Multi-peak positive solutions for nonlinear Schrödinger equations
with critical frequency

早稲田大学院・理工学研究科 佐藤 洋平 (Yohei Sato)
Department of Mathematics, School of Science and Engineering,
Waseda University

0. Introduction

In this report, we consider the following nonlinear Schrödinger equations:

\[ -\epsilon^2 \Delta u + V(x)u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N, \]
\[ u > 0 \quad \text{in} \quad \mathbb{R}^N, \]
\[ u \in H^1(\mathbb{R}^N), \quad (*)_\epsilon \]

where \( \epsilon > 0 \) is a small parameter and \( p \) satisfies \( 1 < p < \infty \) \((N = 1, 2)\), \( 1 < p < \frac{N+2}{N-2} \) \((N \geq 3)\). We are interested in the existence of solutions of \((*)_\epsilon\) for small \( \epsilon > 0 \) and their behavior as \( \epsilon \to 0 \). We assume the potential \( V(x) \) satisfies the following assumptions:

(V.1) \( V(x) \in C^1(\mathbb{R}^N, \mathbb{R}) \) and \( V(x) \geq 0 \) for all \( x \in \mathbb{R}^N \).
(V.2) \( 0 < \lim_{|x| \to \infty} \inf_{x \in \mathbb{R}^N} V(x) \leq \sup_{x \in \mathbb{R}^N} V(x) < \infty \).

Under the above assumptions, the solutions of \((*)_\epsilon\) are characterized as critical points of

\[ \Psi_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2} \left( \epsilon^2 |\nabla u|^2 + V(x)u^2 \right) - \frac{1}{p+1} u_+^{p+1} dx \in C^2(H^1(\mathbb{R}^N), \mathbb{R}). \]

When \( V(x) \) satisfies

\[ \inf_{x \in \mathbb{R}^N} V(x) > 0, \quad (0.1) \]

\((*)_\epsilon\) has a family of single-peak solutions \( u_\epsilon(x) \) concentrating around a local minimum of \( V(x) \) for small \( \epsilon > 0 \) in the following sense: let \( x_\epsilon \) be a unique maximum of \( u_\epsilon(x) \). Then \( x_\epsilon \) approaches to a local minimum \( x_0 \) of \( V(x) \) as \( \epsilon \to 0 \) and

\[ w_\epsilon(y) = u_\epsilon(\epsilon y + x_\epsilon) \quad (0.2) \]

converges to a least energy solution \( w_0(x) \) of the following “limit equation”

\[ -\Delta w + V(x_0)w = w^p, \quad w > 0 \quad \text{in} \quad \mathbb{R}^N, \quad w \in H^1(\mathbb{R}^N). \quad (0.3) \]
In particular, \( u_{\epsilon}(x) \) satisfies
\[
\lim_{\epsilon \to 0} ||u_{\epsilon}||_{L^{\infty}(\mathbb{R}^{N})} = |w_{0}(0)| > 0, \\
\lim_{\epsilon \to 0} \epsilon^{-N} \Psi_{\epsilon}(u_{\epsilon}) = \left( \frac{1}{2} - \frac{1}{p+1} \right) b_{V(x_{0}), \mathbb{R}^{N}} > 0.
\] (0.4) (0.5)

Here \( b_{V(x_{0}), \mathbb{R}^{N}} \) is defined by
\[
b_{V(x_{0}), \mathbb{R}^{N}} = \inf_{u \in H^{1}(\mathbb{R}^{N}) \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^{N}} |\nabla u|^{2} + V(x_{0})u^{2} \, dx}{||u_{+}||_{L^{p+1}(\mathbb{R}^{N})}^{2}} \right)^{\frac{p+1}{p-1}}
\]
and \( \left( \frac{1}{2} - \frac{1}{p+1} \right) b_{V(x_{0}), \mathbb{R}^{N}} \) is the lowest non-trivial critical value of the functional corresponding to (0.3). See [FW, O1, O2, DF] for related results.

Recently, Byeon and Wang [BW1, BW2] have started to study the case
\[
\inf_{x \in \mathbb{R}^{N}} V(x) = 0. 
\] (0.6)

They showed that \((*)_{\epsilon}\) also has single-peak solutions concentrating around an isolated component \( A \) of \( \{x \in \mathbb{R}^{N}; V(x) = 0\} \). The features of their solutions are completely different from the solutions under the condition (0.1), which depend on the behavior of \( V(x) \) around \( A \). More precisely, their solutions \( u_{\epsilon}(x) \) satisfy
\[
\lim_{\epsilon \to 0} ||u_{\epsilon}||_{L^{\infty}(\mathbb{R}^{N})} = 0, \\
\lim_{\epsilon \to 0} \epsilon^{-N} \Psi_{\epsilon}(u_{\epsilon}) = 0.
\]

These are in contrast with (0.4)–(0.5). We can also see \( \lim \inf_{\epsilon \to 0} \epsilon^{-\frac{2}{p-1}} ||u_{\epsilon}||_{L^{\infty}(\mathbb{R}^{N})} \in (0, \infty] \) and \( \lim \inf_{\epsilon \to 0} \epsilon^{-\frac{2(p+1)}{p-1}} \Psi_{\epsilon}(u_{\epsilon}) \in (0, \infty] \). (We remark that \( p < \frac{N+2}{N-2} \) implies \( \frac{2(p+1)}{p-1} > N \).) Under additional conditions on the behavior of \( V(x) \) near \( A \), we can introduce a rescaled function — which is different from (0.2) — to observe the behavior of \( u_{\epsilon}(x) \) around \( A \). More precisely, under one of the conditions \( (L.2)-(L.4) \) (see p.5 below), we define \( g(\epsilon) > 0 \) as in the table 0.1 below and set
\[
w_{\epsilon}(y) = \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2}{p-1}} u_{\epsilon}(g(\epsilon)y + x_{\epsilon})
\]
for a suitable point \( x_{\epsilon} \) around \( A \), then \( w_{\epsilon}(y) \) approaches to the least energy solution of
\[
-\Delta w + V_{0}(y)w = w^{p}, \quad w > 0 \quad \text{in} \quad \Omega_{0}, \quad w \in H_{0}^{1}(\Omega_{0}). 
\] (L)_{V_{0}, \Omega_{0}}
Here $\Omega_0 \subset \mathbb{R}^N$ and $V_0(x) \in C(\Omega_0, \mathbb{R})$ are also given in the table 0.1 below. In particular, $u_\epsilon(x)$ satisfies

$$
\lim_{\epsilon \to 0} \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \Psi_\epsilon(u_\epsilon) \in (0, \infty)
$$

and the decaying rate of $\Psi_\epsilon(u_\epsilon)$ depends on the behavior of $V(x)$ around $A$.

In this report, we study the multi-peak solutions of $(\ast)_\epsilon$ combining various types of peaks. We assume that $A_1, \cdots, A_k \subset \mathbb{R}^N$ satisfy

(V.3) $A_1, \cdots, A_k \subset \mathbb{R}^N$ are bounded open sets satisfying $A_i \cap A_j = \emptyset$ $(i \neq j)$ and

$$
m_i = \inf_{x \in A_i} V(x) < \inf_{x \in \partial A_i} V(x) < \infty \quad (i = 1, \cdots, k).
$$

Here we remark that $m_i$ may be 0. In what follows, we write

$$A_i = \{x \in A_i; V(x) = m_i\} \quad (i = 1, \cdots, k).$$

Under the assumption (V.3), we will show that $(\ast)_\epsilon$ has $k$-peak solutions which have 1-peak in each $A_i$ and it concentrates around a local minimum of $V(x)$ in $A_i$. We remark that the behavior of the desired solution $u_\epsilon(x)$ in $A_i$ depends on the behavior of $V(x)$ around $A_i$ and just single scaling (0.2) is not enough to describe the behavior of $u_\epsilon(x)$. Moreover the localized critical level $\Psi_{\epsilon, A_i}(u_\epsilon)$ — which is defined as (1.1) below — approaches to 0 with different rate with respect to $\epsilon$. This makes our problems difficult.

It seems that the existence of multi-peak solutions joining solutions which have different scales is not well studied. In our knowledge, there is only one work by Byeon and Oshita [BO] in which they constructed multi-peak solutions by a Lyapunov-Schmidt reduction method under the assumptions of non-degeneracy of solutions of limit problems.

Recently, in [S], we constructed the multi-peak solutions under the assumptions (V.1)–(V.3). We used a variational approach and it does not need the assumptions of non-degeneracy of solutions of limit problems. In the following sections, we will introduce main results and an outline of the proof in [S]. We also study about the asymptotic behavior of solutions and consider other examples than [S].

1. Main Results

Firstly, we introduce main theorems in [S].

(a) Existence of multi-peak solutions

In this section, we consider the existence of multi-peak solutions of $(\ast)_\epsilon$. To state theorems, we set

$$
\Psi_{\epsilon, A_i}(u) = \int_{A_i} \frac{1}{2}(\epsilon^2 |\nabla u|^2 + V(x)u^2) - \frac{1}{p+1} u^{p+1} dx.
$$

(1.1)
and define
\[ c_{\epsilon, \Lambda_{i}} = \inf_{u \in H_{0}^{1} \Lambda_{i} \setminus \{0\}} \left( \frac{\int_{\Lambda_{i}} \epsilon^{2} |\nabla u|^{2} + V(x)u^{2} \, dx}{||u_{+}||_{L^{p+1} \Lambda_{i}}^{2}} \right)^{\frac{p+1}{p-1}}. \] (1.2)

Under the assumptions (V.1)–(V.3), we can observe
\[ \lim_{\epsilon \to 0} \epsilon^{-N} c_{\epsilon, \Lambda_{i}} < \infty, \quad \liminf_{\epsilon \to 0} \epsilon^{-\frac{2(p+1)}{p-1}} c_{\epsilon, \Lambda_{i}} \in (0, \infty]. \]

We remark that \( \left( \frac{1}{2} - \frac{1}{p+1} \right) c_{\epsilon, \Lambda_{i}} \) is the lowest non-trivial critical value of the functional which corresponds to the equation
\[ -\epsilon^{2} \Delta u + V(x)u = u^{p}, \quad u > 0 \quad \text{in} \ \Lambda_{i}, \quad u \in H_{0}^{1} \Lambda_{i}. \] (1.3)

Our first theorem is

**Theorem 1.1** ([S]). Assume that (V.1)–(V.2) hold and \( \Lambda_{1}, \cdots, \Lambda_{k} \subset \mathbb{R}^{N} \) satisfy (V.3). Then there exists \( \epsilon_{0} > 0 \) such that for any \( \epsilon \in (0, \epsilon_{0}) \), there exists a positive solution \( u_{\epsilon}(x) \) of \((*)_{\epsilon}\) which satisfies
\[ \Psi_{\epsilon, \Lambda_{i}}(u_{\epsilon}) = \left( \frac{1}{2} - \frac{1}{p+1} \right) c_{\epsilon, \Lambda_{i}} + o(\epsilon^{2(p+1)} \frac{p+1}{p-1}) \quad (i = 1, \cdots, k), \] (1.4)
\[ \Psi_{\epsilon, \mathbb{R}^{N} \setminus \bigcup_{i=1}^{k} \Lambda_{i}}(u_{\epsilon}) = o(\epsilon^{2(p+1)} \frac{p+1}{p-1}). \] (1.5)

Moreover for some constants \( C, c > 0 \), \( u_{\epsilon}(x) \) satisfies
\[ u_{\epsilon}(x) \leq C \exp \left( -\frac{c \ \text{dist}(x, \bigcup_{i=1}^{k} \Lambda_{i})}{\epsilon} \right) \quad \text{for} \ \epsilon \in \mathbb{R}^{N} \setminus \left( \bigcup_{i=1}^{k} \Lambda_{i} \right). \] (1.6)

We remark that under the assumptions of Theorem 1.1, for any non-empty subset \( \{i_{1}, \cdots, i_{\ell}\} \subset \{1, \cdots, k\}, \Lambda_{i_{1}}, \cdots, \Lambda_{i_{\ell}} \) also satisfy (V.3). Thus we have

**Corollary 1.2** ([S]). There exists \( \epsilon_{0} > 0 \) such that for any \( \epsilon \in (0, \epsilon_{0}) \), there exists a positive solution \( u_{\epsilon}(x) \) of \((*)_{\epsilon}\) such that
\[ \Psi_{\epsilon, \Lambda_{j}}(u_{\epsilon}) = \left( \frac{1}{2} - \frac{1}{p+1} \right) c_{\epsilon, \Lambda_{j}} + o(\epsilon^{2(p+1)} \frac{p+1}{p-1}) \quad (j = 1, \cdots, \ell), \] (1.7)
\[ \Psi_{\epsilon, \mathbb{R}^{N} \setminus \left( \bigcup_{j=1}^{\ell} \Lambda_{j} \right)}(u_{\epsilon}) = o(\epsilon^{2(p+1)} \frac{p+1}{p-1}). \] (1.8)

Moreover for some constants \( C, c > 0 \), \( u_{\epsilon}(x) \) satisfies
\[ u_{\epsilon}(x) \leq C \exp \left( -\frac{c \ \text{dist}(x, \bigcup_{j=1}^{\ell} \Lambda_{j})}{\epsilon} \right) \quad \text{for} \ \epsilon \in \mathbb{R}^{N} \setminus \left( \bigcup_{i=1}^{\ell} \Lambda_{i_{j}} \right). \]
Especially, for small $\epsilon > 0$, $(\ast)_\epsilon$ has at least $2^k-1$ positive solutions.

(b) **Asymptotic behavior of solutions**

In this section, we consider the asymptotic behavior of solutions of $(\ast)_\epsilon$ obtained in Theorem 1.1 and Corollary 1.2. For some $i \in \{1, \cdots, k\}$, we assume that $\Lambda_i$ satisfies one of the following assumptions:

(L.1) $\inf_{x \in \Lambda_i} V(x) = m_i > 0$

(L.2) $V(x)$ has a unique local minimum $x_i$ in $\Lambda_i$ and $V(x)$ is represented as

$$V(x) = P(x - x_i) + Q(x - x_i).$$

Here $P(x)$ is a $m$-homogeneous positive function for some $m > 0$, that is,

$$P(x) > 0 \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\},$$

$$P(tx) = t^m P(x) \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}^N,$$

and $\lim_{|x| \to 0} |x|^{-m} Q(x) = 0$.

(L.3) $V(x)$ has a unique local minimum $x_i$ in $\Lambda_i$ and $V(x)$ is represented as

$$V(x) = \exp\{-r(x - x_i)\ell - Q(x - x_i)\}.$$  

Here $\ell > 0$ and $r(x) : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ is a positive continuous function such that

$$r(x) > 0 \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\},$$

$$r(tx) = \frac{1}{t} r(x) \quad \text{for all } (t, x) \in (0, \infty) \times (\mathbb{R}^N \setminus \{0\}).$$

We also assume $\Omega \equiv \{x \in \mathbb{R}^N; r(x) > 1\}$ is strictly star-shaped with respect to $0$ and $\lim_{|x| \to 0} |x|^{-\ell} Q(x)$ is strictly star-shaped with respect to $0$ and $\lim_{|x| \to 0} |x|^{-\ell} Q(x) = 0$.

(L.4) $\Omega \equiv int\{x \in \Lambda_i; V(x) = 0\}$ is a non-empty connected bounded set with smooth boundary.

In [BW1], (L.2) is called finite case, (L.3) is infinite case and (L.4) is flat case. Now we have

**Theorem 1.3 ([S])**. Suppose that the assumptions of Theorem 1.1 is satisfied and let $u_\epsilon(x)$ be a positive solution obtained in Theorem 1.1. Assume also, for some $i \in \{1, \cdots, k\}$, $\Lambda_i$ satisfies one of the assumptions (L.1)–(L.4). Set $g(\epsilon) > 0$, $\Omega_0 \subset \mathbb{R}^N$ and $V_0(x) \in C(\Omega_0, \mathbb{R}^N)$ as in the table 0.1 below.
(i) $c_{\epsilon, \Lambda_{i}}$ given in (1.2) satisfies
\[
\lim_{\epsilon \to 0} \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda_{i}} = b_{V_{0}, \Omega_{0}}.
\]
Here, $b_{V_{0}, \Omega_{0}}$ is defined by
\[
b_{V_{0}, \Omega_{0}} = \inf_{u \in H_{0}^{1}(\Omega_{0}) \setminus \{0\}} \left( \frac{\int_{\Omega_{0}} |\nabla u|^{2} + V_{0}(x)u^{2} \, dx}{||u_{+}||_{L^{p+1}(\Omega_{0})}^{2}} \right)^{\frac{p+1}{p-1}}.
\] (1.9)

(ii) After extracting a subsequence $\epsilon_{n} \to 0$, there exists a sequence $x_{n} \in \Lambda_{i}$ and a least energy solution $w_{0}(x)$ of $(L)_{V_{0}, \Omega_{0}}$ such that the rescaled function
\[
w_{\epsilon_{n}}(y) = \left( \frac{g(\epsilon_{n})}{\epsilon_{n}} \right)^{\frac{2}{p-1}} u_{\epsilon_{n}}(g(\epsilon_{n})y + x_{n})
\] (1.10)
converges to $w_{0}(x)$ in the following sense
\[
x_{n} \to x_{i} \in A_{i},
\] (1.11)
\[
||w_{\epsilon_{n}} - w_{0}||_{H^{1}(O_{i,n})} \to 0.
\] (1.12)

Here we set $O_{i,n} = \{y \in \mathbb{R}^{N}; g(\epsilon_{n})y + x_{n} \in \Lambda_{i}\}$.

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<th>(L.1)</th>
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<td>$g(\epsilon)$</td>
<td>$\epsilon$</td>
<td>$\epsilon^{\frac{2}{m+2}}$</td>
<td>$(\log \epsilon^{-2})^{-\frac{1}{2}}$</td>
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<tr>
<td>$V_{0}(x)$</td>
<td>$m$</td>
<td>$P(x)$</td>
<td>0</td>
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<td>$\Omega_{0}$</td>
<td>$\mathbb{R}^{N}$</td>
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**Remark 1.4.** Byeon and Oshita [BO] constructed multi-peak solutions by a Lyapunov-Schmidt reduction method; More precisely, they showed the existence of multi-peak solutions under the following situations:

- $V(x) \in C(\mathbb{R}^{N}, \mathbb{R})$ and $V(x)$ satisfies $V(x) \geq 0$ for all $x \in \mathbb{R}^{N}$ and (V.2)–(V.3).
- All $\Lambda_{i}$ ($i = 1, \cdots, k$) satisfy one of the assumptions (L.2)–(L.4) and assumptions of the non-degeneracy of least energy solutions of limit equations.
Under the additional conditions $V(x) \in C^4(\mathbb{R}^N, \mathbb{R})$ and $u \to |u|^{p-1}u \in C^4(\mathbb{R})$, they also showed the existence of multi-peak solutions $u_\epsilon(x)$ which join peaks concentrating local minima satisfying (L.2)–(L.4) and topologically non-trivial critical points $x_i$ of $V(x)$ in $\Lambda_i$ with $V(x_i) > 0$. We remark that they also study the situation $\lim_{|x| \to \infty} V(x) = \infty$.

(c) Other limit equations

In this section, we consider about more general assumptions than (L.2)–(L.3). Firstly, we give some examples.

Examples.

(i) Suppose that $N = 1$ and $\Lambda = (t_1, t_2)$ for $t_1 < 0 < t_2$. We also assume that for $a > b > 1$,

$$V(x) = \begin{cases} |x|^a & \text{if } x \in (t_1, 0], \\ |x|^b & \text{if } x \in (0, t_2). \end{cases}$$

Then, setting $g(\epsilon) = \epsilon^{\frac{2}{a+b}}, \Omega_0 = (0, \infty)$ and $V_0(x) = |x|^a$, after extracting subsequence $\epsilon_n \to 0$, there exists a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function $w_\epsilon(x)$, which defined as (1.10), converges to $w_0(x)$.

(ii) Suppose that $N = 2$ and $0 \in \Lambda$. We also assume that for $a > b > 1$,

$$V(x_1, x_2) = |x_1|^a + |x_2|^b \quad \text{for } (x_1, x_2) \in \Lambda.$$

Then, setting $g(\epsilon) = \epsilon^{\frac{2}{b+a}}, \Omega_0 = \mathbb{R}^2$ and $V_0(x_1, x_2) = |x_2|^b$, after extracting subsequence $\epsilon_n \to 0$, there exists a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function $w_\epsilon(x)$, which defined as (1.10), converges to $w_0(x)$.

In what follows, we give a condition that there exist limit problems. This condition contains (L.2)–(L.3) and two examples above. We fix a $i_0 \in \{1, \cdots, k\}$ and we write $\Lambda_{i_0}$ by $\Lambda$. Without loss of generality, we assume that $0 \in \Lambda$ and 0 is a local minimum of $V(x)$. We set

$$V_\epsilon(x) = \left(\frac{g(\epsilon)}{\epsilon}\right)^2 V(g(\epsilon)x),
\Lambda_{g(\epsilon)} = \{y \in \mathbb{R}^N; g(\epsilon)y \in \Lambda\}.$$ 

Now, we consider the following assumption.

(L*) 0 is a unique local minimum of $V(x)$ in $\Lambda$. Moreover, there exist $g(\epsilon) > 0$ such that $g(\epsilon) \to 0$ as $\epsilon \to 0$ and setting

$$V_0(x) = \lim_{\epsilon \to 0} V_\epsilon(x) \in [0, \infty].
\Omega_0 = \text{int}\{x \in \mathbb{R}^N | V_0(x) < \infty\},$$

We also assume that for $a > b > 1$,
it holds the following:

(i) For any compact set $D \subset \subset \Omega_0$,

$$\lim_{\epsilon \to 0} \sup_{x \in D} |V_\epsilon(x) - V_0(x)| = 0. \quad (1.13)$$

(ii) For any closed set $E \supset \supset \Omega_0$,

$$\lim_{\epsilon \to 0} \inf_{x \in \Lambda_{g(\epsilon)} \setminus E} V_\epsilon(x) = \infty. \quad (1.14)$$

(iii) If $\Omega_0 = \mathbb{R}^N$, then for any $\delta \in (0,1)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that for $\epsilon \in (0, \epsilon_0)$,

$$V_\epsilon(x) \geq (1 - \delta)V_0(x) \quad \text{for all } x \in \Lambda_{g(\epsilon)}, \quad (1.15)$$

Moreover, for $k \in \mathbb{N}$ with $1 \leq k \leq N$, $V_0(x)$ satisfies

$$V_0(x_1, \ldots, x_k, x_{k+1}, \ldots, x_N) = V_0(x_1, \ldots, x_k, 0, \ldots, 0) \quad \text{for all } x \in \mathbb{R}^N \quad (1.16)$$

$$\lim_{|x_1| + \cdots + |x_k| \to \infty} V_0(x_1, \ldots, x_k, 0, \ldots, 0) = \infty. \quad (1.17)$$

If $\Omega_0 \neq \mathbb{R}^N$, then $\partial \Omega_0$ is smooth and $V_0(x)$ satisfies

$$\lim_{|x| \to \infty} V_0(x) = \infty \quad (1.18)$$

We remark that $(L)_{V_0,\Omega_0}$ has a least energy solution under the condition $(1.16)-(1.17)$ or $(1.18)$. Here we have

**Theorem 1.5.** Suppose that the assumptions of Theorem 1.1 is satisfied and let $u_\epsilon(x)$ be a positive solution obtained in Theorem 1.1. Assume also $\Lambda$ satisfies the assumptions $(L^*)$.

(i) $c_{\epsilon,\Lambda}$ given in (1.2) satisfies

$$\lim_{\epsilon \to 0} \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} = b_{V_0,\Omega_0}. \quad (1.19)$$

Here, $b_{V_0,\Omega_0}$ is defined by (1.9).

(ii) After extracting a subsequence $\epsilon_n \to 0$, there exists a sequence $x_n \in \Lambda_i$ and a least energy solution $w_0(x)$ of $(L)_{V_0,\Omega_0}$ such that the rescaled function

$$w_\epsilon(y) = \left( \frac{g(\epsilon_n)}{\epsilon_n} \right)^{\frac{2}{p-1}} u_{\epsilon_n}(g(\epsilon_n)y + x_n) \quad (1.19)$$
converges to \( w_0(x) \) in the following sense

\[
x_n \to 0, \\
\| w_{\epsilon_n} - w_0 \|_{H^1(O_{i,n})} \to 0.
\]

(1.20) \hspace{1cm} (1.21)

Here we set \( O_{i,n} = \{ y \in \mathbb{R}^N; g(\epsilon_n)y + x_n \in \Lambda_i \} \).

In next sections, we will give a outline of the proof of main theorems.

2. Variational formulation

In this section, we give our variational formulation of \((*)_{\epsilon}\), which used in [S]. We will reduce to a variational problem defined on infinite dimensional torus \( \Sigma_{\epsilon,\Lambda_1} \times \cdots \times \Sigma_{\epsilon,\Lambda_k} \).

(a) Preliminary

We use the following notation: for an open subset \( D \subset \mathbb{R}^N \),

\[
< u, v >_{\epsilon,D} = \int_D \epsilon^2 \nabla u \cdot \nabla v + V(x)uv \, dx \quad \text{for } u, v \in H^1(D),
\]

\[
\| u \|_{\epsilon,D}^2 = < u, u >_{\epsilon,D} \quad \text{for } u \in H^1(D),
\]

\[
\| f \|_{\epsilon,D}^* = \sup_{u \in H^1(D), \| u \|_{\epsilon,D} \leq 1} |f(u)| \quad \text{for } f \in (H^1(D))^*.
\]

For an open subset \( D \subset \mathbb{R}^N \) and \( W(x) \in C(D, \mathbb{R}) \), we also set

\[
E_{W,D} = \{ v \in H^1(D); \int_D W(x)v^2 \, dx < \infty \},
\]

\[
< u, v >_{W,D} = \int_D \nabla u \cdot \nabla v + W(x)uv \, dx \quad \text{for } u, v \in E_{W,D},
\]

\[
\| u \|_{W,D}^2 = < u, u >_{W,D} \quad \text{for } u \in E_{W,D}.
\]

In what follows, we assume that \( \Lambda_i \) has smooth boundary. We set \( \Lambda_* = \bigcup_{i=1}^k \Lambda_i \). By the following proposition, for subsets \( D = \Lambda_1, \cdots, \Lambda_k \) or \( \mathbb{R}^N \setminus \Lambda_* \), norm \( \| \cdot \|_{\epsilon,D} \) is equivalent to \( \| \cdot \|_{H^1(D)} \).

**Proposition 2.1.** There exists \( C_1 > 0 \) independent of \( \epsilon \in (0,1) \) such that for subsets \( D = \Lambda_1, \cdots, \Lambda_k \) or \( \mathbb{R}^N \setminus \Lambda_* \),

\[
\| u \|_{L^2(D)}^2 \leq \frac{C_1}{\epsilon^2} \| u \|_{\epsilon,D}^2 \quad \text{for } u \in H^1(D).
\]

(2.1)

**Proof.** It follows from the Poincaré inequality. (See Proposition 1.1 of [S].) \( \square \)

From Proposition 2.1, we can easily show the following lemma.
Lemma 2.2. There exists $\nu_0 > 0$ independent of $\epsilon \in (0, 1)$ such that for subsets $D = \Lambda_1, \cdots, \Lambda_k$ or $\mathbb{R}^N \setminus \Lambda_*$,

$$\frac{1}{2}||u||^2_{\epsilon,D} \leq ||u||^2_{\epsilon,D} - 2\nu_0 \epsilon^2 ||u||^2_{L^2(D)} \quad \text{for } u \in H^1(D). \quad (2.2)$$

Proof. Setting $\nu_0 = \frac{1}{4C_1}$ for the constant $C_1$ appeared in (2.1), we get inequality (2.2).

In what follows, we choose $\Lambda'_i$ such that

$$A_i \subset \subset \Lambda'_i \subset \subset \Lambda_i,$$

and set $\Lambda_*' = \bigcup_{i=1}^k \Lambda'_i$. We also use the following lemma.

Lemma 2.3. There exists a bounded linear operator $P : H^1(\Lambda_* \setminus \Lambda_*') \to H^1(\mathbb{R}^N \setminus \Lambda_*')$ such that for some $C_2 > 0$ independent of $\epsilon \in (0, 1)$,

$$(Pu)(x) = u(x) \quad \text{for } x \in \Lambda_* \setminus \Lambda_*' \text{ and } u \in H^1(\Lambda_* \setminus \Lambda_*'),

||Pu||_{\epsilon,\mathbb{R}^N \setminus \Lambda_*'} \leq C_2 ||u||_{\epsilon,\Lambda_* \setminus \Lambda_*'} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*'). \quad (2.3)$$

Proof. By a standard way, there exists a bounded linear operator $P : H^1(\Lambda_* \setminus \Lambda_*') \to H^1(\mathbb{R}^N \setminus \Lambda_*')$ such that for some $C > 0$,

$$(Pu)(x) = u(x) \quad \text{for } x \in \Lambda_* \setminus \Lambda_*' \text{ and } u \in H^1(\Lambda_* \setminus \Lambda_*'),

||Pu||_{L^2(\mathbb{R}^N \setminus \Lambda_*')} \leq C ||u||_{L^2(\Lambda_* \setminus \Lambda_*')} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*'),

||Pu||_{H^1(\mathbb{R}^N \setminus \Lambda_*')} \leq C ||u||_{H^1(\Lambda_* \setminus \Lambda_*')} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*').$$

Thus, noting $\inf_{x \in \Lambda_* \setminus \Lambda_*'} V(x) > 0$, we have

$$||Pu||^2_{\epsilon,\mathbb{R}^N \setminus \Lambda_*'} \leq \epsilon^2 ||Pu||^2_{H^1(\mathbb{R}^N \setminus \Lambda_*')} + \left( \sup_{x \in \mathbb{R}^N} V(x) \right) ||Pu||^2_{L^2(\mathbb{R}^N \setminus \Lambda_*')} \leq C^2 \epsilon^2 ||u||^2_{H^1(\Lambda_* \setminus \Lambda_*')} + C^2 \left( \sup_{x \in \mathbb{R}^N} V(x) \right) ||u||^2_{L^2(\Lambda_* \setminus \Lambda_*')} \leq C^2 \int_{\Lambda_* \setminus \Lambda_*'} \epsilon^2 |\nabla u|^2 \, dx + C^2 \left( \frac{\sup_{x \in \mathbb{R}^N} V(x)}{\inf_{x \in \Lambda_* \setminus \Lambda_*'} V(x)} \right) \int_{\Lambda_* \setminus \Lambda_*'} V(x) u^2 \, dx \leq C^2 \left( 1 + \frac{1 + \sup_{x \in \mathbb{R}^N} V(x)}{\inf_{x \in \Lambda_* \setminus \Lambda_*'} V(x)} \right) ||u||_{\epsilon,\Lambda_* \setminus \Lambda_*'}.$$

This is nothing but (2.3).
(b) Functional setting

Firstly, to find critical points of $\Psi_{\epsilon}(u)$, we modify the nonlinearity $|u|^{p-1}u$. We use a local mountain pass approach introduced by del Pino and Felmer [DF]. We choose $f_{\epsilon} \in C^{1}([0, \infty), \mathbb{R})$ such that for some $0 < \ell_{1} < \ell_{2}$ and $\alpha - 2 > \frac{2(p+1)}{p-1} - N > 0$

\[
f_{\epsilon}(\xi) = \begin{cases} 
\xi^{p} & \text{for } \xi \in (0, \epsilon^{\frac{\alpha}{p-1}}\ell_{1}), \\
\epsilon^{\alpha}\nu_{0}\xi & \text{for } \xi \in (\epsilon^{\frac{\alpha}{p-1}}\ell_{2}, \infty), \\
0 & \text{for } \xi \in [0, \infty).
\end{cases}
\] (2.4)

Here $\nu_{0}$ is given in Lemma 2.2. We set

\[
g_{\epsilon}(x, \xi) = \begin{cases} 
\xi^{p} & \text{if } x \in \Lambda_{*} \text{ and } \xi \geq 0, \\
f_{\epsilon}(\xi) & \text{if } x \not\in \Lambda_{*} \text{ and } \xi \geq 0, \\
0 & \text{if } \xi \leq 0,
\end{cases}
\] (2.5)

\[
G_{\epsilon}(x, \xi) = \int_{0}^{\xi}g_{\epsilon}(x, \tau)d\tau, \quad F_{\epsilon}(\xi) = \int_{0}^{\xi}f_{\epsilon}(\tau)d\tau.
\] (2.6)

Now, we define a functional $\Phi_{\epsilon}(u)$ by

\[
\Phi_{\epsilon}(u) = \frac{1}{2}\|u\|_{C^{1}(\mathbb{R}^{N}, \mathbb{R})}^{2} - \int_{\mathbb{R}^{N}}G_{\epsilon}(x, u)dx \in C^{2}(H^{1}(\mathbb{R}^{N}), \mathbb{R}).
\] (2.7)

We remark that $\Phi_{\epsilon}(u)$ satisfies (PS)-condition. (cf.[DT]) We also note that if $u_{\epsilon}(x)$ is a critical point of $\Phi_{\epsilon}(u)$ satisfying

\[
0 \leq u_{\epsilon}(x) \leq \epsilon^{\frac{\alpha}{p-1}}\ell_{1} \quad \text{for } x \in \mathbb{R}^{N}\setminus\Lambda_{*},
\] (2.8)

then $u_{\epsilon}(x)$ is a critical point of $\Psi_{\epsilon}(u)$. Thus, in what follows, we will find critical points of $\Phi_{\epsilon}(u)$ which satisfy (2.7).

Next, we reduce our problem to a problem on $H^{1}(\Lambda_{*})$. For given $u \in H^{1}(\Lambda_{*})$, we consider the following minimizing problem:

\[
I_{\epsilon}(u) = \inf_{\varphi \in H_{0}^{1}(\mathbb{R}^{N}\setminus\Lambda_{*})} \Phi_{\epsilon}(Pu + \varphi).
\] (2.9)

Remark 2.4. Letting $\varphi_{\epsilon}(u)$ be a minimizer of (2.8), then $Pu + \varphi_{\epsilon}(u)$ is a minimizer of $\inf_{v \in E_{u}} \Phi_{\epsilon}(v)$, where $E_{u} = \{v \in H^{1}(\mathbb{R}^{N}) : v = u \text{ on } \Lambda_{*}\}$. Thus $w(x) = (Pu + \varphi_{\epsilon}(u))|_{\mathbb{R}^{N}\setminus\Lambda_{*}}(x)$ satisfies the following boundary value problem:

\[-\epsilon^{2}\Delta w + V(x)w = f_{\epsilon}(w) \quad \text{in } \mathbb{R}^{N}\setminus\Lambda_{*}, \quad w = u \quad \text{on } \partial\Lambda_{*}.
\] (2.10)

For the minimizing problem (2.8), we have the following proposition.
Proposition 2.5. For any $u \in H^1(\Lambda_*)$ and $\epsilon \in (0, 1)$, the minimizing problem (2.8) has a unique minimizer $\varphi_{\epsilon}(u)$ which satisfies

(i) $u \mapsto \varphi_{\epsilon}(u): H^1(\Lambda_*) \to H^1_0(\mathbb{R}^N \setminus \Lambda_*)$ is of class $C^1$.

(ii) $I_\epsilon(u) = \Phi_{\epsilon}(Pu + \varphi_{\epsilon}(u)): H^1(\Lambda_*) \to \mathbb{R}$ is of class $C^2$.

(iii) $u \in H^1(\Lambda_*)$ is a critical point of $I_\epsilon(u)$ if and only if $Pu + \varphi_{\epsilon}(u)$ is a critical point of $\Phi_{\epsilon}(u)$.

(iv) $I_\epsilon(u)$ satisfies (PS)-condition.

Proof. We outline the proof. (See Proposition 1.6 of [S].) We consider the functional $J(\varphi) = \Phi_{\epsilon}(Pu + \varphi): H^1_0(\mathbb{R}^N \setminus \Lambda_*) \to \mathbb{R}$. Then $J(\varphi)$ is strongly convex and coercive, that is, $J(\varphi)$ satisfies

$$J''(\varphi)(h, h) \geq \frac{1}{2}||h||_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2$$

for any $\varphi, h \in H^1_0(\mathbb{R}^N \setminus \Lambda_*)$. (2.10)

From (2.10), (2.8) has a unique minimizer $\varphi_{\epsilon}(u)$. (i) follows from the implicit function theorem. (ii)-(iv) follows from the fact that

$$\Phi_{\epsilon}'(Pu + \varphi_{\epsilon}(u))h = 0 \quad \text{for all } h \in H^1_0(\mathbb{R}^N \setminus \Lambda_*)$$

and

$$\varphi_{\epsilon}'(u)\zeta \in H^1_0(\mathbb{R}^N \setminus \Lambda_*) \quad \text{for all } \zeta \in H^1(\Lambda_*)$$

that is,

$$I_\epsilon'(u)\zeta = \Phi_{\epsilon}'(Pu + \varphi_{\epsilon}(u))(P\zeta) \quad \text{for all } \zeta \in H^1(\Lambda_*)$$

Setting $Q_{\epsilon}(u) = (Pu + \varphi_{\epsilon}(u))|_{\mathbb{R}^N \setminus \Lambda_*}$, by Remark 2.4, $Q_{\epsilon}(u)$ satisfies (2.9). In what follows, we identify $H^1(\Lambda_*)$ and $H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k)$, Moreover, for functions $u_i \in H^1(\Lambda_i)$ ($i = 1, \ldots, k$), we write $u = (u_1, \ldots, u_k)$ if $u \in H^1(\Lambda_*)$ satisfies $u_i = u|_{\Lambda_i}$ ($i = 1, \ldots, k$). When $u = (u_1, \ldots, u_k)$, we also write

$$I_\epsilon(u_1, \ldots, u_k) \quad \text{for } I_\epsilon(u),$$

$$Q_\epsilon(u_1, \ldots, u_k) \quad \text{for } Q_\epsilon(u).$$

Moreover, we can write

$$I_\epsilon(u_1, \ldots, u_k) = \sum_{i=1}^{k} \left( \frac{1}{2}||u_i||_{\epsilon, \Lambda_i}^2 - \frac{1}{p+1}||u_i||_{L^{p+1}(\Lambda_i)}^{p+1} \right)$$

$$+ \frac{1}{2}||Q_\epsilon(u_1, \ldots, u_k)||_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 - \int_{\mathbb{R}^N \setminus \Lambda_*} F_\epsilon(Q_\epsilon(u_1, \ldots, u_k)) \ dx.$$
For \( h_{\epsilon}(x) \in L^\infty(\mathbb{R}^N \setminus \Lambda_*) \) satisfying \( ||h_{\epsilon}||_{L^\infty(\mathbb{R}^N \setminus \Lambda_*)} \leq 2\epsilon^\alpha \nu_0 \), we consider the following linear boundary value problem:

\[
-\epsilon^2 \Delta v + V(x)v = h_{\epsilon}(x)v \quad \text{in} \quad \mathbb{R}^N \setminus \Lambda_* , \quad v = u \quad \text{on} \quad \Lambda_* \setminus \Lambda'_* .
\]  

(2.13)

In a similar way to Proposition 2.5, we can easily observe (2.13) has a unique solution. We denote the unique solution of (2.13) by \( Q_{h_{\epsilon},\epsilon}(u_1, \cdots, u_k)(x) \). Then we can write

\[
Q_{\epsilon}(u_1, \cdots, u_k)(x)=Q_{f(Q_{\epsilon}(u_1,\cdots,u_k))/Q_{\epsilon}(u_1,\cdots,u_k),\epsilon}(u_1, \cdots, u_k)(x).
\]

Here, we define the auxiliary functionals

\[
I_{\epsilon,\Lambda_i}(u_i) = \frac{1}{2}||u_i||_{\epsilon,\Lambda_i}^2 - \frac{1}{p+1}||u_{i+1}||_{L^{p+1}(\Lambda_i)}^{p+1} + \frac{1}{2}||Q_{0,\epsilon}(0, \cdots, u_i, \cdots, 0)||_{\epsilon,\mathbb{R}^N \setminus \Lambda_*}^2
\]

\[
\in C^2(H^1(\Lambda_i), \mathbb{R}) \quad (i=1, \cdots, k).
\]

The following proposition gives estimates of difference between \( I_{\epsilon}(u_1, \cdots, u_k) \) and auxiliary functional \( \sum_{i=1}^{k}I_{\epsilon,\Lambda_i}(u_i) \) and it plays an important role in our proof.

**Proposition 2.6.** There exists \( C_3 > 0 \) such that for \( \epsilon \in (0, 1) \) and \( (u_1, \cdots, u_k) \in H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k) \),

\[
|I_{\epsilon}(u_1, \cdots, u_k) - \sum_{i=1}^{k}I_{\epsilon,\Lambda_i}(u_i)| \leq \epsilon^{\alpha-2}C_3 \sum_{i=1}^{k}||u_i||_{\epsilon,\Lambda_i}^2.
\]

(2.14)

To prove Proposition 2.6, we show the following lemma.

**Lemma 2.7.** There exists \( C_4 > 0 \) such that for \( \epsilon \in (0, 1) \) and \( (u_1, \cdots, u_k) \in H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k) \),

\[
||Q_{h_{\epsilon},\epsilon}(u_1, \cdots, u_k)||_{\epsilon,\mathbb{R}^N \setminus \Lambda_*} \leq C_4 ||u||_{\epsilon,\Lambda_* \setminus \Lambda'_*} ,
\]

(2.15)

\[
||Q_{\epsilon}(u_1, \cdots, u_k)||_{\epsilon,\mathbb{R}^N \setminus \Lambda_*} \leq C_4 ||u||_{\epsilon,\Lambda_* \setminus \Lambda'_*} ,
\]

(2.16)

\[
||Q_{\epsilon}(u_1, \cdots, u_k) - Q_{0,\epsilon}(u_1, \cdots, u_k)||_{\epsilon,\mathbb{R}^N \setminus \Lambda} \leq \epsilon^{\alpha} 2C_4 ||u||_{\epsilon,\Lambda_* \setminus \Lambda'_*} .
\]

(2.17)

**Proof.** First of all, we set \( v(x) = Q_{0,\epsilon}(u_1, \cdots, u_k)(x) \) and \( w(x) = Q_{\epsilon}(u_1, \cdots, u_k)(x) \). From Remark 2.4, we remark that \( w \) is a unique minimizer of the minimizing problem \( K_{\epsilon}(w) = \inf_{u \in E_u} K_{\epsilon}(u) \), where \( K_{\epsilon}(u) = \int_{\mathbb{R}^N \setminus \Lambda_*} \frac{1}{2}(\epsilon^2|\nabla u|^2 + V(x)u^2) - F_{\epsilon}(u) \) \( dx \). Thus by Lemma 2.2 and Lemma 2.3, we have

\[
\frac{1}{4}||w||_{\epsilon,\mathbb{R}^N \setminus \Lambda_*}^2 \leq K_{\epsilon}(w) \leq K_{\epsilon}(Pu) \leq ||Pu||_{\epsilon,\mathbb{R}^N \setminus \Lambda_*}^2 \leq C_2^2 ||u||_{\epsilon,\Lambda_* \setminus \Lambda'_*}^2.
\]

(2.18)

Thus (2.16) follows from (2.18). We can show (2.15) in a similar way. Next we show (2.17). By variational characterizations for \( w(x) \) and \( v(x) \) \( w - v \) satisfies

\[
<w - v, \varphi>_{\epsilon,\mathbb{R}^N \setminus \Lambda_*} = \int_{\mathbb{R}^N \setminus \Lambda_*} f_{\epsilon}(w)\varphi \ dx \quad \text{for all} \quad \varphi \in H^1_0(\mathbb{R}^N \setminus \Lambda_*).
\]

(2.19)
Since \( w - v \in H^1_0(\mathbb{R}^N \setminus \Lambda_*) \), setting \( \varphi = w - v \) in (2.19), we have

\[
\|w - v\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 = \int_{\mathbb{R}^N \setminus \Lambda_*} f_{\epsilon}(w)(w - v) \, dx
\]

\[
\leq \epsilon^\alpha \nu_0 \|w\|_{L^2(\mathbb{R}^N \setminus \Lambda_*)} \|w - v\|_{L^2(\mathbb{R}^N \setminus \Lambda_*)}
\]

\[
\leq \epsilon^{\alpha-2} \nu_0 C_1 \|w\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*} \|w - v\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}.
\]

(2.20)

Here we used Proposition 2.1. By (2.20), we get

\[
\|w - v\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*} \leq \epsilon^{\alpha-2} \nu_0 C_1 \|w\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}.
\]

(2.21)

From (2.16) and (2.21), for some \( C_4 > 0 \) which is independent of \( u = (u_1, \cdots, u_k) \) and \( \epsilon \in (0, 1) \), we obtain

\[
\|w - v\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*} \leq \epsilon^{\alpha-2} C_4 \|u\|_{\epsilon, \Lambda_* \setminus \Lambda_*}.
\]

Proof of Proposition 2.6. From (2.12), we have

\[
I_{\epsilon}(u_1, \cdots, u_k) - \sum_{i=1}^{k} I_{\epsilon, \Lambda_i}(u_i)
\]

\[
= \frac{1}{2} (\|Q_{\epsilon}(u_1, \cdots, u_k)\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 - \sum_{i=1}^{k} \|Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0)\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2) - \int_{\mathbb{R}^N \setminus \Lambda_*} F_{\epsilon}(Q_{\epsilon}(u_1, \cdots, u_k)) \, dx
\]

\[
= \frac{1}{2} (I) + (II).
\]

Since \( Q_{0, \epsilon}(u_1, \cdots, u_k) \) is a solution of the linear equation, we have \( Q_{0, \epsilon}(u_1, \cdots, u_k) = \sum_{i=1}^{k} Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0) \). Thus,

\[
(I) = \|Q_{\epsilon}(u_1, \cdots, u_k)\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2 - \|Q_{0, \epsilon}(u_1, \cdots, u_k)\|_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}^2
\]

\[
+ 2 \sum_{1 \leq i < j \leq k} < Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0), Q_{0, \epsilon}(0, \cdots, u_j, \cdots, 0) >_{\epsilon, \mathbb{R}^N \setminus \Lambda_*}.
\]

(2.22)

Now we remark that, for \( i \neq j \), the following estimate holds:

\[
| < Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0), Q_{0, \epsilon}(0, \cdots, u_j, \cdots, 0) >_{\epsilon, \mathbb{R}^N \setminus \Lambda_*} | \leq C e^{-a} \|u_i\|_{\epsilon, \Lambda_i} \|u_j\|_{\epsilon, \Lambda_j},
\]

(2.23)

where constants \( C, a > 0 \) are independent of \( (u_1, \cdots, u_k) \in H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k) \) and \( \epsilon \in (0, 1) \). It is showed in [ST]. The key of the proof is the subsolution estimate in [Si]. (Also see [GT].) From Lemma 2.7, (2.22) and (2.23), it follows that

\[
| (I) | \leq 2 \epsilon^{\alpha-2} C_4^2 \|u\|_{\epsilon, \Lambda_*}^2 + 2 C e^{-a} 2 \sum_{1 \leq i < j \leq k} \|u_i\|_{\epsilon, \Lambda_i} \|u_j\|_{\epsilon, \Lambda_j}
\]

(2.24)
On the other hand, since $F(\xi) \leq \frac{1}{2} \epsilon^\alpha \nu_0 \xi^2$, we have

\begin{align*}
|\langle II \rangle| & \leq \frac{1}{2} \epsilon^\alpha \nu_0 \|Q_\epsilon(u_1, \cdots, u_k)\|^2_{L^2(\mathbb{R}^N \setminus \Lambda_\epsilon)} \\
& \leq \frac{1}{2} \epsilon^\alpha - 2 \nu_0 C_1 \|Q_\epsilon(u_1, \cdots, u_k)\|^2_{\epsilon, \mathbb{R}^N \setminus \Lambda_\epsilon} \\
& \leq \frac{1}{2} \epsilon^\alpha - 2 \nu_0 C_1 C_4^2 \|u\|^2_{\epsilon, \Lambda_\epsilon}.
\end{align*}

(2.25)

Here we use Proposition 2.1 and Lemma 2.7. We get (2.14) from (2.24) and (2.25).

We use the following notation: for $u_i, v_i \in H^1(\Lambda_i)$ ($i = 1, \cdots, k$),

\begin{align*}
<u_i, v_i>_{\epsilon, \Lambda_i} &= <u_i, v_i>_{\epsilon, \Lambda_\epsilon} \\
+ \langle Q_{0, \epsilon}(0, \cdots, u_i, \cdots, 0), Q_{0, \epsilon}(0, \cdots, v_i, \cdots, 0) \rangle_{\epsilon, \mathbb{R}^N \setminus \Lambda_\epsilon} (i = 1, \cdots, k),
\end{align*}

$\|u_i\|^2_{\epsilon, \Lambda_i, \#} = <u_i, u_i>_{\epsilon, \Lambda_i, \#}$ ($i = 1, \cdots, k$).

By Lemma 2.7, we easily get

$$\|u_i\|_{\epsilon, \Lambda_i} \leq \|u_i\|_{\epsilon, \Lambda_i, \#} \leq (1 + C_4) \|u_i\|_{\epsilon, \Lambda_\epsilon} \text{ for all } u_i \in H^1(\Lambda_i).$$

Thus $\| \cdot \|_{\epsilon, \Lambda_i, \#}$ is equivalent to $\| \cdot \|_{\epsilon, \Lambda_i}$ for each $i$. With this notation, $I_{\epsilon, \Lambda_i}(u)$ can be written as

$$I_{\epsilon, \Lambda_i}(u_i) = \frac{1}{2} \|u_i\|_{\epsilon, \Lambda_i, \#}^2 - \frac{1}{p+1} \|u_{i+}\|_{L^{p+1}(\Lambda_i)}^{p+1} : H^1(\Lambda_i) \to \mathbb{R}. \quad (i = 1, \cdots, k)$$

We can easily see that $I_{\epsilon, \Lambda_i}(u_i)$ ($i = 1, \cdots, k$) has a mountain pass geometry and satisfies $(PS)_c$-condition for all $c \in \mathbb{R}$ in a standard way.

(c) Reduction to a problem on $\Sigma_{\epsilon, \Lambda_1} \times \cdots \times \Sigma_{\epsilon, \Lambda_k}$

In this section, we reduce our problem to a variational problem on an infinite dimensional torus $\Sigma_{\epsilon, \Lambda_1} \times \cdots \times \Sigma_{\epsilon, \Lambda_k}$, where

$$\Sigma_{\epsilon, \Lambda_i} = \{v_i \in H^1(\Lambda_i); \|v_i\|^2_{\epsilon, \Lambda_i, \#} = d_{\epsilon, \Lambda_i}\} \quad (i = 1, \cdots, k).$$

(2.26)

Here we define $d_{\epsilon, \Lambda_i}$ by

$$d_{\epsilon, \Lambda_i} = \inf_{v_i \in H^1(\Lambda_i) \setminus \{0\}} \left( \frac{\|v_i\|_{\epsilon, \Lambda_i, \#}^{2(p+1)}}{\|v_{i+}\|_{L^{p+1}(\Lambda_i)}} \right)^{\frac{2(p+1)}{p-1}} \quad (i = 1, \cdots, k).$$

(2.27)

Then $d_{\epsilon, \Lambda_i}$ satisfies the following decay estimates.
Lemma 2.8. (i) If \( \inf_{x \in \Lambda} V(x) = m_i > 0 \), then \( \lim_{\epsilon \to 0} \epsilon^{-N} d_{\epsilon, \Lambda_i} = b_{m_i, \mathbb{R}^N} > 0 \).

(ii) If \( \inf_{x \in \Lambda} V(x) = 0 \), then \( \lim_{\epsilon \to 0} \epsilon^{-N} d_{\epsilon, \Lambda_i} = 0 \) and \( \lim_{\epsilon \to 0} \epsilon \frac{2(p+1)}{p-1} d_{\epsilon, \Lambda_i} \in (0, \infty] \).

Proof. Lemma 2.8 can be shown by a similar way of Proposition 4.2 below.

We consider the auxiliary problems constrained on sphere \( \Sigma_{\epsilon, \Lambda_i} \):

\[
J_{\epsilon, \Lambda_i}(v_i) = \sup_{t > 0} J_{\epsilon, \Lambda_i}(tv_i) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{d_{\epsilon, \Lambda_i}}{||v_i||_{L^{p+1}(\Lambda_i)}^{2}} \right)^{\frac{p+1}{p-1}} : \Sigma_{\epsilon, \Lambda} \to \mathbb{R} \ (i = 1, \ldots, k).
\]

For \( v_i \in \Sigma_{\epsilon, \Lambda_i} \), we can see that \( t \mapsto I_{\epsilon, \Lambda_i}(tv_i) : [0, \infty) \to \mathbb{R} \) takes a global maximum at

\[
t = t_{\epsilon, \Lambda_i}(v_i) = d_{\epsilon, \Lambda_i}^{\frac{1}{p-1}} ||v_i||_{L^{p+1}(\Lambda_i)}^{-\frac{p}{p+1}}.
\] (2.28)

and

\[
J_{\epsilon, \Lambda_i}(v_i) = I_{\epsilon, \Lambda_i}(t_{\epsilon, \Lambda_i}(v_i)v_i) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{d_{\epsilon, \Lambda_i}}{||v_i||_{L^{p+1}(\Lambda_i)}^{2}} \right)^{\frac{p+1}{p-1}}.
\]

We choose \( r > 0 \) such that \( 1 - p(1 + r)^{-(p-1)} < 0 \) and set

\[
N_{\epsilon, \Lambda_i} = \{ v_i \in \Sigma_{\epsilon, \Lambda_i} : ||v_i||_{p+1}^{p+1} \geq (1+r)^{-\frac{p+1}{p-1}} \} \ (i = 1, \ldots, k).
\] (2.29)

\( N_{\epsilon, \Lambda_i} \) is a neighborhood of least energy critical points of \( J_{\epsilon, \Lambda_i}(v_i) \). In fact, we can easily get the following lemma.

Lemma 2.9. For any \( \epsilon \in (0, 1) \),

\[
\left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon, \Lambda_i} = \inf_{v_i \in N_{\epsilon, \Lambda_i}} J_{\epsilon, \Lambda_i}(v_i) \ (i = 1, \ldots, k),
\]

\[
(1+r)\left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon, \Lambda_i} \leq \inf_{v_i \in \partial N_{\epsilon, \Lambda_i}} J_{\epsilon, \Lambda_i}(v_i) \ (i = 1, \ldots, k).
\]

Proof. By a direct computation, we can easily see that

\[
J_{\epsilon, \Lambda_i}(v_i) \leq (1+r)\left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon, \Lambda_i} \text{ if and only if } ||v_i||_{L^{p+1}(\Lambda_i)}^{p+1} \geq (1+r)^{-\frac{p+1}{p-1}} d_{\epsilon, \Lambda_i}, \ (2.30)
\]

Thus Lemma 2.9 follows.

For minimizing sequences of \( J_{\epsilon, \Lambda_i} \), we have the following estimates.
Lemma 2.10. If a sequence $v_{i,\epsilon} \in \Sigma_{\epsilon,\Lambda_{i}}$ ($\epsilon \to 0$) satisfies

$$J_{\epsilon,\Lambda_{i}}(v_{i,\epsilon}) = \left(\frac{1}{2} - \frac{1}{p+1}\right)d_{\epsilon,\Lambda_{i}} + o(\epsilon^{\frac{2(p+1)}{p-1}}),$$

(2.31)

then we have

$$||v_{i,\epsilon+}||_{L^{p+1}(\Lambda_{i})}^{p+1} = d_{\epsilon,\Lambda_{i}} + o(\epsilon^{\frac{2(p+1)}{p-1}}).$$

(2.32)

Proof. From (2.31) and (2.30), we have

$$||v_{i,\epsilon+}||_{L^{p+1}(\Lambda_{i})}^{p+1} = \left(1 + \frac{o(\epsilon^{\frac{2(p+1)}{p-1}})}{d_{\epsilon,\Lambda_{i}}}ight)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_{i}}.$$

Using a Taylor expansion, for small $|r|$, we find $(1 + r)^{-\frac{p-1}{2}} = 1 + O(r)$. Thus we get

$$||v_{i,\epsilon+}||_{L^{p+1}(\Lambda_{i})}^{p+1} = \left(1 + \frac{o(\epsilon^{\frac{2(p+1)}{p-1}})}{d_{\epsilon,\Lambda_{i}}}ight) d_{\epsilon,\Lambda_{i}} + o(\epsilon^{\frac{2(p+1)}{p-1}}).$$

We set a subset $N_{\epsilon}$ as follows:

$$N_{\epsilon} = \{(v_{1},\cdots,v_{k}) \in \Sigma_{\epsilon,\Lambda_{1}} \times \cdots \times \Sigma_{\epsilon,\Lambda_{k}}; ||v_{i+}||_{L^{p+1}(\Lambda_{i})}^{p+1} \geq (1 + r)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_{i}} (i = 1,\cdots,k)\}.$$  

(2.33)

From Lemma 2.9, we can see $N_{\epsilon} \neq \emptyset$. We try to find a critical point of $J_{\epsilon}(v_{1},\cdots,v_{k}) : N_{\epsilon} \to (0,\infty]$ which is defined by

$$J_{\epsilon}(v_{1},\cdots,v_{k}) = \sup_{s_{1},\cdots,s_{k} \geq 0} I_{\epsilon}(s_{1}v_{1},\cdots,s_{k}v_{k}).$$

(2.34)

For simplicity, in what follows, we use notation: $v = (v_{1},\cdots,v_{k}) \in N_{\epsilon}$, $s = (s_{1},\cdots,s_{k}) \in [0,\infty)^{k}$, $sv = (s_{1}v_{1},\cdots,s_{k}v_{k})$.

The following proposition is important.

Proposition 2.11. There exists $\epsilon_{1} \in (0,1)$ such that for all $\epsilon \in (0,\epsilon_{1})$ we have

(i) There exist constants $R_{2} > R_{1} > 0$ independent of $\epsilon$ such that for any $v \in N_{\epsilon}$, $s \mapsto I_{\epsilon}(sv)$ has a unique maximizer $s_{\epsilon}(v) = (s_{1,\epsilon}(v),\cdots,s_{k,\epsilon}(v))$ in $[R_{1},R_{2}]^{k}$.

(ii) $v \mapsto s_{\epsilon}(v) : N_{\epsilon} \to \mathbb{R}^{k}$ is of class of $C^{1}$.

(iii) $J_{\epsilon}(v) : N_{\epsilon} \to \mathbb{R}$ is of class of $C^{1}$.

(iv) $J_{\epsilon}(v) : N_{\epsilon} \to \mathbb{R}$ satisfies (PS)-condition.

(v) If $v \in N_{\epsilon}$ is a critical point of $J_{\epsilon}(v)$, then $s_{\epsilon}(v)v \in H^{1}(\Lambda_{1}) \oplus \cdots \oplus H^{1}(\Lambda_{k})$ is a critical point of $I_{\epsilon}(u_{1},\cdots,u_{k})$. 
(vi) For all \( v = (v_1, \ldots, v_k) \in N_\varepsilon \), we have
\[
J_\varepsilon(v) = \sum_{i=1}^{k} J_{\varepsilon, \Lambda_i}(v_i) + o(\varepsilon^{\frac{2(p+1)}{p-1}}).
\]
where \( o(\varepsilon^{\frac{2(p+1)}{p-1}}) \) is uniformly for \( v \in N_\varepsilon \).

**Proof.** See Proposition 1.13 and Proposition 1.14 of [S].

3. Outline of the proof of Theorem 1.1.

In this section, we will show Theorem 1.1. We define
\[
c_\varepsilon = \inf_{v \in N_\varepsilon} J_\varepsilon(v).
\]
Then we show the following proposition.

**Proposition 3.1.** For small \( \varepsilon \in (0, \varepsilon_1) \), we have
(i) \( c_\varepsilon = \left( \frac{1}{2} - \frac{1}{p+1} \right) \sum_{i=1}^{k} d_{\varepsilon, \Lambda_i} + o(\varepsilon^{\frac{2(p+1)}{p-1}}) \).
(ii) \( \inf_{v \in N_\varepsilon} J_\varepsilon(v) < \inf_{v \in \partial N_\varepsilon} J_\varepsilon(v) \).
(iii) \( c_\varepsilon \) is a critical value of \( J_\varepsilon(v) \). Moreover corresponding critical points lie in \( N_\varepsilon \).

**Proof.** From (2.35), we recall
\[
J_\varepsilon(v) = \sum_{i=1}^{k} J_{\varepsilon, \Lambda_i}(v_i) + o(\varepsilon^{\frac{2(p+1)}{p-1}})
\]
for all \( v \in N_\varepsilon \), where \( o(\varepsilon^{\frac{2(p+1)}{p-1}}) \) is uniformly for \( v \in N_\varepsilon \). Thus from Lemma 2.9, it follows
\[
c_\varepsilon = \inf_{v \in N_\varepsilon} J_\varepsilon(v) = \sum_{i=1}^{k} \inf_{v_i \in N_{\varepsilon, \Lambda_i}} J_{\varepsilon, \Lambda_i}(v_i) + o(\varepsilon^{\frac{2(p+1)}{p-1}})
\]
\[
= \left( \frac{1}{2} - \frac{1}{p+1} \right) \sum_{i=1}^{k} d_{\varepsilon, \Lambda_i} + o(\varepsilon^{\frac{2(p+1)}{p-1}}).
\]
Next we show (ii). We note \( \partial N_\varepsilon = \bigcup_{j=1}^{k} (N_{\varepsilon, \Lambda_1} \times \cdots \times \partial N_{\varepsilon, \Lambda_j} \times \cdots \times N_{\varepsilon, \Lambda_k}) \). Again, from Lemma 2.9, we get for each \( j \)
\[
\inf_{v \in N_{\varepsilon, \Lambda_1} \times \cdots \times \partial N_{\varepsilon, \Lambda_j} \times \cdots \times N_{\varepsilon, \Lambda_k}} J_\varepsilon(v) = \sum_{i \neq j} \inf_{v_i \in N_{\varepsilon, \Lambda_i}} J_{\varepsilon, \Lambda_i}(v_i) + \inf_{v_j \in \partial N_{\varepsilon, \Lambda_j}} J_{\varepsilon, \Lambda_j}(v_j) + o(\varepsilon^{\frac{2(p+1)}{p-1}})
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{p+1} \right) \sum_{i \neq j} d_{\varepsilon, \Lambda_i} + (1+r) \left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\varepsilon, \Lambda_j} + o(\varepsilon^{\frac{2(p+1)}{p-1}})
\]
\[
= c_\varepsilon + r \left( \frac{1}{2} - \frac{1}{p+1} \right) d_{\varepsilon, \Lambda_j} + o(\varepsilon^{\frac{2(p+1)}{p-1}}).
\]
(ii) follows from (3.2)-(3.3). Since \( J_\varepsilon(v) \) satisfies (PS)-condition, we can see \( c_\varepsilon \) is a critical value of \( J_\varepsilon(v) \) in a standard way. 

Corollary 3.2. Minimzer $v_\epsilon = (v_{1,\epsilon}, \cdots, v_{k,\epsilon})$ of (3.1) satisfies

$$J_{\epsilon,\Lambda_i}(v_{i,\epsilon}) = \left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad (i = 1, \cdots, k).$$

Proof. From (2.35), we have

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon,\Lambda_i} \leq J_{\epsilon,\Lambda_i}(v_{i,\epsilon}) = J_\epsilon(v_\epsilon) - \sum_{\ell \neq i} J_{\epsilon,\Lambda_\ell}(v_{\ell,\epsilon}) + o(\epsilon^{\frac{2(p+1)}{p-1}})$$

$$\leq c_\epsilon - \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{\ell \neq i} d_{\epsilon,\Lambda_\ell} + o(\epsilon^{\frac{2(p+1)}{p-1}})$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}).$$

Thus we get Corollary 3.2.

From the a minimizer of (3.1), we get a critical point of $\Phi_\epsilon(u)$ by the following:

Proposition 3.3. Let $v_\epsilon = (v_{1,\epsilon}, \cdots, v_{k,\epsilon})$ be a minimizer of (3.1). Then

$$u_\epsilon(x) = \begin{cases} s_{i,\epsilon}(v_\epsilon) v_{i,\epsilon}(x) & \text{for } x \in \Lambda_i \quad (i = 1, \cdots, k), \\ Q_\epsilon(s_{1,\epsilon}(v_\epsilon) v_{1,\epsilon}, \cdots, s_{k,\epsilon}(v_\epsilon) v_{k,\epsilon})(x) & \text{for } x \in \mathbb{R}^N \setminus \Lambda_* \end{cases} \quad (3.4)$$

is a critical point of $\Phi_\epsilon(u)$. Here $(s_{1,\epsilon}(v), \cdots, s_{k,\epsilon}(v))$ are given in Proposition 2.11.

Proof. It follows from Proposition 2.5, Proposition 2.11 and Proposition 3.1.

Outline of the proof of Theorem 1.1. We can show that $u_\epsilon(x)$ defined by (3.4) has an exponential decay on $\mathbb{R}^N \setminus \Lambda_*$ as $\epsilon \to 0$. (See [S].) Thus $u_\epsilon(x)$ satisfies (2.7) and $u_\epsilon(x)$ is a critical point of original functional $\Psi_\epsilon(u)$. Moreover, we have the following detailed estimate of $u_\epsilon(x)$ (See [S]):

$$||u_{\epsilon+}||^{p+1}_{L^{p+1}(\Lambda_i)} = c_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}), \quad (3.5)$$

$$||u_\epsilon||^2_{L^2(\Lambda_i)} = c_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}). \quad (3.6)$$

Thus (1.4)–(1.5) follow from (3.5)–(3.6). We get (1.6) by a standard way. The proof of Theorem 1.1 is completed.

4. Asymptotic profile of solutions $u_\epsilon(x)$.

In this section, we will prove Theorem 1.5. By a similar way, we can prove Theorem 1.3. (See Section 3 of [S].) First of all, we note the following.
Remark 4.1. When assumption \((L^*)\) holds, there exists a constant \(C_5 > 0\) independent of \(\epsilon \in (0, 1)\) such that
\[
||w||_{H^1(\Lambda_{g(\epsilon)})} \leq C_5 ||w||_{V_\epsilon, \Lambda_{g(\epsilon)}} \quad \text{for all } w \in H^1(\Lambda_{g(\epsilon)}).
\] (4.1)

In fact we can see that, for some \(\ell > 0\) and \(\delta > 0\) independent of \(\epsilon \in (0, 1)\), \(V_\epsilon(y)\) satisfies
\[
V_\epsilon(y) \geq \begin{cases} 
0 & \text{for } y \in (-\ell, \ell)^k \times \mathbb{R}^{N-k} \subset \mathbb{R}^N, \\
\delta & \text{elsewhere.}
\end{cases}
\]

Thus (4.1) can be shown as Proposition 2.1. (See Lemma 1.2 of [S].)

In the following arguments, \(d_{\epsilon, \Lambda}\) defined in (2.27) and \(b_{V_0, \Omega_0}\) defined in (1.9) will play important roles.

Proposition 4.2. \(b_{V_0, \Omega_0}\) is achieved by some \(w(x) \in H^1_0(\Omega_0)\) and
\[
\left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} d_{\epsilon, \Lambda} \rightarrow b_{V_0, \Omega_0} \quad \text{as } \epsilon \rightarrow 0.
\] (4.2)

Proof. From Lemma 2.11 of [S], it suffice to show
\[
\left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} \rightarrow b_{V_0, \Omega_0} \quad \text{as } \epsilon \rightarrow 0.
\] (4.3)

Firstly we show
\[
\left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} \leq b_{V_0, \Omega_0} + o(1).
\] (4.4)

Suppose that \(w(x) \in H^1_0(\Omega)\) achieves \(b_{V_0, \Omega}\). We choose a function \(\varphi \in C^1_0(\mathbb{R}^N, [0, 1])\) such that
\[
\varphi(x) = \begin{cases} 
1 & \text{for } x \in \Lambda' \\
0 & \text{for } x \in \mathbb{R}^N \setminus \Lambda
\end{cases}
\]
and set
\[
v_\epsilon(x) = \varphi(x) \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} w \left( \frac{x}{g(\epsilon)} \right), \quad \psi_\epsilon(x) = \varphi(g(\epsilon)x).
\]

Then by direct computations, we have
\[
||v_\epsilon||^2_{z, \Lambda} = \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N \left[ \int_{\Lambda_{g(\epsilon)}} \psi_\epsilon(x)^2 \{ |\nabla w(x)|^2 + V_\epsilon(x)w(x)^2 \} dx \\
+ \int_{\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)}} |\nabla \psi_\epsilon(x)|^2 w(x)^2 + 2 \nabla \psi_\epsilon(x) \cdot \nabla w(x) \psi_\epsilon(x) w(x) dx \right]
\]
\[
= \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N ((I) + (II)).
\]
Here we have

$$ |(II)| \leq C^2 \|w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)})}^2 + 2C \|\nabla w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)})} \|w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)})} \to 0$$

as $\epsilon \to 0$. Thus, from (1.13), we find

$$ \|v_\epsilon\|_{\epsilon,\Lambda}^2 = \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N (\|w\|_{V_{0,\Omega_0}}^2 + o(1)). \quad (4.5) $$

By a similar way, we get

$$ \|v_{\epsilon+}\|_{L^{p+1}(\Lambda)}^{p+1} = \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N (\|w_{+}\|_{L^{p+1}(\Omega_0)}^{p+1} + o(1)). \quad (4.6) $$

Thus from definition of $c_{\epsilon,\Lambda}$ and (4.5)-(4.6),

$$ \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} \leq \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{\|v_\epsilon\|_{\epsilon,\Lambda}}{\|v_{\epsilon+}\|_{L^{p+1}(\Lambda)}} \right)^{\frac{2(p+1)}{p-1}}$$

$$ = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{\|w\|_{V_{0,\Omega_0}} + o(1)}{\|w_{+}\|_{L^{p+1}(\Omega_0)} + o(1)} \right)^{\frac{2(p+1)}{p-1}}$$

$$ = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{\|w\|_{V_{0,\Omega_0}}}{\|w_{+}\|_{L^{p+1}(\Omega_0)}} \right)^{\frac{2(p+1)}{p-1}} + o(1) \quad (4.7) $$

Next, we show

$$ b_{V_{0,\Omega_0}} \leq \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} + o(1). \quad (4.8) $$

Let $v_\epsilon(x) \in H_0^1(\Lambda)$ attains $c_{\epsilon,\Lambda}$. We may assume $\|v_\epsilon\|_{\epsilon,\mathbb{R}^N}^2 = \left( \frac{\epsilon}{g(\epsilon)} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N$. Then, from (4.4), we see that

$$ \lim_{\epsilon \to 0} \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \|v_{\epsilon+}\|_{L^{p+1}(\Lambda)} > 0 \quad (4.9) $$

We set

$$ v_\epsilon(x) = \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2}{p-1}} v_\epsilon(g(\epsilon)x). $$

Then $\|v_\epsilon\|_{V_{0,\mathbb{R}^N}} = 1$ and $\lim_{\epsilon \to 0} \|v_\epsilon\|_{L^{p+1}(\mathbb{R}^N)} > 0$. We need to divide in two cases:

$\Omega_0 = \mathbb{R}^N$ or $\Omega_0 \neq \mathbb{R}^N$. Firstly, we consider the case $\Omega = \mathbb{R}^N$. By direct computations, we have

$$ \|w_\epsilon\|_{V_{0,\mathbb{R}^N}}^2 = \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \epsilon^2 \|\nabla v_\epsilon\|^2 + \left( \frac{\epsilon}{g(\epsilon)} \right)^2 V_0 \left( \frac{x}{g(\epsilon)} \right) v_\epsilon^2(x) dx. $$
Thus, from (1.15), for any $\delta \in (0,1)$, we obtain
\[ ||w_{\epsilon}||^2_{(1-\delta)V_0,\mathbb{R}^N} \leq \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} ||v_{\epsilon}||^2_{\epsilon,\mathbb{R}^N} + o(1). \] (4.10)

On the other hand, we have
\[ ||w_{\epsilon}||^{p+1}_{L^{p+1}(\mathbb{R}^N)} = \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} ||v_{\epsilon}||^{p+1}_{L^{p+1}(\Lambda)}. \] (4.11)

From (4.10)–(4.11) we get
\[ b_{(1-\delta)V_0,\mathbb{R}^N} \leq \left( \frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} + o(1). \] (4.8)

(4.8) follows from the fact $b_{(1-\delta)V_0,\mathbb{R}^N} \to b_{V_0,\mathbb{R}^N}$ as $\delta \to 0$. Next, we consider the case $\Omega_0 \neq \mathbb{R}^N$. Since $||w_{\epsilon}||_{V_\epsilon,\mathbb{R}^N} = 1$, we can choose a subsequence $\epsilon_n \to 0$ and $w_0(x) \in H^1(\mathbb{R}^N)$ such that
\[ w_{\epsilon_n}(x) \to w_0(x) \text{ weakly in } H^1(\mathbb{R}^N) \text{ and strongly in } L^{p+1}_{\text{loc}}(\mathbb{R}^N). \] (4.12)

From (1.18) and (4.12), we see $w_{\epsilon_n}(x) \to w_0(x)$ strongly in $L^{p+1}(\mathbb{R}^N)$. Moreover, from (1.14), for any closed set $E \supset \supset \Omega_0$ we find
\[ ||w_{\epsilon_n}||^2_{L^2(\Lambda_{g(\epsilon_n)} \setminus E)} \leq \frac{1}{\inf_{x \in \Lambda_{g(\epsilon_n)} \setminus E} V_\epsilon(x)} \int_{\Lambda_{g(\epsilon_n)} \setminus E} V_\epsilon(x) w_{\epsilon_n}(x)^2 \, dx \to 0. \]

In particular, we find $w_0 \in H^1_0(\Omega_0)$ and $w_{0+} \neq 0$. Thus we have
\[ b_{V_0,\Omega_0} \leq \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{||w_0||_{V_0,\Omega_0}}{||w_{0+}||_{L^{p+1}(\Omega_0)}} \right)^{\frac{2(p+1)}{p-1}} \]
\[ \leq \liminf_{\epsilon_n \to 0} \frac{1}{2} - \frac{1}{p+1} \left( \frac{||w_{\epsilon_n}||_{\epsilon_n,\Lambda_{g(\epsilon_n)}}}{||w_{\epsilon_n}||_{L^{p+1}(\Lambda_{g(\epsilon_n)})}} \right)^{\frac{2(p+1)}{p-1}} \]
\[ = \liminf_{\epsilon_n \to 0} \left( \frac{g(\epsilon_n)}{\epsilon_n} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon_n)^{-N} \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{||v_{\epsilon_n}||_{\epsilon_n,\Lambda}}{||v_{\epsilon_n}||_{L^{p+1}(\Lambda)}} \right)^{\frac{2(p+1)}{p-1}} \]
\[ = \liminf_{\epsilon_n \to 0} \left( \frac{g(\epsilon_n)}{\epsilon_n} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon_n)^{-N} c_{\epsilon_n,\Lambda}. \] (4.13)

Since (4.13) is not depend on subsequence, (4.8) holds. From (4.4) and (4.8), we get (4.4) and complete the proof of Proposition 4.2. \[ \blacksquare \]
To prove Theorem 1.5, we define a rescaled functional

$$\Phi_{\Lambda_{g(\epsilon)}}(w) = \frac{1}{2} ||w||_{V_{\epsilon}, \Lambda_{g(\epsilon)}}^{2} - \frac{1}{p+1} ||w_{+}||_{L^{p+1}(\Lambda_{g(\epsilon)})}^{p+1} : H^{1}(\Lambda_{g(\epsilon)}) \rightarrow \mathbb{R}.$$ 

We also define a functional corresponding to the limit problem \((L)_{V_{0}, \Omega_{0}}\) by

$$I_{V_{0}, \Omega_{0}}(w) = \frac{1}{2} ||w||_{V_{0}, \Omega_{0}}^{2} - \frac{1}{p+1} ||w_{+}||_{L^{p+1}(\Omega_{0})}^{p+1} : E_{V_{0}, \Omega_{0}} \cap H_{0}^{1}(\Omega_{0}) \rightarrow \mathbb{R}.$$ 

Now we have the following proposition.

**Proposition 4.3.** Suppose \((w_{\epsilon}(y)) \subset H^{1}(\Lambda_{g(\epsilon)})\) satisfies

\[
||w_{\epsilon}||_{L^{p+1}(\Lambda_{g(\epsilon)})}^{p+1} = b_{V_{0}, \Omega_{0}} + o(1),
\]

\[
||w_{\epsilon}||_{V_{\epsilon}, \Lambda_{g(\epsilon)}}^{2} \leq b_{V_{0}, \Omega_{0}} + o(1),
\]

\[
\Phi_{\Lambda_{g(\epsilon)}}'(w_{\epsilon}) \varphi = 0 \quad \text{for all } \varphi \in H_{0}^{1}(\Lambda_{g(\epsilon)}).
\]

If \((L^{*})\) holds, then after extracting a subsequence \(\epsilon_{n} \rightarrow 0\), there exist \(x_{n} \in \Lambda_{\epsilon_{n}}\) and a least energy solution \(w_{0} \in H_{0}^{1}(\Omega_{0})\) of \((L)_{V_{0}, \Omega_{0}}\) such that

\[
\epsilon_{n} x_{n} \rightarrow 0,
\]

\[
||w_{\epsilon_{n}} - w_{0}(\cdot - x_{n})||_{H^{1}(\Lambda_{\epsilon_{n}})} \rightarrow 0 \quad \text{as } \epsilon_{n} \rightarrow 0,
\]

\[
I_{V_{0}, \Omega_{0}}(w_{0}) = \left(\frac{1}{2} - \frac{1}{p+1}\right) b_{V_{0}, \Omega_{0}}, \quad I_{V_{0}, \Omega_{0}}'(w_{0}) = 0.
\]

To prove Proposition 4.3, we will use the following lemma.

**Lemma 4.4.** Suppose \(w \in H_{0}^{1}(\Omega_{0}) \setminus \{0\}\) satisfies \(I'_{V_{0}, \Omega_{0}}(w)w \leq 0\). Then

\[
||w||_{V_{0}, \Omega_{0}}^{2} \geq b_{V_{0}, \Omega_{0}}.
\]

**Proof.** \(I'_{V_{0}, \Omega_{0}}(w)w \leq 0\) implies \(||w||_{V_{0}, \Omega_{0}}^{2} \leq ||w_{+}||_{L^{p+1}(\Omega_{0})}^{p+1}\). Thus, from the definition of \(b_{V_{0}, \Omega_{0}}\), we have

\[
b_{V_{0}, \Omega_{0}} \leq \left(\frac{||w||_{V_{0}, \Omega_{0}}}{||w_{+}||_{L^{p+1}(\Omega_{0})}}\right)^{2(p+1)} \leq ||w||_{V_{0}, \Omega_{0}}^{2}.
\]

**Proof of Proposition 4.3.** For the case \(\Omega_{0} = \mathbb{R}^{N}\), we show the proposition. For the case \(\Omega_{0} \neq \mathbb{R}^{N}\), we can show it by a similar way. We use concentration compactness argument. From (4.14)–(4.16) and (1.15)–(1.17), we can easily see that there exists \(x_{\epsilon} = (0, \cdots, 0, x_{\epsilon,k}, \cdots, x_{\epsilon,N}) \in \mathbb{R}^{N}\) such that for large \(R > 0\), \(\liminf_{\epsilon \rightarrow 0} ||w_{\epsilon}||_{L^{p+1}(B_{R}(x_{\epsilon}))} > 0\).
From (4.15)-(4.16), there exist a subsequence $\epsilon_n \to 0$ and $w_0 \in H^1(\mathbb{R}^N)$ such that for any bounded set $D \subset \mathbb{R}^N$,

$$v_{\epsilon_n}(x) = w_{\epsilon_n}(x + x_{\epsilon_n}) \to w_0(x) \text{ weakly in } H^1(D) \text{ and strongly in } L^{p+1}(D). \quad (4.17)$$

For any $\varphi \in C_0^1(\Omega_0)$ we have

$$\int_{\Lambda_{g(\epsilon_n)} + x_{\epsilon_n}} \nabla v_{\epsilon_n} \cdot \nabla \varphi + V_{\epsilon_n}(x + x_{\epsilon_n})v_{\epsilon_n}\varphi \, dx - \int_{\Lambda_{g(\epsilon_n)} + x_{\epsilon_n}} v_{\epsilon_n+}^{p}\varphi \, dx = 0.$$

Since $\text{supp} \varphi$ is compact, from (1.13), (1.15)-(1.17), $w_0(x)$ satisfies

$$\int_{\mathbb{R}^N} \nabla w_0 \cdot \nabla \varphi + V_0(x)w_0\varphi \, dx - \int_{\mathbb{R}^N} w_0+^{p}\varphi \, dx \leq 0,$$

that is, $I'_{V_0,\Omega_0}(w_0)w_0 \leq 0$. From Lemma 4.4 and (4.15), it follows that

$$b_{V_0,\mathbb{R}^N} \leq ||w_0||_{V_0,\mathbb{R}^N}^2 \leq \lim_{\epsilon_n \to 0} \inf_{\epsilon_n} ||v_{\epsilon_n}||_{V_{\epsilon_n},\Lambda_{g(\epsilon_n)} + x_{\epsilon_n}}^2 \leq b_{V_0,\mathbb{R}^N}. \quad (4.18)$$

By (4.17) and (4.18), we can see

$$\lim_{\epsilon_n \to 0} ||v_{\epsilon_n} - w_0||_{V_{\epsilon_n},\Lambda_{g(\epsilon_n)} + x_{\epsilon_n}}^2 = 0.$$

From (4.1), we also obtain $\lim_{\epsilon_n \to 0} ||v_{\epsilon_n} - w_0||_{H^1(\Lambda_{g(\epsilon_n)} + x_{\epsilon_n})} = 0$. Therefore we have $I'_{V_0,\Omega_0}(w_0) = 0$ and

$$\tilde{\Phi}_{\Lambda_{g(\epsilon_n)}}(w_{\epsilon_n}) \to I_{V_0,\Omega_0}(w_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right)b_{V_0,\Omega_0}.$$

We complete the proof of Proposition 4.3.

**End of proof of Theorem 1.5.** (i) of Theorem 1.5 follows from (4.3). Let $u_\epsilon(x)$ be a critical point of $\Psi_\epsilon(u)$ obtained in Theorem 1.1. We set $w_\epsilon(x) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{rac{1}{p-1}} u_\epsilon(g(\epsilon)x)$. Then from Proposition 4.2 and (3.5)-(3.6), $w_\epsilon|_{\Lambda_{g(\epsilon)}}(x)$ satisfies (4.14)-(4.16). This implies (ii) of Theorem 1.5. Thus we complete the proof of Theorem 1.5.

**References**


