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<td>Author(s)</td>
<td>Sato, Yohei</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録</td>
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<tr>
<td>Issue Date</td>
<td>2007-01</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58902">http://hdl.handle.net/2433/58902</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
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Kyoto University
Multi-peak positive solutions for nonlinear Schrödinger equations
with critical frequency

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0. Introduction
In this report, we consider the following nonlinear Schrödinger equations:

$$-\epsilon^2 \Delta u + V(x)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N,$$
$$u > 0 \quad \text{in } \mathbb{R}^N,$$
$$u \in H^1(\mathbb{R}^N),$$

where $\epsilon > 0$ is small parameter and $p$ satisfies $1 < p < \infty$ ($N = 1, 2$), $1 < p < \frac{N+2}{N-2}$ ($N \geq 3$). We are interested in the existence of solutions of $(\ast)_\epsilon$ for small $\epsilon > 0$ and their behavior as $\epsilon \to 0$. We assume the potential $V(x)$ satisfies the following assumptions:

(V.1) $V(x) \in C^1(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$.
(V.2) $0 < \liminf_{|x| \to \infty} V(x) \leq \sup_{x \in \mathbb{R}^N} V(x) < \infty$.

Under the above assumptions, the solutions of $(\ast)_\epsilon$ are characterized as critical points of

$$\Psi_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2} (\epsilon^2 |\nabla u|^2 + V(x)u^2) - \frac{1}{p+1} u_+^{p+1} dx \in C^2(H^1(\mathbb{R}^N), \mathbb{R}).$$

When $V(x)$ satisfies

$$\inf_{x \in \mathbb{R}^N} V(x) > 0,$$ (0.1)

$(\ast)_\epsilon$ has a family of single-peak solutions $u_\epsilon(x)$ concentrating around a local minimum of $V(x)$ for small $\epsilon > 0$ in the following sense: let $x_\epsilon$ be a unique maximum of $u_\epsilon(x)$. Then $x_\epsilon$ approaches to a local minimum $x_0$ of $V(x)$ as $\epsilon \to 0$ and

$$w_\epsilon(y) = u_\epsilon(\epsilon y + x_\epsilon)$$ (0.2)

converges to a least energy solution $w_0(x)$ of the following "limit equation"

$$-\Delta w + V(x_0)w = w^p, \quad w > 0 \quad \text{in } \mathbb{R}^N, \quad w \in H^1(\mathbb{R}^N).$$ (0.3)
In particular, $u_\epsilon(x)$ satisfies
\begin{align}
\lim_{\epsilon \to 0} ||u_\epsilon||_{L^\infty(\mathbb{R}^N)} &= |w_0(0)| > 0, \\
\lim_{\epsilon \to 0} \epsilon^{-N}\Psi_\epsilon(u_\epsilon) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)b_{V(x_0),\mathbb{R}^N} > 0.
\end{align}

Here $b_{V(x_0),\mathbb{R}^N}$ is defined by
\begin{equation}
b_{V(x_0),\mathbb{R}^N} = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x_0)u^2 \, dx}{||u||_{L^{p+1}(\mathbb{R}^N)}^2}\right)^{\frac{p+1}{p-1}}
\end{equation}
and $\left(\frac{1}{2} - \frac{1}{p+1}\right)b_{V(x_0),\mathbb{R}^N}$ is the lowest non-trivial critical value of the functional corresponding to (0.3). See [FW, O1, O2, DF] for related results.

Recently, Byeon and Wang [BW1, BW2] have started to study the case
\begin{equation}
\inf_{x \in \mathbb{R}^N} V(x) = 0.
\end{equation}

They showed that $(*)_\epsilon$ also has single-peak solutions concentrating around an isolated component $A$ of $\{x \in \mathbb{R}^N; V(x) = 0\}$. The features of their solutions are completely different from the solutions under the condition (0.1), which depend on the behavior of $V(x)$ around $A$. More precisely, their solutions $u_\epsilon(x)$ satisfy
\begin{align}
\lim_{\epsilon \to 0} ||u_\epsilon||_{L^\infty(\mathbb{R}^N)} &= 0, \\
\lim_{\epsilon \to 0} \epsilon^{-N}\Psi_\epsilon(u_\epsilon) &= 0.
\end{align}

These are in contrast with (0.4)-(0.5). We can also see $\lim_{\epsilon \to 0} \epsilon^{-\frac{2}{p-1}}||u_\epsilon||_{L^\infty(\mathbb{R}^N)} \in (0, \infty]$ and $\lim_{\epsilon \to 0} \epsilon^{-\frac{2(p+1)}{p-1}}\Psi_\epsilon(u_\epsilon) \in (0, \infty]$. (We remark that $p < \frac{N+2}{N-2}$ implies $\frac{2(p+1)}{p-1} > N$.) Under additional conditions on the behavior of $V(x)$ near $A$, we can introduce a rescaled function — which is different from (0.2) — to observe the behavior of $u_\epsilon(x)$ around $A$. More precisely, under one of the conditions (L.2)-(L.4) (see p.5 below), we define $g(\epsilon) > 0$ as in the table 0.1 below and set
\begin{equation}w_\epsilon(y) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2}{p-1}} u_\epsilon(g(\epsilon)y + x_\epsilon)
\end{equation}
for a suitable point $x_\epsilon$ around $A$, then $w_\epsilon(y)$ approaches to the least energy solution of
\begin{equation}
-\Delta w + V_0(y)w = w^p, \quad w > 0 \quad \text{in} \quad \Omega_0, \quad w \in H^1_0(\Omega_0). \quad (L)_{V_0,\Omega_0}
\end{equation}
Here $\Omega_0 \subset \mathbb{R}^N$ and $V_0(x) \in C(\Omega_0, \mathbb{R})$ are also given in the table 0.1 below. In particular, $u_\epsilon(x)$ satisfies

$$
\lim_{\epsilon \to 0} \left( g(\epsilon)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \Psi_\epsilon(u_\epsilon) \right) \in (0, \infty)
$$

and the decaying rate of $\Psi_\epsilon(u_\epsilon)$ depends on the behavior of $V(x)$ around $A$.

In this report, we study the multi-peak solutions of $(*)_\epsilon$ combining various types of peaks. We assume that $\Lambda_1, \cdots, \Lambda_k \subset \mathbb{R}^N$ satisfy

(V.3) $\Lambda_1, \cdots, \Lambda_k \subset \mathbb{R}^N$ are bounded open sets satisfying $\Lambda_i \cap \Lambda_j = \emptyset \ (i \neq j)$ and

$$
m_i = \inf_{x \in \Lambda_i} V(x) < \inf_{x \in \partial \Lambda_i} V(x) < \infty \ (i = 1, \cdots, k).
$$

Here we remark that $m_i$ may be 0. In what follows, we write $A_i = \{x \in \Lambda_i; V(x) = m_i\} \ (i = 1, \cdots, k)$.

Under the assumption (V.3), we will show that $(*)_\epsilon$ has $k$-peak solutions which have 1-peak in each $\Lambda_i$ and it concentrates around a local minimum of $V(x)$ in $\Lambda_i$. We remark that the behavior of the desired solution $u_\epsilon(x)$ in $\Lambda_i$ depends on the behavior of $V(x)$ around $A_i$ and just single scaling (0.2) is not enough to describe the behavior of $u_\epsilon(x)$. Moreover the localized critical level $\Psi_{\epsilon, \Lambda_i}(u_\epsilon) = \Psi_\epsilon(u_\epsilon)$ which is defined as (1.1) below approaches to 0 with different rate with respect to $\epsilon$. This makes our problems difficult.

It seems that the existence of multi-peak solutions joining solutions which have different scales is not well studied. In our knowledge, there is only one work by Byeon and Oshita [BO] in which they constructed multi-peak solutions by a Lyapunov-Schmidt reduction method under the assumptions of non-degeneracy of solutions of limit problems.

Recently, in [S], we constructed the multi-peak solutions under the assumptions (V.1)--(V.3). We used a variational approach and it does not need the assumptions of non-degeneracy of solutions of limit problems. In the following sections, we will introduce main results and an outline of the proof in [S]. We also study about the asymptotic behavior of solutions and consider other examples than [S].

1. Main Results

Firstly, we introduce main theorems in [S].

(a) Existence of multi-peak solutions

In this section, we consider the existence of multi-peak solutions of $(*)_\epsilon$. To state theorems, we set

$$
\Psi_{\epsilon, \Lambda_i}(u) = \int_{\Lambda_i} \frac{1}{2} (\epsilon^2 |\nabla u|^2 + V(x) u^2) - \frac{1}{p+1} u^{p+1} dx.
$$

(1.1)
and define
\[ c_{\epsilon, \Lambda_{i}} = \inf_{u \in H_{0}^{1}(\Lambda_{i}) \setminus \{0\}} \left( \frac{\int_{\Lambda_{i}} \epsilon^{2} |\nabla u|^{2} + V(x)u^{2} \, dx}{||u_{+}||_{L^{p+1}(\Lambda_{i})}^{2}} \right)^{\frac{p+1}{p-1}}. \]  \\
(1.2)

Under the assumptions (V.1)-(V.3), we can observe
\[ \lim_{\epsilon \to 0} \epsilon^{-N} c_{\epsilon, \Lambda_{i}} < \infty, \quad \liminf_{\epsilon \to 0} \epsilon^{-\frac{2(p+1)}{p-1}} c_{\epsilon, \Lambda_{i}} \in (0, \infty]. \]

We remark that \((\frac{1}{2} - \frac{1}{p+1})c_{\epsilon, \Lambda_{i}}\) is the lowest non-trivial critical value of the functional which corresponds to the equation
\[ -\epsilon^{2} \Delta u + V(x)u = u^{p}, \quad u > 0 \text{ in } \Lambda_{i}, \quad u \in H_{0}^{1}(\Lambda_{i}). \]

Our first theorem is

**Theorem 1.1 ([S]).** Assume that (V.1)-(V.2) hold and \(\Lambda_{1}, \cdots, \Lambda_{k} \subset \mathbb{R}^{N}\) satisfy (V.3). Then there exists \(\epsilon_{0} > 0\) such that for any \(\epsilon \in (0, \epsilon_{0})\), there exists a positive solution \(u_{\epsilon}(x)\) of (*) \(_{\epsilon}\) which satisfies
\[ \Psi_{\epsilon, \Lambda_{i}}(u_{\epsilon}) = \left( \frac{1}{2} - \frac{1}{p+1} \right)c_{\epsilon, \Lambda_{i}} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad (i = 1, \cdots, k), \]
\[ \Psi_{\epsilon, \mathbb{R}^{N} \setminus \bigcup_{i=1}^{k} \Lambda_{i}}(u_{\epsilon}) = o(\epsilon^{\frac{2(p+1)}{p-1}}). \]

Moreover for some constants \(C, c > 0\), \(u_{\epsilon}(x)\) satisfies
\[ u_{\epsilon}(x) \leq C \exp \left(-\frac{c \operatorname{dist}(x, \bigcup_{i=1}^{k} A_{i})}{\epsilon} \right) \quad \text{for } x \in \mathbb{R}^{N} \setminus \bigcup_{i=1}^{k} \Lambda_{i}. \]

We remark that under the assumptions of Theorem 1.1, for any non-empty subset \(\{i_{1}, \cdots, i_{\ell}\} \subset \{1, \cdots, k\}\), \(\Lambda_{i_{1}}, \cdots, \Lambda_{i_{\ell}}\) also satisfy (V.3). Thus we have

**Corollary 1.2 ([S]).** There exists \(\epsilon_{0} > 0\) such that for any \(\epsilon \in (0, \epsilon_{0})\), there exists a positive solution \(u_{\epsilon}(x)\) of (*) \(_{\epsilon}\) such that
\[ \Psi_{\epsilon, \Lambda_{j}}(u_{\epsilon}) = \left( \frac{1}{2} - \frac{1}{p+1} \right)c_{\epsilon, \Lambda_{j}} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad (j = 1, \cdots, \ell), \]
\[ \Psi_{\epsilon, \mathbb{R}^{N} \setminus \bigcup_{j=1}^{\ell} \Lambda_{i_{j}}}(u_{\epsilon}) = o(\epsilon^{\frac{2(p+1)}{p-1}}). \]

Moreover for some constants \(C, c > 0\), \(u_{\epsilon}(x)\) satisfies
\[ u_{\epsilon}(x) \leq C \exp \left(-\frac{c \operatorname{dist}(x, \bigcup_{j=1}^{\ell} A_{i_{j}})}{\epsilon} \right) \quad \text{for } x \in \mathbb{R}^{N} \setminus \bigcup_{j=1}^{\ell} \Lambda_{i_{j}}. \]
Especially, for small $\epsilon > 0$, $(\ast)_\epsilon$ has at least $2^k - 1$ positive solutions.

(b) Asymptotic behavior of solutions

In this section, we consider the asymptotic behavior of solutions of $(\ast)_\epsilon$ obtained in Theorem 1.1 and Corollary 1.2. For some $i \in \{1, \cdots, k\}$, we assume that $\Lambda_i$ satisfies one of the following assumptions:

(L.1) $\inf_{x \in \Lambda_i} V(x) = m_i > 0$

(L.2) $V(x)$ has a unique local minimum $x_i$ in $\Lambda_i$ and $V(x)$ is represented as

$$V(x) = P(x - x_i) + Q(x - x_i).$$

Here $P(x)$ is a $m$-homogeneous positive function for some $m > 0$, that is,

$$P(x) > 0 \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\},$$

$$P(tx) = t^m P(x) \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}^N,$$

and $\lim_{|x| \to 0} |x|^{-m}Q(x) = 0$.

(L.3) $V(x)$ has a unique local minimum $x_i$ in $\Lambda_i$ and $V(x)$ is represented as

$$V(x) = \exp\{-r(x-x_i)^{\ell} - Q(x-x_i)\}.$$

Here $\ell > 0$ and $r(x) : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ is a positive continuous function such that

$$r(x) > 0 \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\},$$

$$r(tx) = \frac{1}{t} r(x) \quad \text{for all } (t, x) \in (0, \infty) \times (\mathbb{R}^N \setminus \{0\}).$$

We also assume $\Omega \equiv \{x \in \mathbb{R}^N ; r(x) > 1\}$ is strictly star-shaped with respect to 0 and $\lim_{|x| \to 0} |x|^{\ell}Q(x) = 0$.

(L.4) $\Omega \equiv \text{int}\{x \in \Lambda_i ; V(x) = 0\}$ is a non-empty connected bounded set with smooth boundary.

In [BW1], (L.2) is called finite case, (L.3) is infinite case and (L.4) is flat case. Now we have

**Theorem 1.3 ([S])**. Suppose that the assumptions of Theorem 1.1 is satisfied and let $u_\epsilon(x)$ be a positive solution obtained in Theorem 1.1. Assume also, for some $i \in \{1, \cdots, k\}$, $\Lambda_i$ satisfies one of the assumptions (L.1)–(L.4). Set $g(\epsilon) > 0$, $\Omega_0 \subset \mathbb{R}^N$ and $V_0(x) \in C(\Omega_0, \mathbb{R}^N)$ as in the table 0.1 below.
(i) \( c_{\epsilon, \Lambda_i} \) given in (1.2) satisfies
\[
\lim_{\epsilon \rightarrow 0} \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda_i} = b_{V_0, \Omega_0}.
\]
Here, \( b_{V_0, \Omega_0} \) is defined by
\[
\begin{align*}
b_{V_0, \Omega_0} &= \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \left( \frac{\int_{\Omega_0} |\nabla u|^2 + V_0(x) u^2 \, dx}{||u_+||_{L^{p+1}(\Omega_0)}^{2(p+1)}} \right)^{\frac{p+1}{p-1}}.
\end{align*}
\]

(ii) After extracting a subsequence \( \epsilon_n \rightarrow 0 \), there exists a sequence \( x_n \in \Lambda_i \) and a least energy solution \( w_0(x) \) of \((L)_{V_0, \Omega_0} \) such that the rescaled function
\[
w_{\epsilon_n}(y) = \left( \frac{g(\epsilon_n)}{\epsilon_n} \right)^{\frac{2}{p-1}} u_{\epsilon_n}(g(\epsilon_n)y + x_n)
\]
converges to \( w_0(x) \) in the following sense
\[
\begin{align*}
x_n &\rightarrow x_i \in A_i, \quad (1.11) \\
||w_{\epsilon_n} - w_0||_{H^1(O_{i,n})} &\rightarrow 0. \quad (1.12)
\end{align*}
\]
Here we set \( O_{i,n} = \{y \in \mathbb{R}^N; g(\epsilon_n)y + x_n \in \Lambda_i\} \).

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<td>( \epsilon^{\frac{2}{m+2}} )</td>
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<td>( V_0(x) )</td>
<td>( m )</td>
<td>( P(x) )</td>
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<td>( \mathbb{R}^N )</td>
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**Table 0.1**

**Remark 1.4.** Byeon and Oshita [BO] constructed multi-peak solutions by a Lyapunov-Schmidt reduction method; More precisely, they showed the existence of multi-peak solutions under the following situations:

- \( V(x) \in C(\mathbb{R}^N, \mathbb{R}) \) and \( V(x) \) satisfies \( V(x) \geq 0 \) for all \( x \in \mathbb{R}^N \) and \((V.2)-(V.3)\).
- All \( \Lambda_i \) (\( i = 1, \cdots, k \)) satisfy one of the assumptions \((L.2)-(L.4)\) and assumptions of the non-degeneracy of least energy solutions of limit equations.
Under the additional conditions $V(x) \in C^4(\mathbb{R}^N, \mathbb{R})$ and $u \to |u|^{p-1}u \in C^4(\mathbb{R})$, they also showed the existence of multi-peak solutions $u_\epsilon(x)$ which join peaks concentrating local minima satisfying (L.2)–(L.4) and topologically non-trivial critical points $x_i$ of $V(x)$ in $\Lambda_i$ with $V(x_i) > 0$. We remark that they also study the situation $\lim_{|x| \to \infty} V(x) = \infty$.

(c) Other limit equations

In this section, we consider about more general assumptions than (L.2)–(L.3). Firstly, we give some examples.

Examples.

(i) Suppose that $N = 1$ and $\Lambda = (t_1, t_2)$ for $t_1 < 0 < t_2$. We also assume that for $a > b > 1$,

$$V(x) = \begin{cases} |x|^a & \text{if } x \in (t_1, 0], \\ |x|^b & \text{if } x \in (0, t_2). \end{cases}$$

Then, setting $g(\epsilon) = \epsilon^{\frac{2}{a+b+2}}$, $\Omega_0 = (0, \infty)$ and $V_0(x) = |x|^a$, after extracting subsequence $\epsilon_n \to 0$, there exists a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function $w_{\epsilon_n}(y)$, which defined as (1.10), converges to $w_0(x)$.

(ii) Suppose that $N = 2$ and $0 \in \Lambda$. We also assume that for $a > b > 1$,

$$V(x_1, x_2) = |x_1|^a + |x_2|^b \text{ for } (x_1, x_2) \in \Lambda.$$

Then, setting $g(\epsilon) = \epsilon^{\frac{2}{a+b+2}}$, $\Omega_0 = \mathbb{R}^2$ and $V_0(x_1, x_2) = |x_2|^b$, after extracting subsequence $\epsilon_n \to 0$, there exists a a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function $w_{\epsilon_n}(y)$, which defined as (1.10), converges to $w_0(x)$.

In what follows, we give a condition that there exist limit problems. This condition contains (L.2)–(L.3) and two examples above. We fix a $i_0 \in \{1, \cdots, k\}$ and we write $\Lambda_{i_0}$ by $\Lambda$. Without loss of generality, we assume that $0 \in \Lambda$ and 0 is a local minimum of $V(x)$. We set

$$V_\epsilon(x) = \left(\frac{g(\epsilon)}{\epsilon} \right)^2 V(g(\epsilon)x),$$

$$\Lambda_{g(\epsilon)} = \{y \in \mathbb{R}^N; g(\epsilon)y \in \Lambda\}.$$  

Now, we consider the following assumption.

(L*) 0 is a unique local minimum of $V(x)$ in $\Lambda$. Moreover, there exist $g(\epsilon) > 0$ such that $g(\epsilon) \to 0$ as $\epsilon \to 0$ and setting

$$V_0(x) = \lim_{\epsilon \to 0} V_\epsilon(x) \in [0, \infty],$$

$$\Omega_0 = \text{int}\{x \in \mathbb{R}^N | V_0(x) < \infty\},$$
it holds the following:

(i) For any compact set $D \subset \subset \Omega_0$,

$$\lim_{\epsilon \to 0} \sup_{x \in D} |V_\epsilon(x) - V_0(x)| = 0. \quad (1.13)$$

(ii) For any closed set $E \supset \supset \Omega_0$,

$$\lim_{\epsilon \to 0} \inf_{x \in \Lambda(g(\epsilon)) \setminus E} V_\epsilon(x) = \infty. \quad (1.14)$$

(iii) If $\Omega_0 = \mathbb{R}^N$, then for any $\delta \in (0, 1)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that for $\epsilon \in (0, \epsilon_0)$,

$$V_\epsilon(x) \geq (1 - \delta) V_0(x) \quad \text{for all } x \in \Lambda(g(\epsilon)), \quad (1.15)$$

Moreover, for $k \in \mathbb{N}$ with $1 \leq k \leq N$, $V_0(x)$ satisfies

$$V_0(x_1, \ldots, x_k, x_{k+1}, \ldots, x_N) = V_0(x_1, \ldots, x_k, 0, \ldots, 0) \quad \text{for all } x \in \mathbb{R}^N. \quad (1.16)$$

$$\lim_{|x_1| + \cdots + |x_k| \to \infty} V_0(x_1, \ldots, x_k, 0, \ldots, 0) = \infty. \quad (1.17)$$

If $\Omega_0 \neq \mathbb{R}^N$, then $\partial \Omega_0$ is smooth and $V_0(x)$ satisfies

$$\lim_{|x| \to \infty} V_0(x) = \infty \quad (1.18)$$

We remark that $(L)_{V_0, \Omega_0}$ has a least energy solution under the condition (1.16)–(1.17) or (1.18). Here we have

**Theorem 1.5.** Suppose that the assumptions of Theorem 1.1 is satisfied and let $u_\epsilon(x)$ be a positive solution obtained in Theorem 1.1. Assume also $\Lambda$ satisfies the assumptions $(L^*)$.

(i) $c_{\epsilon, \Lambda_i}$ given in (1.2) satisfies

$$\lim_{\epsilon \to 0} \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda_i} = b_{V_0, \Omega_0}. \quad (1.19)$$

Here, $b_{V_0, \Omega_0}$ is defined by (1.9).

(ii) After extracting a subsequence $\epsilon_n \to 0$, there exists a sequence $x_n \in \Lambda_i$ and a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function

$$w_\epsilon(y) = \left( \frac{g(\epsilon_n)}{\epsilon_n} \right)^{\frac{p-2}{2}} u_{\epsilon_n}(g(\epsilon_n)y + x_n)$$

satisfies

$$w_\epsilon(y) \to w_0(x) \quad \text{as } \epsilon \to 0. \quad (1.19)$$
converges to \( w_0(x) \) in the following sense

\[
x_n \to 0, \quad \|w_{\epsilon_n} - w_0\|_{H^1(O_{i,n})} \to 0.
\]

Here we set \( O_{i,n} = \{ y \in \mathbb{R}^N; g(\epsilon_n)y + x_n \in \Lambda_i \} \).

In next sections, we will give an outline of the proof of main theorems.

2. Variational formulation

In this section, we give our variational formulation of \((*)_\epsilon\), which used in [S]. We will reduce to a variational problem defined on infinite dimensional torus \( \Sigma_{\epsilon, \Lambda_1} \times \cdots \times \Sigma_{\epsilon, \Lambda_k} \).

(a) Preliminary

We use the following notation: for an open subset \( D \subset \mathbb{R}^N \),

\[
\langle u, v \rangle_{\epsilon,D} = \int_D \epsilon^2 \nabla u \cdot \nabla v + V(x)uv \, dx \quad \text{for } u, v \in H^1(D),
\]

\[
\|u\|_{\epsilon,D}^2 = \langle u, u \rangle_{\epsilon,D} \quad \text{for } u \in H^1(D),
\]

\[
\|f\|_{\epsilon,D}^* = \sup_{u \in H^1(D), \|u\|_{\epsilon,D} \leq 1} |f(u)| \quad \text{for } f \in (H^1(D))^*.
\]

For an open subset \( D \subset \mathbb{R}^N \) and \( W(x) \in C(D, \mathbb{R}) \), we also set

\[
E_{W,D} = \{ v \in H^1(D); \int_D W(x)v^2 \, dx < \infty \},
\]

\[
\langle u, v \rangle_{W,D} = \int_D \nabla u \cdot \nabla v + W(x)uv \, dx \quad \text{for } u, v \in E_{W,D},
\]

\[
\|u\|_{W,D}^2 = \langle u, u \rangle_{W,D} \quad \text{for } u \in E_{W,D}.
\]

In what follows, we assume that \( \Lambda_i \) has smooth boundary. We set \( \Lambda_* = \bigcup_{i=1}^k \Lambda_i \). By the following proposition, for subsets \( D = \Lambda_1, \cdots, \Lambda_k \) or \( \mathbb{R}^N \setminus \Lambda_* \), norm \( \| \cdot \|_{\epsilon,D} \) is equivalent to \( \| \cdot \|_{H^1(D)} \).

**Proposition 2.1.** There exists \( C_1 > 0 \) independent of \( \epsilon \in (0,1) \) such that for subsets \( D = \Lambda_1, \cdots, \Lambda_k \) or \( \mathbb{R}^N \setminus \Lambda_* \),

\[
\|u\|_{L^2(D)}^2 \leq \frac{C_1}{\epsilon^2} \|u\|_{\epsilon,D}^2 \quad \text{for } u \in H^1(D).
\]

**Proof.** It follows from the Poincaré inequality. (See Proposition 1.1 of [S].)

From Proposition 2.1, we can easily show the following lemma.
Lemma 2.2. There exists $\nu_0 > 0$ independent of $\epsilon \in (0,1)$ such that for subsets $D = \Lambda_1, \cdots, \Lambda_k$ or $\mathbb{R}^N \setminus \Lambda_*$,

$$\frac{1}{2} ||u||_{2,D}^2 \leq ||u||_{2,D}^2 - 2\nu_0 \epsilon^2 ||u||_{L^2(D)}^2 \quad \text{for } u \in H^1(D).$$

(2.2)

**Proof.** Setting $\nu_0 = \frac{1}{4C_1}$ for the constant $C_1$ appeared in (2.1), we get inequality (2.2).

In what follows, we choose $\Lambda_i'$ such that

$$A_i \subset \subset \Lambda_i' \subset \subset \Lambda_i,$$

and set $\Lambda_*' = \bigcup_{i=1}^{k} \Lambda_i'$. We also use the following lemma.

**Lemma 2.3.** There exists a bounded linear operator $P : H^1(\Lambda_* \setminus \Lambda_*') \to H^1(\mathbb{R}^N \setminus \Lambda'_*)$ such that for some $C_2 > 0$ independent of $\epsilon \in (0,1)$,

$$(P u)(x) = u(x) \quad \text{for } x \in \Lambda_* \setminus \Lambda_*' \text{ and } u \in H^1(\Lambda_* \setminus \Lambda_*'),$$

$$||Pu||_{L^2(\mathbb{R}^N \setminus \Lambda_*') \to H^1(\mathbb{R}^N \setminus \Lambda_*')} \leq C_2 ||u||_{L^2(\Lambda_* \setminus \Lambda'_*)} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*').$$

(2.3)

**Proof.** By a standard way, there exists a bounded linear operator $P : H^1(\Lambda_* \setminus \Lambda_*') \to H^1(\mathbb{R}^N \setminus \Lambda_*')$ such that for some $C > 0$,

$$(P u)(x) = u(x) \quad \text{for } x \in \Lambda_* \setminus \Lambda_*' \text{ and } u \in H^1(\Lambda_* \setminus \Lambda_*'),$$

$$||Pu||_{L^2(\mathbb{R}^N \setminus \Lambda_*') \to H^1(\mathbb{R}^N \setminus \Lambda_*')} \leq C ||u||_{L^2(\Lambda_* \setminus \Lambda_*')} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*'),$$

$$||Pu||_{H^1(\mathbb{R}^N \setminus \Lambda_*')} \leq C ||u||_{H^1(\Lambda_* \setminus \Lambda_*')} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*').$$

Thus, noting $\inf_{x \in \Lambda_* \setminus \Lambda_*'} V(x) > 0$, we have

$$||Pu||_{L^2(\mathbb{R}^N \setminus \Lambda_*')}^2 \leq \epsilon^2 ||Pu||_{H^1(\mathbb{R}^N \setminus \Lambda_*')}^2 + \left( \sup_{x \in \mathbb{R}^N} V(x) \right) ||Pu||_{L^2(\mathbb{R}^N \setminus \Lambda_*')}^2$$

$$\leq C^2 \epsilon^2 ||u||_{H^1(\Lambda_* \setminus \Lambda_*')}^2 + C^2 \left( \sup_{x \in \mathbb{R}^N} V(x) \right) ||u||_{L^2(\Lambda_* \setminus \Lambda_*')}^2$$

$$\leq C^2 \int_{\Lambda_* \setminus \Lambda_*'} \epsilon^2 |\nabla u|^2 dx + C^2 \left( \sup_{x \in \mathbb{R}^N} V(x) \right) \int_{\Lambda_* \setminus \Lambda_*'} V(x) u^2 dx$$

$$\leq C^2 \left( 1 + \frac{1}{\inf_{x \in \Lambda_* \setminus \Lambda_*'} V(x)} \right) ||u||_{L^2(\Lambda_* \setminus \Lambda_*')}. $$

This is nothing but (2.3).
(b) Functional setting

Firstly, to find critical points of $\Psi_{\epsilon}(u)$, we modify the nonlinearity $|u|^{p-1}u$. We use a local mountain pass approach introduced by del Pino and Felmer [DF]. We choose $f_{\epsilon} \in C^{1}([0, \infty), \mathbb{R})$ such that for some $0 < \ell_1 < \ell_2$ and $\alpha - 2 > \frac{2(p+1)}{p-1} - N > 0$

$$f_{\epsilon}(\xi) = \begin{cases} \xi^p & \text{for } \xi \in (0, \epsilon^{\frac{\alpha}{p-1}}\ell_1), \\ e^{\alpha} \nu_0 \xi & \text{for } \xi \in (\epsilon^{\frac{\alpha}{p-1}}\ell_2, \infty), \end{cases} \quad 0 \leq f'_{\epsilon}(\xi) \leq 2e^\alpha \nu_0 \text{ for } \xi \in [0, \infty). \quad (2.4)$$

Here $\nu_0$ is given in Lemma 2.2. We set

$$g_{\epsilon}(x, \xi) = \begin{cases} \xi^p & \text{if } x \in \Lambda_\ast \text{ and } \xi \geq 0, \\ f_{\epsilon}(\xi) & \text{if } x \notin \Lambda_\ast \text{ and } \xi \geq 0, \\ 0 & \text{if } \xi \leq 0, \end{cases} \quad (2.5)$$

$$G_{\epsilon}(x, \xi) = \int_0^\xi g_{\epsilon}(x, \tau) \, d\tau, \quad F_{\epsilon}(\xi) = \int_0^\xi f_{\epsilon}(\tau) \, d\tau.$$

Now, we define a functional $\Phi_{\epsilon}(u)$ by

$$\Phi_{\epsilon}(u) = \frac{1}{2}||u||_{\epsilon, \mathbb{R}^N}^2 - \int_{\mathbb{R}^N} G_{\epsilon}(x, u) \, dx \in C^{2}(H^1(\mathbb{R}^N), \mathbb{R}). \quad (2.6)$$

We remark that $\Phi_{\epsilon}(u)$ satisfies (PS)-condition. (cf.[DT]) We also note that if $u_{\epsilon}(x)$ is a critical point of $\Phi_{\epsilon}(u)$ satisfying

$$0 \leq u_{\epsilon}(x) \leq \epsilon^{\frac{\alpha}{p-1}}\ell_1 \quad \text{for } x \in \mathbb{R}^N \setminus \Lambda_\ast, \quad (2.7)$$

then $u_{\epsilon}(x)$ is a critical point of $\Psi_{\epsilon}(u)$. Thus, in what follows, we will find critical points of $\Phi_{\epsilon}(u)$ which satisfy (2.7).

Next, we reduce our problem to a problem on $H^1(\Lambda_\ast)$. For given $u \in H^1(\Lambda_\ast)$, we consider the following minimizing problem:

$$I_{\epsilon}(u) = \inf_{\varphi \in H^1_0(\mathbb{R}^N \setminus \Lambda_\ast)} \Phi_{\epsilon}(Pu + \varphi). \quad (2.8)$$

**Remark 2.4.** Letting $\varphi_{\epsilon}(u)$ be a minimizer of (2.8), then $Pu + \varphi_{\epsilon}(u)$ is a minimizer of $\inf_{v \in E_u} \Phi_{\epsilon}(v)$, where $E_u = \{ v \in H^1(\mathbb{R}^N) \colon v = u \text{ on } \Lambda_\ast \}$. Thus $w(x) = (Pu + \varphi_{\epsilon}(u))|_{\mathbb{R}^N \setminus \Lambda_\ast}(x)$ satisfies the following boundary value problem:

$$-\epsilon^2 \Delta w + V(x)w = f_{\epsilon}(w) \quad \text{in } \mathbb{R}^N \setminus \Lambda_\ast, \quad w = u \quad \text{on } \partial \Lambda_\ast. \quad (2.9)$$

For the minimizing problem (2.8), we have the following proposition.
Proposition 2.5. For any $u \in H^1(\Lambda_*)$ and $\epsilon \in (0, 1)$, the minimizing problem (2.8) has a unique minimizer $\varphi_{\epsilon}(u)$ which satisfies

(i) $u \mapsto \varphi_{\epsilon}(u): H^1(\Lambda_*) \to H^1_0(\mathbb{R}^N \setminus \Lambda_*)$ is of class $C^1$.

(ii) $I_{\epsilon}(u) = \Phi_{\epsilon}(Pu + \varphi_{\epsilon}(u)): H^1(\Lambda_*) \to \mathbb{R}$ is of class $C^2$.

(iii) $u \in H^1(\Lambda_*)$ is a critical point of $I_{\epsilon}(u)$ if and only if $Pu + \varphi_{\epsilon}(u)$ is a critical point of $\Phi_{\epsilon}(u)$.

(iv) $I_{\epsilon}(u)$ satisfies (PS)-condition.

Proof. We outline the proof. (See Proposition 1.6 of [S].) We consider the functional $J(\varphi) = \Phi_{\epsilon}(Pu + \varphi): H^1_0(\mathbb{R}^N \setminus \Lambda_*) \to \mathbb{R}$. Then $J(\varphi)$ is strongly convex and coercive, that is, $J(\varphi)$ satisfies

$$J''(\varphi)(h, h) \geq \frac{1}{2}||h||_{L^2(\mathbb{R}^N \setminus \Lambda_*)}^2$$

for any $\varphi, h \in H^1_0(\mathbb{R}^N \setminus \Lambda_*)$. (2.10)

From (2.10), (2.8) has a unique minimizer $\varphi_{\epsilon}(u)$. (i) follows from the implicit function theorem. (ii)–(iv) follows from the fact that

$$\Phi_{\epsilon}'(Pu + \varphi_{\epsilon}(u))h = 0 \quad \text{for all } h \in H^1_0(\mathbb{R}^N \setminus \Lambda_*)$$

and

$$\varphi_{\epsilon}(u)\zeta \in H^1_0(\mathbb{R}^N \setminus \Lambda_*) \quad \text{for all } \zeta \in H^1(\Lambda_*)$$

that is,

$$I_{\epsilon}'(u)\zeta = \Phi_{\epsilon}'(Pu + \varphi_{\epsilon}(u))(P\zeta) \quad \text{for all } \zeta \in H^1(\Lambda_*)$$

Setting $Q_{\epsilon}(u) = (Pu + \varphi_{\epsilon}(u))|_{\mathbb{R}^N \setminus \Lambda_*}$, by Remark 2.4, $Q_{\epsilon}(u)$ satisfies (2.9). In what follows, we identify $H^1(\Lambda_*)$ and $H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k)$, Moreover, for functions $u_i \in H^1(\Lambda_i)$ $(i = 1, \cdots, k)$, we write $u = (u_1, \cdots, u_k)$ if $u \in H^1(\Lambda_*)$ satisfies $u_i = u|_{\Lambda_i}$ $(i = 1, \cdots, k)$. When $u = (u_1, \cdots, u_k)$, we also write

$$I_{\epsilon}(u_1, \cdots, u_k) \quad \text{for } I_{\epsilon}(u)$$

$$Q_{\epsilon}(u_1, \cdots, u_k) \quad \text{for } Q_{\epsilon}(u)$$

Moreover, we can write

$$I_{\epsilon}(u_1, \cdots, u_k) = \sum_{i=1}^{k} \left( \frac{1}{2}||u_i||_{L^2(\Lambda_i)}^2 - \frac{1}{p+1}||u_i||_{L^{p+1}(\Lambda_i)}^{p+1} \right)
+ \frac{1}{2}||Q_{\epsilon}(u_1, \cdots, u_k)||_{L^2(\mathbb{R}^N \setminus \Lambda_*)}^2 - \int_{\mathbb{R}^N \setminus \Lambda_*} F_{\epsilon}(Q_{\epsilon}(u_1, \cdots, u_k)) \, dx.$$
For $h_{\epsilon}(x) \in L^\infty(\mathbb{R}^N \setminus \Lambda_*)$ satisfying $||h_{\epsilon}||_{L^\infty(\mathbb{R}^N \setminus \Lambda_*)} \leq 2\epsilon^\alpha\nu_0$, we consider the following linear boundary value problem:

$$-\epsilon^2 \Delta v + V(x)v = h_{\epsilon}(x)v \quad \text{in} \quad \mathbb{R}^N \setminus \Lambda_*, \quad v = u \quad \text{on} \quad \Lambda_* \setminus \Lambda_*. \quad (2.13)$$

In a similar way to Proposition 2.5, we can easily observe (2.13) has a unique solution. We denote the unique solution of (2.13) by $Q_{h_{\epsilon}}(u_1, \cdots, u_k)(x)$. Then we can write as $Q_{\epsilon}(u_1, \cdots, u_k)(x) = Q_{f_{\epsilon}(Q_{\epsilon}(u_1, \cdots, u_k))/Q_{\epsilon}(u_1, \cdots, u_k), \epsilon}(u_1, \cdots, u_k)(x)$. Here, we define the auxiliary functionals

$$I_{\epsilon, \Lambda_{i}}(u_{i}) = \frac{1}{2}||u_{i}||_{\epsilon, \Lambda_{a}}^{2} - \frac{1}{p+1}||u_{i+}||_{L^{p+1}(\Lambda)}^{p+1} + \frac{1}{2}||Q_{0, \epsilon}(0, \cdots, u_{i}, \cdots, 0)||_{\epsilon, \mathbb{R}^N \setminus \Lambda_{*}}^{2}$$

$$\in C^2(H^{1}(\Lambda_{i}), \mathbb{R}) \quad (i = 1, \cdots, k).$$

The following proposition gives estimates of difference between $I_{\epsilon}(u_1, \cdots, u_k)$ and auxiliary functional $\sum_{i=1}^{k}I_{\epsilon, \Lambda_{i}}(u_{i})$ and it plays an important role in our proof.

**Proposition 2.6.** There exists $C_3 > 0$ such that for $\epsilon \in (0, 1)$ and $(u_1, \cdots, u_k) \in H^{1}(\Lambda_{1}) \oplus \cdots \oplus H^{1}(\Lambda_{k})$,

$$|I_{\epsilon}(u_1, \cdots, u_k) - \sum_{i=1}^{k}I_{\epsilon, \Lambda_{i}}(u_{i})| \leq \epsilon^{\alpha-2}C_3 \sum_{i=1}^{k}||u_{i}||_{\epsilon, \Lambda_{a}}^{2}. \quad (2.14)$$

To prove Proposition 2.6, we show the following lemma.

**Lemma 2.7.** There exists $C_4 > 0$ such that for $\epsilon \in (0, 1)$ and $(u_1, \cdots, u_k) \in H^{1}(\Lambda_{1}) \oplus \cdots \oplus H^{1}(\Lambda_{k})$,

$$||Q_{h_{\epsilon}}(u_1, \cdots, u_k)||_{\epsilon, \mathbb{R}^N \setminus \Lambda_{*}} \leq C_4||u||_{\epsilon, \Lambda_{*} \setminus \Lambda_{*}'}, \quad (2.15)$$

$$||Q_{\epsilon}(u_1, \cdots, u_k)||_{\epsilon, \mathbb{R}^N \setminus \Lambda_{*}} \leq C_4||u||_{\epsilon, \Lambda_{*} \setminus \Lambda_{*}'}, \quad (2.16)$$

$$||Q_{\epsilon}(u_1, \cdots, u_k) - Q_{0, \epsilon}(u_1, \cdots, u_k)||_{\epsilon, \mathbb{R}^N \setminus \Lambda_{*}} \leq \epsilon^{\alpha}2C_4||u||_{\epsilon, \Lambda_{*} \setminus \Lambda_{*}'} \quad (2.17)$$

**Proof.** First of all, we set $v(x) = Q_{0, \epsilon}(u_1, \cdots, u_k)(x)$ and $w(x) = Q_{\epsilon}(u_1, \cdots, u_k)(x)$. From Remark 2.4, we remark that $w$ is a unique minimizer of the minimizing problem $K_{\epsilon}(w) = \inf_{u \in E_{u}}K_{\epsilon}(u)$, where $K_{\epsilon}(u) = \int_{\mathbb{R}^N \setminus \Lambda_{*}} \frac{1}{2}(\epsilon^2|\nabla u|^2 + V(x)u^2) - F_{\epsilon}(u) dx$. Thus by Lemma 2.2 and Lemma 2.3, we have

$$\frac{1}{4}||w||_{\epsilon, \mathbb{R}^N \setminus \Lambda_{*}}^2 \leq K_{\epsilon}(w) \leq K_{\epsilon}(Pu) \leq ||Pu||_{\epsilon, \mathbb{R}^N \setminus \Lambda_{*}}^2 \leq C_2^2||u||_{\epsilon, \Lambda_{*} \setminus \Lambda_{*}'}^2. \quad (2.18)$$

Thus (2.16) follows from (2.18). We can show (2.15) in a similar way. Next we show (2.17). By variational characterizations for $w(x)$ and $v(x)$ $w - v$ satisfies

$$< w - v, \varphi >_{\epsilon, \mathbb{R}^N \setminus \Lambda_{*}} = \int_{\mathbb{R}^N \setminus \Lambda_{*}} f_{\epsilon}(w)\varphi dx \quad \text{for all} \quad \varphi \in H_{0}^{1}(\mathbb{R}^N \setminus \Lambda_{*}). \quad (2.19)$$
Since \( w - v \in H^1_0(\mathbb{R}^N \setminus \Lambda_*) \), setting \( \varphi = w - v \) in (2.19), we have

\[
\|w - v\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 = \int_{\mathbb{R}^N \setminus \Lambda_*} f_\epsilon(w)(w - v) \, dx \\
\leq \epsilon^\alpha \nu_0 \|w\|_{L^2(\mathbb{R}^N \setminus \Lambda_*)} \|w - v\|_{L^2(\mathbb{R}^N \setminus \Lambda_*)} \\
\leq \epsilon^{\alpha-2} \nu_0 C_1 \|w\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*} \|w - v\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}.
\]  

(2.20)

Here we used Proposition 2.1. By (2.20), we get

\[
\|w - v\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*} \leq \epsilon^{\alpha-2} \nu_0 C_1 \|w\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}.
\]  

(2.21)

From (2.16) and (2.21), for some \( C_4 > 0 \) which is independent of \( u = (u_1, \cdots, u_k) \) and \( \epsilon \in (0, 1) \), we obtain

\[
\|w - v\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*} \leq \epsilon^{\alpha-2} C_4 \|u\|_{\epsilon, \Lambda_* \setminus \Lambda_*'}.
\]

Proof of Proposition 2.6. From (2.12), we have

\[
I_\epsilon(u_1, \cdots, u_k) - \sum_{i=1}^{k} I_{\epsilon, \Lambda_i}(u_i) \\
= \frac{1}{2} \left( \|Q_\epsilon(u_1, \cdots, u_k)\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 - \sum_{i=1}^{k} \|Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0)\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 \right) \\
- \int_{\mathbb{R}^N \setminus \Lambda_*} F_\epsilon(Q_\epsilon(u_1, \cdots, u_k)) \, dx \\
= \frac{1}{2} (I) + (II).
\]

Since \( Q_{0, \epsilon}(u_1, \cdots, u_k) \) is a solution of the linear equation, we have \( Q_{0, \epsilon}(u_1, \cdots, u_k) = \sum_{i=1}^{k} Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0) \). Thus,

\[
(I) = \|Q_\epsilon(u_1, \cdots, u_k)\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 - \|Q_{0, \epsilon}(u_1, \cdots, u_k)\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 \\
+ 2 \sum_{1 \leq i < j \leq k} < Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0), Q_{0, \epsilon}(0, \cdots, u_j, \cdots, 0) >_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}.
\]  

(2.22)

Now we remark that, for \( i \neq j \), the following estimate holds:

\[
| < Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0), Q_{0, \epsilon}(0, \cdots, u_j, \cdots, 0) >_{\epsilon, \mathbb{R}^N \setminus \Lambda_*} | \leq Ce^{-\frac{a}{\epsilon}} \|u_i\|_{\epsilon, \Lambda_i} \|u_j\|_{\epsilon, \Lambda_j},
\]  

(2.23)

where constants \( C, a > 0 \) are independent of \( (u_1, \cdots, u_k) \in H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k) \) and \( \epsilon \in (0, 1) \). It is showed in [ST]. The key of the proof is the subsolution estimate in [Si]. (Also see [GT].) From Lemma 2.7, (2.22) and (2.23), it follows that

\[
|I| \leq 2\epsilon^{\alpha-2} C_4^2 \|u\|_{\epsilon, \Lambda_*}^2 + 2Ce^{-\frac{a}{\epsilon}} \sum_{1 \leq i < j \leq k} \|u_i\|_{\epsilon, \Lambda_i} \|u_j\|_{\epsilon, \Lambda_j}
\]  

(2.24)
On the other hand, since $F(\xi) \leq \frac{1}{2} \epsilon^\alpha \nu_0 \xi^2$, we have

\[
|\langle II \rangle| \leq \frac{1}{2} \epsilon^\alpha \nu_0 ||Q_\epsilon(u_1, \ldots, u_k)||^2_{L^2(\mathbb{R}^N \setminus \Lambda)}
\leq \frac{1}{2} \epsilon^{\alpha-2} \nu_0 C_1 ||Q_\epsilon(u_1, \ldots, u_k)||^2_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}
\leq \frac{1}{2} \epsilon^{\alpha-2} \nu_0 C_1^2 ||u||^2_{\epsilon, \Lambda_*}.
\]

(2.25)

Here we use Proposition 2.1 and Lemma 2.7. We get (2.14) from (2.24) and (2.25).

We use the following notation: for $u_i, v_i \in H^1(\Lambda_i) \ (i = 1, \cdots, k)$,

\[
< u_i, v_i >_{\epsilon, \Lambda_i} = < u_i, v_i >_{\epsilon, \Lambda_i} + < Q_0, u_i >_{\epsilon, \Lambda_i}
\]

\[
||u||^2_{\epsilon, \Lambda_i} = < u_i, u_i >_{\epsilon, \Lambda_i} \quad (i = 1, \cdots, k).
\]

By Lemma 2.7, we easily get

\[
||u||_{\epsilon, \Lambda} \leq ||u||_{\epsilon, \Lambda_i, \#} \leq (1 + C_4) ||u||_{\epsilon, \Lambda_i} \quad \text{for all } u_i \in H^1(\Lambda_i).
\]

Thus $|| \cdot ||_{\epsilon, \Lambda_i, \#}$ is equivalent to $|| \cdot ||_{\epsilon, \Lambda_i}$ for each $i$. With this notation, $I_{\epsilon, \Lambda_i}(u)$ can be written as

\[
I_{\epsilon, \Lambda_i}(u_i) = \frac{1}{2} ||u||^2_{\epsilon, \Lambda_i, \#} - \frac{1}{p+1} ||u||^p_{L^{p+1}(\Lambda_i)} : H^1(\Lambda_i) \to \mathbb{R}. \quad (i = 1, \cdots, k)
\]

We can easily see that $I_{\epsilon, \Lambda_i}(u_i) \ (i = 1, \cdots, k)$ has a mountain pass geometry and satisfies $(PS)_c$-condition for all $c \in \mathbb{R}$ in a standard way.

(c) Reduction to a problem on $\Sigma_{\epsilon, \Lambda_1} \times \cdots \times \Sigma_{\epsilon, \Lambda_k}$

In this section, we reduce our problem to a variational problem on an infinite dimensional torus $\Sigma_{\epsilon, \Lambda_1} \times \cdots \times \Sigma_{\epsilon, \Lambda_k}$, where

\[
\Sigma_{\epsilon, \Lambda_i} = \{ u_i \in H^1(\Lambda_i); ||u||^2_{\epsilon, \Lambda_i, \#} = d_{\epsilon, \Lambda_i} \} \quad (i = 1, \cdots, k).
\]

(2.26)

Here we define $d_{\epsilon, \Lambda_i}$ by

\[
d_{\epsilon, \Lambda_i} = \inf_{u_i \in H^1(\Lambda_i) \setminus \{0\}} \left( \frac{||u||_{\epsilon, \Lambda_i, \#}^{2(p+1)}}{||u||^p_{L^{p+1}(\Lambda)}} \right)^{\frac{p-1}{p}} \quad (i = 1, \cdots, k).
\]

(2.27)

Then $d_{\epsilon, \Lambda_i}$ satisfies the following decay estimates.
Lemma 2.8. (i) If $\inf_{x \in \Lambda_i} V(x) = m_i > 0$, then $\lim_{\epsilon \to 0} \epsilon^{-N} d_{\epsilon, \Lambda_i} = b_{m_i, \mathbb{R}^N} > 0$.
(ii) If $\inf_{x \in \Lambda_i} V(x) = 0$, then $\lim_{\epsilon \to 0} \epsilon^{-N} d_{\epsilon, \Lambda_i} = 0$ and $\liminf_{\epsilon \to 0} \epsilon^{-\frac{2(p+1)}{p-1}} d_{\epsilon, \Lambda_i} \in (0, \infty]$.

Proof. Lemma 2.8 can be shown by a similar way of Proposition 4.2 below.

We consider the auxiliary problems constrained on sphere $\Sigma_{\epsilon, \Lambda_i}$:

$$J_{\epsilon, \Lambda_i}(v_i) = \sup_{t > 0} I_{\epsilon, \Lambda_i}(tv_i) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{d_{\epsilon, \Lambda_i}}{||v_i||_{L^{p+1}(\Lambda_i)}^2}\right)^\frac{p+1}{p-1} : \Sigma_{\epsilon, \Lambda_i} \to \mathbb{R} \ (i = 1, \cdots, k).$$

For $v_i \in \Sigma_{\epsilon, \Lambda_i}$, we can see that $t \mapsto I_{\epsilon, \Lambda_i}(tv_i) : [0, \infty) \to \mathbb{R}$ takes a global maximum at

$$t = t_{\epsilon, \Lambda_i}(v_i) = d_{\epsilon, \Lambda_i}^\frac{1}{p-1} ||v_i||_{L^{p+1}(\Lambda_i)}^{-\frac{p}{p-1}}$$

and

$$J_{\epsilon, \Lambda_i}(v_i) = I_{\epsilon, \Lambda_i}(t_{\epsilon, \Lambda_i}(v_i)v_i) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{d_{\epsilon, \Lambda_i}}{||v_i||_{L^{p+1}(\Lambda_i)}^2}\right)^\frac{p+1}{p-1}.$$ 

We choose $r > 0$ such that $1 - p(1+r)^-(p-1) < 0$ and set

$$N_{\epsilon, \Lambda_i} = \{v_i \in \Sigma_{\epsilon, \Lambda_i}; ||v_i||_{L^{p+1}(\Lambda_i)}^p \geq (1+r)^{-\frac{p+1}{p-1}} d_{\epsilon, \Lambda_i}\} \ (i = 1, \cdots, k).$$  

$N_{\epsilon, \Lambda_i}$ is a neighborhood of least energy critical points of $J_{\epsilon, \Lambda_i}(v_i)$. In fact, we can easily get the following lemma.

Lemma 2.9. For any $\epsilon \in (0, 1)$,

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon, \Lambda_i} = \inf_{v_i \in N_{\epsilon, \Lambda_i}} J_{\epsilon, \Lambda_i}(v_i) \ (i = 1, \cdots, k),$$

$$(1 + r)\left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon, \Lambda_i} \leq \inf_{v_i \in \partial N_{\epsilon, \Lambda_i}} J_{\epsilon, \Lambda_i}(v_i) \ (i = 1, \cdots, k).$$

Proof. By a direct computation, we can easily see that

$$J_{\epsilon, \Lambda_i}(v_i) \leq (1 + r)\left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon, \Lambda_i} \text{ if and only if } ||v_i||_{L^{p+1}(\Lambda_i)}^p \geq (1 + r)^{-\frac{p+1}{p-1}} d_{\epsilon, \Lambda_i}. \ (2.30)$$

Thus Lemma 2.9 follows.

For minimizing sequences of $J_{\epsilon, \Lambda_i}$, we have the following estimates.
Lemma 2.10. If a sequence $v_{i,\epsilon} \in \Sigma_{\epsilon,\Lambda_{i}} (\epsilon \to 0)$ satisfies

\[
J_{\epsilon,\Lambda_{i}}(v_{i,\epsilon}) = \left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon,\Lambda_{i}} + o(\epsilon^{\frac{2(p+1)}{p-1}}),
\]

then we have

\[
||v_{i,\epsilon+}||_{L^{p+1}(\Lambda_{i})}^{p+1} = d_{\epsilon,\Lambda_{i}} + o(\epsilon^{\frac{2(p+1)}{p-1}}).
\]

**Proof.** From (2.31) and (2.30), we have

\[
||v_{i,\epsilon+}||_{L^{p+1}(\Lambda_{i})}^{p+1} = \left( 1 + \frac{o(\epsilon^{\frac{2(p+1)}{p-1}})}{d_{\epsilon,\Lambda_{i}}} \right)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_{i}}.
\]

Using a Taylor expansion, for small $|r|$, we find $(1+r)^{-\frac{p-1}{2}} = 1 + O(r)$. Thus we get

\[
||v_{i,\epsilon+}||_{L^{p+1}(\Lambda_{i})}^{p+1} = \left( 1 + \frac{o(\epsilon^{\frac{2(p+1)}{p-1}})}{d_{\epsilon,\Lambda_{i}}} \right) d_{\epsilon,\Lambda_{i}} + o(\epsilon^{\frac{2(p+1)}{p-1}}).
\]

We set a subset $N_{\epsilon}$ as follows:

\[
N_{\epsilon} = \{ (v_{1}, \cdots, v_{k}) \in \Sigma_{\epsilon,\Lambda_{1}} \times \cdots \times \Sigma_{\epsilon,\Lambda_{k}}; ||v_{i,\epsilon+}||_{L^{p+1}(\Lambda_{i})}^{p+1} \geq (1+r)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_{i}} (i = 1, \cdots, k) \}.
\]

From Lemma 2.9, we can see $N_{\epsilon} \neq \emptyset$. We try to find a critical point of $J_{\epsilon}(v_{1}, \cdots, v_{k}) : N_{\epsilon} \to (0, \infty]$ which is defined by

\[
J_{\epsilon}(v_{1}, \cdots, v_{k}) = \sup_{s_{1}, \cdots, s_{k} \geq 0} I_{\epsilon}(s_{1}v_{1}, \cdots, s_{k}v_{k}).
\]

For simplicity, in what follows, we use notation: $v = (v_{1}, \cdots, v_{k}) \in N_{\epsilon}$, $s = (s_{1}, \cdots, s_{k}) \in [0, \infty)^{k}$, $sv = (s_{1}v_{1}, \cdots, s_{k}v_{k})$.

The following proposition is important.

**Proposition 2.11.** There exists $\epsilon_{1} \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon_{1})$ we have

(i) There exist constants $R_{2} > R_{1} > 0$ independent of $\epsilon$ such that for any $v \in N_{\epsilon}$, $s \mapsto I_{\epsilon}(sv)$ has a unique maximizer $s_{\epsilon}(v) = (s_{1,\epsilon}(v), \cdots, s_{k,\epsilon}(v))$ in $[R_{1}, R_{2}]^{k}$.

(ii) $v \mapsto s_{\epsilon}(v) : N_{\epsilon} \to \mathbb{R}^{k}$ is of class of $C^{1}$.

(iii) $J_{\epsilon}(v) : N_{\epsilon} \to \mathbb{R}$ is of class of $C^{1}$.

(iv) $J_{\epsilon}(v) : N_{\epsilon} \to \mathbb{R}$ satisfies (PS)-condition.

(v) If $v \in N_{\epsilon}$ is a critical point of $J_{\epsilon}(v)$, then $s_{\epsilon}(v)v \in H^{1}(\Lambda_{1}) \oplus \cdots \oplus H^{1}(\Lambda_{k})$ is a critical point of $I_{\epsilon}(u_{1}, \cdots, u_{k})$. 


(vi) For all \( \mathbf{v} = (v_1, \ldots, v_k) \in N_\epsilon \), we have
\[
  J_\epsilon(\mathbf{v}) = \sum_{i=1}^{k} J_{\epsilon,\Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}}).
\]
where \( o(\epsilon^{\frac{2(p+1)}{p-1}}) \) is uniformly for \( \mathbf{v} \in N_\epsilon \).

Proof. See Proposition 1.13 and Proposition 1.14 of [S].

3. Outline of the proof of Theorem 1.1.

In this section, we will show Theorem 1.1. We define
\[
c_\epsilon = \inf_{\mathbf{v} \in N_\epsilon} J_\epsilon(\mathbf{v}).
\]
(3.1)

Then we show the following proposition.

Proposition 3.1. For small \( \epsilon \in (0, \epsilon_1) \), we have
(i) \( c_\epsilon = \left( \frac{1}{2} - \frac{1}{p+1} \right) \sum_{i=1}^{k} d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \).
(ii) \( \inf_{\mathbf{v} \in N_\epsilon} J_\epsilon(\mathbf{v}) < \inf_{\mathbf{v} \in \partial N_\epsilon} J_\epsilon(\mathbf{v}) \).
(iii) \( c_\epsilon \) is a critical value of \( J_\epsilon(\mathbf{v}) \). Moreover corresponding critical points lie in \( N_\epsilon \).

Proof. From (2.35), we recall
\[
  J_\epsilon(\mathbf{v}) = \sum_{i=1}^{k} J_{\epsilon,\Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad \text{for all } \mathbf{v} \in N_\epsilon,
\]
where \( o(\epsilon^{\frac{2(p+1)}{p-1}}) \) is uniformly for \( \mathbf{v} \in N_\epsilon \). Thus from Lemma 2.9, it follows
\[
  c_\epsilon = \inf_{\mathbf{v} \in N_\epsilon} J_\epsilon(\mathbf{v}) = \sum_{i=1}^{k} \inf_{v_i \in N_{\epsilon,\Lambda_i}} J_{\epsilon,\Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}})
  = \left( \frac{1}{2} - \frac{1}{p+1} \right) \sum_{i=1}^{k} d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}).
\]
(3.2)

Next we show (ii). We note \( \partial N_\epsilon = \bigcup_{j=1}^{k} (N_{\epsilon,\Lambda_1} \times \cdots \times \partial N_{\epsilon,\Lambda_j} \times \cdots \times N_{\epsilon,\Lambda_k}) \). Again, from Lemma 2.9, we get for each \( j \)
\[
  \inf_{\mathbf{v} \in N_{\epsilon,\Lambda_1} \times \cdots \times \partial N_{\epsilon,\Lambda_j} \times \cdots \times N_{\epsilon,\Lambda_k}} J_\epsilon(\mathbf{v}) = \sum_{i \neq j} \inf_{v_i \in N_{\epsilon,\Lambda_i}} J_{\epsilon,\Lambda_i}(v_i) + \inf_{v_j \in \partial N_{\epsilon,\Lambda_j}} J_{\epsilon,\Lambda_j}(v_j) + o(\epsilon^{\frac{2(p+1)}{p-1}})
  \geq \left( \frac{1}{2} - \frac{1}{p+1} \right) \sum_{i \neq j} d_{\epsilon,\Lambda_i} + (1+r)\left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon,\Lambda_j} + o(\epsilon^{\frac{2(p+1)}{p-1}})
  = c_\epsilon + r\left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon,\Lambda_j} + o(\epsilon^{\frac{2(p+1)}{p-1}}).
\]
(3.3)

(ii) follows from (3.2)–(3.3). Since \( J_\epsilon(\mathbf{v}) \) satisfies (PS)-condition, we can see \( c_\epsilon \) is a critical value of \( J_\epsilon(\mathbf{v}) \) in a standard way.
Corollary 3.2. Minimizer $v_{\epsilon} = (v_{1,\epsilon}, \cdots, v_{k,\epsilon})$ of (3.1) satisfies

$$J_{\epsilon, \Lambda_{i}}(v_{i,\epsilon}) = \left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon, \Lambda_{i}} + o(\epsilon^\frac{2(p+1)}{p-1}) \quad (i = 1, \cdots, k).$$

**Proof.** From (2.35), we have

$$\left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon, \Lambda_{i}} \leq J_{\epsilon, \Lambda_{i}}(v_{i,\epsilon}) = J_{\epsilon}(v_{\epsilon}) - \sum_{\ell \neq i} J_{\epsilon, \Lambda_{\ell}}(v_{\ell,\epsilon}) + o(\epsilon^\frac{2(p+1)}{p-1})$$

$$\leq c_{\epsilon} - \left( \frac{1}{2} - \frac{1}{p+1} \right) \sum_{\ell \neq i} d_{\epsilon, \Lambda_{\ell}} + o(\epsilon^\frac{2(p+1)}{p-1}).$$

Thus we get Corollary 3.2.

From the a minimizer of (3.1), we get a critical point of $\Phi_{\epsilon}(u)$ by the following:

**Proposition 3.3.** Let $v_{\epsilon} = (v_{1,\epsilon}, \cdots, v_{k,\epsilon})$ be a minimizer of (3.1). Then

$$u_{\epsilon}(x) = \begin{cases} s_{i,\epsilon}(v_{\epsilon})v_{i,\epsilon}(x) & \text{for } x \in \Lambda_{i} \quad (i = 1, \cdots, k), \\ Q_{\epsilon}(s_{1,\epsilon}(v_{\epsilon})v_{1,\epsilon}, \cdots, s_{k,\epsilon}(v_{\epsilon})v_{k,\epsilon})(x) & \text{for } x \in \mathbb{R}^{N} \setminus \Lambda_{*}. \end{cases}$$

(3.4)

is a critical point of $\Phi_{\epsilon}(u)$. Here $(s_{1,\epsilon}(v), \cdots, s_{k,\epsilon}(v))$ are given in Proposition 2.11.

**Proof.** It follows from Proposition 2.5, Proposition 2.11 and Proposition 3.1.

Here we outline the proof of Theorem 1.1

**Outline of the proof of Theorem 1.1.** We can show that $u_{\epsilon}(x)$ defined by (3.4) has an exponential decay on $\mathbb{R}^{N} \setminus \Lambda_{*}$ as $\epsilon \to 0$. (See [S].) Thus $u_{\epsilon}(x)$ satisfies (2.7) and $u_{\epsilon}(x)$ is a critical point of original functional $\Psi_{\epsilon}(u)$. Moreover, we have the following detailed estimate of $u_{\epsilon}(x)$ (See [S]):

$$\|u_{\epsilon+}\|_{L^{p+1}(\Lambda_{i})}^{p+1} = c_{\epsilon, \Lambda_{i}} + o(\epsilon^\frac{2(p+1)}{p-1}),$$

$$\|u_{\epsilon}\|_{\epsilon, \Lambda_{i}}^{2} = c_{\epsilon, \Lambda_{i}} + o(\epsilon^\frac{2(p+1)}{p-1}).$$

(3.5)

(3.6)

Thus (1.4)–(1.5) follow from (3.5)–(3.6). We get (1.6) by a standard way. The proof of Theorem 1.1 is completed.

4. Asymptotic profile of solutions $u_{\epsilon}(x)$.

In this section, we will prove Theorem 1.5. By a similar way, we can prove Theorem 1.3. (See Section 3 of [S].) First of all, we note the following.
Remark 4.1. When assumption \((L^*)\) holds, there exists a constant \(C_5 > 0\) independent of \(\epsilon \in (0,1)\) such that
\[
||w||_{H^1(\Lambda_{g(\epsilon)})} \leq C_5 ||w||_{V_\epsilon, \Lambda_{g(\epsilon)}} \quad \text{for all } w \in H^1(\Lambda_{g(\epsilon)}).
\] (4.1)

In fact we can see that, for some \(\ell > 0\) and \(\delta > 0\) independent of \(\epsilon \in (0,1)\), \(V_\epsilon(y)\) satisfies
\[
V_\epsilon(y) \geq \begin{cases} 
0 & \text{for } y \in (-\ell, \ell)^k \times \mathbb{R}^{N-k} \subset \mathbb{R}^N, \\
\delta & \text{elsewhere.}
\end{cases}
\]

Thus (4.1) can be shown as Proposition 2.1. (See Lemma 1.2 of \([S]\).)

In the following arguments, \(d_{\epsilon, \Lambda}\) defined in (2.27) and \(b_{V_0, \Omega_0}\) defined in (1.9) will play important roles.

Proposition 4.2. \(b_{V_0, \Omega_0}\) is achieved by some \(w(x) \in H_0^1(\Omega)\) and
\[
\left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} \to b_{V_0, \Omega_0} \quad \text{as } \epsilon \to 0.
\] (4.2)

Proof. From Lemma 2.11 of \([S]\), it suffice to show
\[
\left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} \to b_{V_0, \Omega_0} \quad \text{as } \epsilon \to 0.
\] (4.3)

Firstly we show
\[
\left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} \leq b_{V_0, \Omega_0} + o(1).
\] (4.4)

Suppose that \(w(x) \in H_0^1(\Omega)\) achieves \(b_{V_0, \Omega}\). We choose a function \(\varphi \in C^1_0(\mathbb{R}^N, [0,1])\) such that
\[
\varphi(x) = \begin{cases} 
1 & \text{for } x \in \Lambda', \\
0 & \text{for } x \in \mathbb{R}^N \setminus \Lambda
\end{cases}
\]
and set
\[
v_\epsilon(x) = \varphi(x) \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2}{p-1}} w \left( \frac{x}{g(\epsilon)} \right), \quad \psi_\epsilon(x) = \varphi(g(\epsilon)x).
\]

Then by direct computations, we have
\[
||v_\epsilon||_{L, \Lambda}^2 = \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N \left[ \int_{\Lambda_{g(\epsilon)}} \psi_\epsilon(x)^2 \{ |\nabla w(x)|^2 + V_\epsilon(x)w(x)^2 \} dx \\
+ \int_{\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)}} |\nabla \psi_\epsilon(x)|^2 w(x)^2 + 2 \nabla \psi_\epsilon(x) \cdot \nabla w(x) \psi_\epsilon(x) w(x) dx \right]
\]
\[
= \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N ((I) + (II)).
\]
Here we have
\[ |(II)| \leq C^2 \|w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g'(\epsilon)})}^2 + 2C \|\nabla w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g'(\epsilon)})} \|w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g'(\epsilon)})} \to 0 \]
as \(\epsilon \to 0\). Thus, from (1.13), we find
\[ \|v_\epsilon\|_{\epsilon,\Lambda}^2 = \left( \frac{\epsilon}{g(\epsilon)} \right) \left( \frac{2(p+1)}{p-1} \right) g(\epsilon)^N (\|w\|_{V_0,\Omega_0}^2 + o(1)). \] (4.5)

By a similar way, we get
\[ \|v_{\epsilon+}\|_{L^{p+1}(\Lambda)}^{p+1} = \left( \frac{\epsilon}{g(\epsilon)} \right) \left( \frac{2(p+1)}{p-1} \right) g(\epsilon)^N (\|w_+\|_{L^{p+1}(\Omega_0)}^{p+1} + o(1)). \] (4.6)

Thus from definition of \(c_{\epsilon,\Lambda}\) and (4.5)-(4.6),
\[ \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} \leq \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{\|v_\epsilon\|_{\epsilon,\Lambda}}{\|v_{\epsilon+}\|_{L^{p+1}(\Lambda)}} \right)^{\frac{2(p+1)}{p-1}} \]
\[ = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{\|w\|_{V_0,\Omega_0} + o(1)}{\|w_+\|_{L^{p+1}(\Omega_0)} + o(1)} \right)^{\frac{2(p+1)}{p-1}} \]
\[ = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{\|w\|_{V_0,\Omega_0}}{\|w_+\|_{L^{p+1}(\Omega_0)}} \right)^{\frac{2(p+1)}{p-1}} + o(1) \]
\[ = b_{V_0,\Omega_0} + o(1). \] (4.7)

Next, we show
\[ b_{V_0,\Omega_0} \leq \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} + o(1). \] (4.8)

Let \(v_\epsilon(x) \in H^1_0(\Lambda)\) attains \(c_{\epsilon,\Lambda}\). We may assume \(\|v_\epsilon\|_{\epsilon,\Lambda}^{2} = \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N\). Then, from (4.4), we see that
\[ \lim_{\epsilon \to 0} \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \|v_{\epsilon+}\|_{L^{p+1}(\Lambda)} > 0. \] (4.9)

We set
\[ w_\epsilon(x) = \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2}{p-1}} v_\epsilon(g(\epsilon)x). \]

Then \(\|w_\epsilon\|_{V_0,R^N} = 1\) and \(\lim_{\epsilon \to 0} \|w_\epsilon\|_{L^{p+1}(R^N)} > 0\). We need to divide in two cases: \(\Omega_0 = R^N\) or \(\Omega_0 \neq R^N\). Firstly, we consider the case \(\Omega = R^N\). By direct computations, we have
\[ \|w_\epsilon\|_{V_0,R^N}^2 = \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \int_\Lambda \epsilon^{2} |\nabla v_\epsilon|^2 + \left( \frac{\epsilon}{g(\epsilon)} \right)^2 V_0 \left( \frac{x}{g(\epsilon)} \right) v_\epsilon^2(x) \, dx. \]
Thus, from (1.15), for any $\delta \in (0,1)$, we obtain

$$||w_{\epsilon}||^{2}_{(1-\delta)V_{0}, \mathbb{R}^{N}} \leq \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} ||v_{\epsilon}||^{2}_{\epsilon, \mathbb{R}^{N}} + o(1).$$  (4.10)

On the other hand, we have

$$||w_{\epsilon}||^{p+1}_{L^{p+1}(\mathbb{R}^{N})} = \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} ||v_{\epsilon}||^{p+1}_{L^{p+1}(\Lambda)}.$$  (4.11)

From (4.10)–(4.11) we get

$$b_{(1-\delta)V_{0}, \mathbb{R}^{N}} \leq \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} + o(1).$$  (4.8)

(4.8) follows from the fact $b_{(1-\delta)V_{0}, \mathbb{R}^{N}} \rightarrow b_{V_{0}, \mathbb{R}^{N}}$ as $\delta \rightarrow 0$. Next, we consider the case $\Omega_{0} \neq \mathbb{R}^{N}$. Since $||w_{\epsilon}||_{\epsilon, \mathbb{R}^{N}} = 1$, we can choose a subsequence $\epsilon_{n} \rightarrow 0$ and $w_{0}(x) \in H^{1}(\mathbb{R}^{N})$ such that

$$w_{\epsilon_{n}}(x) \rightarrow w_{0}(x) \text{ weakly in } H^{1}(\mathbb{R}^{N}) \text{ and strongly in } L^{p+1}_{\text{loc}}(\mathbb{R}^{N}).$$  (4.12)

From (1.18) and (4.12), we see $w_{\epsilon_{n}}(x) \rightarrow w_{0}(x)$ strongly in $L^{p+1}(\mathbb{R}^{N})$. Moreover, from (1.14), for any closed set $E \supset \supset \Omega_{0}$ we find

$$||w_{\epsilon_{n}}||^{2}_{L^{2}(\Lambda_{g(\epsilon_{n})}\setminus E)} \leq \inf_{\epsilon_{n}} \int_{\Lambda_{g(\epsilon_{n})}\setminus E} V_{\epsilon}(x)w_{\epsilon_{n}}(x)^{2} dx \rightarrow 0.$$  (4.13)

In particular, we find $w_{0} \in H^{1}_{0}(\Omega_{0})$ and $w_{0+} \neq 0$. Thus we have

$$b_{V_{0}, \Omega_{0}} \leq \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{||w_{0}||_{V_{0}, \Omega_{0}}}{||w_{0+}||_{L^{p+1}(\Omega_{0})}} \right)^{\frac{2(p+1)}{p-1}}$$

$$\leq \liminf_{\epsilon_{n} \rightarrow 0} \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{||w_{\epsilon_{n}}||_{\epsilon_{n}, \Lambda_{g(\epsilon_{n})}}}{||w_{\epsilon_{n+}}||_{L^{p+1}(\Lambda_{g(\epsilon_{n})})}} \right)^{\frac{2(p+1)}{p-1}}$$

$$= \liminf_{\epsilon_{n} \rightarrow 0} \left( \frac{g(\epsilon_{n})}{\epsilon_{n}} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon_{n})^{-N} \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{||v_{\epsilon_{n}}||_{\epsilon_{n}, \Lambda}}{||v_{\epsilon_{n+}}||_{L^{p+1}(\Lambda)}} \right)^{\frac{2(p+1)}{p-1}}$$

$$= \liminf_{\epsilon_{n} \rightarrow 0} \left( \frac{g(\epsilon_{n})}{\epsilon_{n}} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon_{n})^{-N} c_{\epsilon_{n}, \Lambda}.$$  (4.13)

Since (4.13) is not depend on subsequence, (4.8) holds. From (4.4) and (4.8), we get (4.4) and complete the proof of Proposition 4.2.
To prove Theorem 1.5, we define a rescaled functional
\[
\overline{\Phi}_{\Lambda_{g(\epsilon)}}(w) = \frac{1}{2}||w||_{V_{\epsilon},\Lambda_{g(\epsilon)}}^{2} - \frac{1}{p+1}||w_{+}||_{L^{p+1}(\Lambda_{g(\epsilon)})}^{p+1} : H^{1}(\Lambda_{g(\epsilon)}) \rightarrow \mathbb{R}.
\]
We also define a functional corresponding to the limit problem \((L)_{V_{0},\Omega_{0}}\) by

\[
I_{V_{0},\Omega_{0}}(w) = \frac{1}{2}||w||_{V_{0},\Omega_{0}}^{2} - \frac{1}{p+1}||w_{+}||_{L^{p+1}(\Omega_{0})}^{p+1} : \mathbb{R}.
\]

Now we have the following proposition.

**Proposition 4.3.** Suppose \((w_{\epsilon}(y)) \subset H^{1}(\Lambda_{g(\epsilon)})\) satisfies

\[
||w_{\epsilon}||_{L^{p+1}(\Lambda_{g(\epsilon)})}^{p+1} = b_{V_{0},\Omega_{0}} + o(1),
\]

(4.14)

\[
||w_{\epsilon}||_{V_{\epsilon},\Lambda_{g(\epsilon)}}^{2} \leq b_{V_{0},\Omega_{0}} + o(1),
\]

(4.15)

\[
\overline{\Phi}_{\Lambda_{g(\epsilon)}}'(w_{\epsilon})\varphi = 0 \text{ for all } \varphi \in H_{0}^{1}(\Lambda_{g(\epsilon)}).
\]

(4.16)

If \((L^{*})\) holds, then after extracting a subsequence \(\epsilon_{n} \rightarrow 0\), there exist \(x_{n} \in \Lambda_{\epsilon_{n}}\) and a least energy solution \(w_{0} \in H_{0}^{1}(\Omega_{0})\) of \((L)_{V_{0},\Omega_{0}}\) such that

\[
\epsilon_{n}x_{n} \rightarrow 0,
\]

\[
||w_{\epsilon_{n}} - w_{0}(\cdot - x_{n})||_{H^{1}(\Lambda_{\epsilon_{n}})} \rightarrow 0 \text{ as } \epsilon_{n} \rightarrow 0,
\]

\[
I_{V_{0},\Omega_{0}}(w_{0}) = \left(\frac{1}{2} - \frac{1}{p+1}\right)b_{V_{0},\Omega_{0}}, \quad I'_{V_{0},\Omega_{0}}(w_{0}) = 0.
\]

To prove Proposition 4.3, we will use the following lemma.

**Lemma 4.4.** Suppose \(w \in H_{0}^{1}(\Omega_{0}) \setminus \{0\}\) satisfies \(I'_{V_{0},\Omega_{0}}(w)w \leq 0\). Then

\[
||w||_{V_{0},\Omega_{0}}^{2} \geq b_{V_{0},\Omega_{0}}.
\]

**Proof.** \(I'_{V_{0},\Omega_{0}}(w)w \leq 0\) implies \(||w||_{V_{0},\Omega_{0}}^{2} \leq ||w||_{L^{p+1}(\Omega_{0})}^{p+1}\). Thus, from the definition of \(b_{V_{0},\Omega_{0}}\), we have

\[
b_{V_{0},\Omega_{0}} \leq \left(\frac{||w||_{V_{0},\Omega_{0}}}{||w||_{L^{p+1}(\Omega_{0})}}\right)^{\frac{2(p+1)}{p-1}} \leq ||w||_{V_{0},\Omega_{0}}^{2}.
\]

**Proof of Proposition 4.3.** For the case \(\Omega_{0} = \mathbb{R}^{N}\), we show the proposition. For the case \(\Omega_{0} \neq \mathbb{R}^{N}\), we can show it by a similar way. We use concentration compactness argument. From (4.14)-(4.16) and (1.15)-(1.17), we can easily see that there exists \(x_{\epsilon} = (0, \cdots, 0, x_{\epsilon,k}, \cdots, x_{\epsilon,N}) \in \mathbb{R}^{N}\) such that for large \(R > 0\), \(\lim\inf_{\epsilon \rightarrow 0} ||w_{\epsilon}||_{L^{p+1}(B_{R}(x_{\epsilon}))} > 0\).
From (4.15)–(4.16), there exist a subsequence $\epsilon_n \to 0$ and $w_0 \in H^1(\mathbb{R}^N)$ such that for any bounded set $D \subset \mathbb{R}^N$,

$$v_{\epsilon_n}(x) = w_{\epsilon_n}(x + x_{\epsilon_n}) \to w_0(x)$$ weakly in $H^1(D)$ and strongly in $L^{p+1}(D)$.

(4.17)

For any $\varphi \in C_0^1(\Omega_0)$ we have

$$\int_{\Lambda_{g(\epsilon_n)}+x_{\epsilon_n}} \nabla v_{\epsilon_n} \cdot \nabla \varphi + V_{\epsilon_n}(x + x_{\epsilon_n}) v_{\epsilon_n} \varphi \, dx - \int_{\Lambda_{g(\epsilon_n)}+x_{\epsilon_n}} v_{\epsilon_n+}^{p} \varphi \, dx = 0.$$

Since $\text{supp} \varphi$ is compact, from (1.13), (1.15)–(1.17), $w_0(x)$ satisfies

$$\int_{\mathbb{R}^N} \nabla w_0 \cdot \nabla \varphi + V_0(x) w_0 \varphi \, dx - \int_{\mathbb{R}^N} w_0+^{p} \varphi \, dx \leq 0,$$

that is, $I'_{V_0,\Omega_0}(w_0)w_0 \leq 0$. From Lemma 4.4 and (4.15), it follows that

$$b_{V_0,\mathbb{R}^N} \leq ||w_0||_{V_0,\mathbb{R}^N}^2 \leq \liminf_{\epsilon_n \to 0} ||v_{\epsilon_n}||_{V_{\epsilon_n},\Lambda_{g(\epsilon_n)}+x_{\epsilon_n}}^2 \leq b_{V_0,\mathbb{R}^N}.$$  \hspace{1cm} (4.18)

By (4.17) and (4.18), we can see

$$\lim_{\epsilon_n \to 0} ||v_{\epsilon_n} - w_0||_{V_{\epsilon_n},\Lambda_{g(\epsilon_n)}+x_{\epsilon_n}}^2 = 0.$$

From (4.1), we also obtain $\lim_{\epsilon_n \to 0} ||v_{\epsilon_n} - w_0||_{H^1(\Lambda_{g(\epsilon_n)}+x_{\epsilon_n})} = 0$. Therefore we have $I'_{V_0,\Omega_0}(w_0) = 0$ and

$$\tilde{\Phi}_{\Lambda_{g(\epsilon_n)}}(w_{\epsilon_n}) \to I_{V_0,\Omega_0}(w_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right)b_{V_0,\Omega_0}.$$

We complete the proof of Proposition 4.3. \hspace{1cm} \Box

**End of proof of Theorem 1.5.** (i) of Theorem 1.5 follows from (4.3). Let $u_{\epsilon}(x)$ be a critical point of $\Psi_{\epsilon}(u)$ obtained in Theorem 1.1. We set $w_{\epsilon}(x) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2}{p-1}} u_{\epsilon}(g(\epsilon)x)$. Then from Proposition 4.2 and (3.5)–(3.6), $w_{\epsilon}|_{\Lambda_{g(\epsilon)}}(x)$ satisfies (4.14)–(4.16). This implies (ii) of Theorem 1.5. Thus we complete the proof of Theorem 1.5. \hspace{1cm} \Box

**References**


