_\Omega \int_\Omega \rho_\phi(x,y) \mu_{i,n}(x) \mu_{i,n}(y) dxdy \\
&\quad - \int_\Omega \int_\Omega \left(\nabla_x G(x,y) \cdot \nabla\phi(x)\right) \mu_{i,n}(x) \mu_{j,n}(y) dxdy
\end{aligned}$$

To control *the cross term*, add the cases $(i,j) = (1,2), (2,1)$ and we have

$$\begin{aligned}
0 &= \int_\Omega \mu_{1,n}\Delta\phi + \int_\Omega\int_\Omega \rho_\phi(x,y)\mu_{1,n}(x)\mu_{1,n}(y)dxdy \\
&\quad - \boxed{\int_\Omega\int_\Omega \left(\nabla_x G(x,y)\cdot\nabla\phi(x)\right)\mu_{1,n}(x)\mu_{2,n}(y)dxdy} \\
&\quad + \int_\Omega \mu_{2,n}\Delta\phi + \int_\Omega\int_\Omega \rho_\phi(x,y)\mu_{2,n}(x)\mu_{2,n}(y)dxdy \\
&\quad - \boxed{\int_\Omega\int_\Omega \left(\nabla_x G(x,y)\cdot\nabla\phi(x)\right)\mu_{2,n}(x)\mu_{1,n}(y)dxdy} \\
&= \int_\Omega \mu_{1,n}\Delta\phi + \int_\Omega \mu_{2,n}\Delta\phi \\
&\quad + \int_\Omega\int_\Omega \rho_\phi(x,y)\left[\mu_{1,n}(x)\mu_{1,n}(y) - \boxed{2\mu_{1,n}(x)\mu_{2,n}(y)} + \mu_{2,n}(x)\mu_{2,n}(y)\right] dxdy \\
&\longrightarrow 4m_1 + 4m_2 - \frac{1}{2\pi}\left[m_1^2 - 2m_1m_2 + m_2^2\right]
\end{aligned}$$

when $\mu_{i,n} \longrightarrow m_i\delta_0$ for each $i = 1, 2$, i.e., a collision occurs at 0. This is the parabola and we get Proposition 3.1 from (12).

4 On the Trudinger-Moser type inequality

Suppose I is the set of parameters (λ_1, λ_2), where the conclusion holds. We know that $I \neq \emptyset$ because we have the trivial region, see Figure 1. Then the proof of Corollary 1.3 is divided into following steps.

Step 1 We are able to assume that there exists $(\lambda_1^0, \lambda_2^0) \in \partial I \cap [0, 8\pi] \times [0, 8\pi]$ such that

$$t(\lambda_1^0, \lambda_2^0) \notin I \quad \text{for every } t > 1$$

from the monotonicity of $J_{t\lambda_1^0, t\lambda_2^0}$ in $t > 0$.

Step 2 Fix

$$(\lambda_{1,n}, \lambda_{2,n}) = t_n(\lambda_1^0, \lambda_2^0), \quad t_n \uparrow 1.$$

Then we are able to construct a non-compact solution sequence $\{v_n\}$ for $\{(\lambda_{1,n}, \lambda_{2,n})\} \subset I$ by the so-called Ding's Trick (see, e.g., Jost-Wang [17]), which contradicts to Theorem 1.2 if $(\lambda_1^0; \lambda_2^0) \in [0, 8\pi) \times [0, 8\pi)$. Therefore we get $I \subset [0, 8\pi) \times [0, 8\pi)$

Step 3 When $\lambda_1^0 = 8\pi$ or $\lambda_2^0 = 8\pi$, the careful observation of the asymptotic behavior of the sequence of minimizers $\hat{v}_n$ for $J_{\lambda_{1,n}, \lambda_{2,n}}$ enables one to prove

$$\liminf_{n \longrightarrow \infty} J_{\lambda_{1,n}, \lambda_{2,n}}(\hat{v}_n) > -\infty$$

and get $I \subset [0, 8\pi] \times [0, 8\pi]$.

We note that $I \supset [0, 8\pi] \times [0, 8\pi]$, which we mentioned in the context of the optimality of Corollary 1.3, see Section 1.

Omitting the details of the proof, however, we review other inequalities we get from Corollary 1.3.

The conclusion of Corollary 1.3 is equivalent to

$$\inf_{v \in E} J_{8\pi, 8\pi}(v) > -\infty. \tag{14}$$

In fact, this inequality guarantees

$$\inf_{v \in E} J_{\lambda_1, \lambda_2}(v) > -\infty$$

for $0 \leq \lambda_2 \leq \lambda_1 \leq 8\pi$, $\lambda_2 < 8\pi$, because the second term of the right-hand side of

$$\begin{aligned} & \frac{1}{2} \|\nabla v\|_2^2 - \lambda_1 \log \int_M e^v - \lambda_2 \log \int_M e^{-v} \\ &= \frac{\lambda_2}{8\pi} \left(\frac{1}{2} \|\nabla v\|_2^2 - 8\pi \log \int_M e^v - 8\pi \log \int_M e^{-v} \right) \\ &+ \frac{1}{2} \left(1 - \frac{\lambda_2}{8\pi} \right) \|\nabla v\|_2^2 - (\lambda_1 - \lambda_2) \log \int_M e^v \end{aligned} \tag{15}$$

is bounded by

$$0 \leq \frac{\lambda_1 - \lambda_2}{1 - \frac{\lambda_2}{8\pi}} \leq 8\pi.$$

Noting $J_{\lambda_1,\lambda_2}(v) = J_{\lambda_2,\lambda_1}(-v)$, we can infer (6) from (14) for $0 \le \lambda_1, \lambda_2 \le 8\pi$ with $\min \lambda_1, \lambda_2 < 8\pi$. Inequality (14), thus, guarantees all the cases of the conclusion of Corollary 1.3.

Since $J_{8\pi,8\pi}(v+c) = J_{8\pi,8\pi}(v)$ for $v \in H^1(M)$ and $c \in \mathbf{R}$, inequality (14) implies

$$\inf_{v \in H^1(M)} J_{8\pi,8\pi}(v) > -\infty. \tag{16}$$

If $\Omega \subset \mathbf{R}^2$ is a bounded domain, we can take a flat torus M of which cell domain $\hat{\Omega}$ contains Ω. Then, each $v \in H_0^1(\Omega)$ is regarded as an element in $H^1(M)$, denoted by $\hat{v}$, by taking zero extension to $\hat{\Omega}$ and then periodic extension to $\mathbf{R}^2$. In this case, inequality (16) implies

$$\inf_{v \in H_0^1(\Omega)} \tilde{J}_{8\pi}(v) > -\infty \tag{17}$$

for

$$\tilde{J}_\lambda(v) = \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 - \lambda \left(\log \int_\Omega e^v dx + \log \int_\Omega e^{-v} dx \right),$$

because of $J_{8\pi,8\pi}(\hat{v}) \le \tilde{J}_{8\pi}(v)$. Similarly to (14), this inequality (17) guarantees

$$\inf_{v \in H_0^1(\Omega)} \tilde{J}_{\lambda_1,\lambda_2}(v) > -\infty \qquad \text{for } (\lambda_1, \lambda_2) \in [0, 8\pi] \times [0, 8\pi],$$

where

$$\tilde{J}_{\lambda_1,\lambda_2}(v) = \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 - \lambda_1 \log \int_\Omega e^v - \lambda_2 \log \int_\Omega e^{-v}.$$

Inequality (17) is optimal, because the optimality of the standard Trudinger-Moser inequality is improved by

$$\inf_{v \in H_0^1(\Omega), v \ge 0} \tilde{I}_\lambda(v) = -\infty \qquad \text{for } \lambda > 8\pi,$$

where

$$\tilde{I}_\lambda(v) = \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 - \lambda \log \int_\Omega e^v.$$

If $\Omega \subset \mathbf{R}^2$ is a simply-connected domain with C^1 boundary, we have a conformal mapping $\varphi : \Omega \to S_+^2$, where $S_+^2 \subset \mathbf{R}^3$ is a hemi-sphere. In this case, each $v \in H^1(\Omega)$ is regarded as an element in $H^1(S^2)$ by even extension, and then it holds that

$$\inf_{v \in H^1(\Omega)} \tilde{J}_{4\pi}(v) > -\infty \tag{18}$$

by (14). This inequality, however, is proven for general domain $\Omega \subset \mathbf{R}^2$ with smooth boundary $\partial\Omega$, because we can control the behavior of the solution sequence to

$$
\begin{aligned}
&-\Delta v = \lambda_1 \left(\frac{e^v}{\int_\Omega e^v dx} - \frac{1}{|\Omega|} \right) - \lambda_2 \left(\frac{e^{-v}}{\int_\Omega e^{-v} dx} - \frac{1}{|\Omega|} \right) \\
&\left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0, \qquad \int_\Omega v dx = 0
\end{aligned}
$$

similarly to that of (1) by the method of reflection [29, 43]. Inequality (18) is generalized as

$$
\inf_{v \in H^1(\Omega), \int_\Omega v dx = 0} \tilde{J}_{\lambda_1,\lambda_2}(v) > -\infty \qquad \text{for } (\lambda_1, \lambda_2) \in [0, 4\pi] \times [0, 4\pi]
$$

similarly, using Chang-Yang's inequality [8]:

$$
\inf_{v \in H^1(\Omega), \int_\Omega v dx = 0} \tilde{I}_{4\pi}(v) > -\infty.
$$

Inequality (18) is also optimal by itself because of the optimality of (14).

5 On the proof of the existence of solutions

1. Struwe-Tarantello type solution. Similar to the arguments of [41, 23], we first demonstrate that the trivial solution $v \equiv 0$ is local minimum of J_{λ_1,λ_2} for admissible parameters. Indeed, direct computation enables one to get

$$
J''_{\lambda_1,\lambda_2}(0)[h_1, h_2] = \int_M \nabla h_1 \cdot \nabla h_2 - \frac{\lambda_1 + \lambda_2}{|M|} \int_M h_1 h_2
$$

for every h_1, $h_2 \in E$ and

$$
\begin{aligned}
J''_{\lambda_1,\lambda_2}(0)[h, h] &= \int_M |\nabla h|^2 - \frac{\lambda_1 + \lambda_2}{|M|} \int_M h^2 \\
&\geq \int_M |\nabla h|^2 - \frac{\lambda_1 + \lambda_2}{\mu_1 |M|} \int_M |\nabla h|^2 > 0
\end{aligned}
$$

for every $h \in E$ satisfying $h \neq 0$ if $\lambda_1 + \lambda_2 < \mu_1 |M|$, see also [35].

On the other hand, the optimality (7) enable one to choose $w \in E$ such that

$$
J_{\lambda_1,\lambda_2}(w) < J_{\lambda_1,\lambda_2}(0) = -(\lambda_1 + \lambda_2) \log |M|.
$$

Therefore J_{λ_1,λ_2} has the mountain pass structure for admissible parameter (λ_1, λ_2).

Set

$$\Gamma = \{\gamma \in C([0,1], E); \gamma(0) = 0, \gamma(1) = w\}$$

and

$$c(\lambda_1, \lambda_2) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{\lambda_1,\lambda_2}(\gamma(t)).$$

As we mentioned in Section 1, J_{λ_1,λ_2} is not known whether the Palais-Smale condition holds or not. Therefore in is not obvious that this mountain pass value $c(\lambda_1, \lambda_2)$ is really a critical value of J_{λ_1,λ_2}. But this obstruction is overcome by the joint use of the Struwe's monotonicity trick and our Theorem 1.2. The details of the argument is almost the same that for $SU(3)$ Toda system [23] (or [41]) and we omit it.

2. Ding-Jost-Li-Wang type solution. We take an isometric embedding (M, g) into $\mathbf{R}^N$ with sufficiently large N by Nash's theorem (Theorem 4.34 of Aubin [2]) and let

$$m(v) = \frac{\int_M xe^v}{\int_M e^v} \in \mathbf{R}^N$$

denote the center of mass of $v \in E$. The following lemma describes the concentration of a sequence in E satisfying $J_\lambda (= J_{\lambda,0}) \to -\infty$, see [7] (or [28]).

Fact 5.1. *Let $\{v_n\} \subset E$ satisfy $J_\lambda(v_n) \to -\infty$ and $x_n \equiv m(v_n) \to x_\infty \in \mathbf{R}^N$ for $\lambda \in (8\pi, 16\pi)$. Then $x_\infty \in M$ and*

$$\frac{e^{v_n}}{\int_M e^{v_n}} \rightharpoonup \delta_{x_\infty} \qquad \text{weakly-}* \text{ in } \quad \mathcal{M}(M) = C(M)'. \tag{19}$$

Therefore the topology of the level set $J_\lambda = -\infty$ is affected by the topology of M if $8\pi < \lambda < 16\pi$. The above fact originates from the improved Trudinger-Moser inequality of Chen and Li [11] and the further origin of such an inequality is in Aubin [1].

As we see in (15),

$$\begin{aligned} J_{\lambda_1,\lambda_2}(v) &= \frac{\lambda_2}{8\pi} J_{8\pi,8\pi}(v) + \left(1 - \frac{\lambda_2}{8\pi}\right) J_\lambda(v) \\ &\geq C + \left(1 - \frac{\lambda_2}{8\pi}\right) J_\lambda(v) \end{aligned}$$

for some constant C by Corollary 1.3, where

$$\lambda = \frac{\lambda_1 - \lambda_2}{1 - \frac{\lambda_2}{8\pi}}.$$

Therefore, if $0 < \lambda_2 < 8\pi$ and $8\pi < \lambda_1 < 16\pi - \lambda_2$, we have $\lambda \in (8\pi, 16\pi)$ and $J_\lambda(v) \longrightarrow \infty$ if $J_{\lambda_1,\lambda_2}(v) \longrightarrow \infty$. Consequently the topology of the level set $J_{\lambda_1,\lambda_2} = -\infty$ is also affected by the topology of M for these parameters. For the parameter region $0 < \lambda_1 < 8\pi$ and $8\pi < \lambda_2 < 16\pi - \lambda_1$ left yet, we are able to do similar argument.

From now on,

$$D = \{(r, \theta) \mid 0 \leq r < 1, 0 \leq \theta < 2\pi\}$$

denotes the two-dimensional unit disc. Since M has genus ≥ 1, there is a non-contractible Jordan curve $\Gamma_1 \subset M$. Then, we can take a closed curve $\Gamma_2 \subset \mathbf{R}^N \setminus M$ linking Γ_1 (Figure 5). See [40] for the general theory of linking.

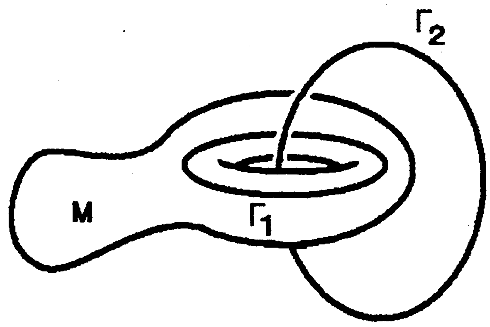

Figure 5: Linking

Now we set

$$\begin{aligned} D_{\lambda_1,\lambda_2} = \{h \in C(D; E) \mid \\ & m(h(\cdot)) \text{ can be extended continuously to } \overline{D}, \\ & m(h(1,\cdot)) : S^1 \to \Gamma_1 \text{ has degree 1}, \\ & \lim_{r\to 1} \sup_{0\leq\theta<2\pi} J_{\lambda_1,\lambda_2}(h(r,\theta)) = -\infty\}. \end{aligned}$$

We are able to see that $D_{\lambda_1,\lambda_2} \neq \emptyset$ from the similar argument of [7, Lemma 3] and we set

$$\alpha_{\lambda_1,\lambda_2} = \inf_{h\in D_{\lambda_1,\lambda_2}} \sup_{(r,\theta)\in D} J_{\lambda_1,\lambda_2}(h(r,\theta)).$$

Thanks to Fact 5.1, we get $\alpha_{\lambda_1,\lambda_2} > -\infty$ if the genus ≥ 1.

Similar to the construction of Struwe-Tarantello type solutions , the lack of the Palais-Smale condition makes it difficult to see this min-max value $\alpha_{\lambda_1,\lambda_2}$ is really a critical value of J_{λ_1,λ_2}. Using also the Struwe's monotonicity trick and our Theorem 1.2, however, we are able to see that $\alpha_{\lambda_1,\lambda_2}$ is a critical value for admissible parameters. The details of the argument is almost the same that for $SU(3)$ Toda system [7] (or [13]) and we omit it.

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