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Multi-peak positive solutions for nonlinear Schrödinger equations with critical frequency

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0. Introduction

In this report, we consider the following nonlinear Schrödinger equations:

$$\begin{aligned} -\epsilon^2\Delta u+V(x)u&=|u|^{p-1}u \quad \text{in } \mathbf{R}^N,\\ u&>0 \quad \text{in } \mathbf{R}^N, \\ u&\in H^1(\mathbf{R}^N), \end{aligned} \tag{$*$}_\epsilon$$

where $\epsilon>0$ is small parameter and p satisfies $1<p<\infty$ $(N=1,2)$, $1<p<\frac{N+2}{N-2}$ $(N\geq 3)$. We are interested in the existence of solutions of $(*)_\epsilon$ for small $\epsilon>0$ and their behavior as $\epsilon\to 0$. We assume the potential $V(x)$ satisfies the following assumptions:

(V.1) $V(x)\in C^1(\mathbf{R}^N,\mathbf{R})$ and $V(x)\geq 0$ for all $x\in\mathbf{R}^N$.

(V.2) $0<\liminf_{|x|\to\infty}V(x)\leq\sup_{x\in\mathbf{R}^N}V(x)<\infty$.

Under the above assumptions, the solutions of $(*)_\epsilon$ are characterized as critical points of

$$\Psi_\epsilon(u)=\int_{\mathbf{R}^N}\frac{1}{2}(\epsilon^2|\nabla u|^2+V(x)u^2)-\frac{1}{p+1}u_+^{p+1}dx\in C^2(H^1(\mathbf{R}^N),\mathbf{R}).$$

When $V(x)$ satisfies

$$\inf_{x\in\mathbf{R}^N}V(x)>0, \tag{0.1}$$

$(*)_\epsilon$ has a family of single-peak solutions $u_\epsilon(x)$ concentrating around a local minimum of $V(x)$ for small $\epsilon>0$ in the following sense: let x_ϵ be a unique maximum of $u_\epsilon(x)$. Then x_ϵ approaches to a local minimum x_0 of $V(x)$ as $\epsilon\to 0$ and

$$w_\epsilon(y)=u_\epsilon(\epsilon y+x_\epsilon) \tag{0.2}$$

converges to a least energy solution $w_0(x)$ of the following "limit equation"

$$-\Delta w+V(x_0)w=w^p,\quad w>0 \quad \text{in } \mathbf{R}^N,\quad w\in H^1(\mathbf{R}^N). \tag{0.3}$$

In particular, $u_\epsilon(x)$ satisfies

$$\lim_{\epsilon\to 0} ||u_\epsilon||_{L^\infty(\mathbf{R}^N)} = |w_0(0)| > 0, \tag{0.4}$$

$$\lim_{\epsilon\to 0} \epsilon^{-N}\Psi_\epsilon(u_\epsilon) = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) b_{V(x_0),\mathbf{R}^N} > 0. \tag{0.5}$$

Here $b_{V(x_0),\mathbf{R}^N}$ is defined by

$$b_{V(x_0),\mathbf{R}^N} = \inf_{u\in H^1(\mathbf{R}^N)\setminus\{0\}} \left(\frac{\int_{\mathbf{R}^N} |\nabla u|^2 + V(x_0)u^2\, dx}{||u_+||^2_{L^{p+1}(\mathbf{R}^N)}} \right)^{\frac{p+1}{p-1}}$$

and $\left(\frac{1}{2} - \frac{1}{p+1}\right) b_{V(x_0),\mathbf{R}^N}$ is the lowest non-trivial critical value of the functional corresponding to (0.3). See [**FW**, **O1**, **O2**, **DF**] for related results.

Recently, Byeon and Wang [**BW1**, **BW2**] have started to study the case

$$\inf_{x\in\mathbf{R}^N} V(x) = 0. \tag{0.6}$$

They showed that $(*)_\epsilon$ also has single-peak solutions concentrating around an isolated component A of $\{x \in \mathbf{R}^N;\ V(x) = 0\}$. The features of their solutions are completely different from the solutions under the condition (0.1), which depend on the behavior of $V(x)$ around A. More precisely, their solutions $u_\epsilon(x)$ satisfy

$$\lim_{\epsilon\to 0} ||u_\epsilon||_{L^\infty(\mathbf{R}^N)} = 0,$$

$$\lim_{\epsilon\to 0} \epsilon^{-N}\Psi_\epsilon(u_\epsilon) = 0.$$

These are in contrast with (0.4)–(0.5). We can also see $\liminf_{\epsilon\to 0} \epsilon^{-\frac{2}{p-1}} ||u_\epsilon||_{L^\infty(\mathbf{R}^N)} \in (0,\infty]$ and $\liminf_{\epsilon\to 0} \epsilon^{-\frac{2(p+1)}{p-1}} \Psi_\epsilon(u_\epsilon) \in (0,\infty]$. (We remark that $p < \frac{N+2}{N-2}$ implies $\frac{2(p+1)}{p-1} > N$.) Under additional conditions on the behavior of $V(x)$ near A, we can introduce a rescaled function — which is different from (0.2) — to observe the behavior of $u_\epsilon(x)$ around A. More precisely, under one of the conditions (L.2)–(L.4) (see p.5 below), we define $g(\epsilon) > 0$ as in the table 0.1 below and set

$$w_\epsilon(y) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2}{p-1}} u_\epsilon(g(\epsilon)y + x_\epsilon)$$

for a suitable point x_ϵ around A, then $w_\epsilon(y)$ approaches to the least energy solution of

$$-\Delta w + V_0(y)w = w^p, \quad w > 0 \quad \text{in } \Omega_0, \qquad w \in H^1_0(\Omega_0). \tag*{$(L)_{V_0,\Omega_0}$}$$

Here $\Omega_0 \subset \mathbf{R}^N$ and $V_0(x) \in C(\Omega_0, \mathbf{R})$ are also given in the table 0.1 below. In particular, $u_\epsilon(x)$ satisfies

$$\lim_{\epsilon \to 0} \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \Psi_\epsilon(u_\epsilon) \in (0, \infty)$$

and the decaying rate of $\Psi_\epsilon(u_\epsilon)$ depends on the behavior of $V(x)$ around A.

In this report, we study the multi-peak solutions of $(*)_\epsilon$ combining various types of peaks. We assume that $\Lambda_1, \cdots, \Lambda_k \subset \mathbf{R}^N$ satisfy

(V.3) $\Lambda_1, \cdots, \Lambda_k \subset \mathbf{R}^N$ are bounded open sets satisfying $\Lambda_i \cap \Lambda_j = \emptyset$ $(i \neq j)$ and

$$m_i = \inf_{x \in \Lambda_i} V(x) < \inf_{x \in \partial \Lambda_i} V(x) < \infty \quad (i = 1, \cdots, k).$$

Here we remark that m_i may be 0. In what follows, we write

$$A_i = \{x \in \Lambda_i;\, V(x) = m_i\} \quad (i = 1, \cdots, k).$$

Under the assumption (V.3), we will show that $(*)_\epsilon$ has k-peak solutions which have 1-peak in each Λ_i and it concentrates around a local minimum of $V(x)$ in Λ_i. We remark that the behavior of the desired solution $u_\epsilon(x)$ in Λ_i depends on the behavior of $V(x)$ around A_i and just single scaling (0.2) is not enough to describe the behavior of $u_\epsilon(x)$. Moreover the localized critical level $\Psi_{\epsilon,\Lambda_i}(u_\epsilon)$ — which is defined as (1.1) below — approaches to 0 with different rate with respect to ϵ. This makes our problems difficult.

It seems that the existence of multi-peak solutions joining solutions which have different scales is not well studied. In our knowledge, there is only one work by Byeon and Oshita [**BO**] in which they constructed multi-peak solutions by a Lyapunov-Schmidt reduction method under the assumptions of non-degeneracy of solutions of limit problems.

Recently, in [**S**], we constructed the multi-peak solutions under the assumptions (V.1)–(V.3). We used a variational approach and it does not need the assumptions of non-degeneracy of solutions of limit problems. In the following sections, we will introduce main results and an outline of the proof in [**S**]. We also study about the asymptotic behavior of solutions and consider other examples than [**S**].

1. Main Results

Firstly, we introduce main theorems in [**S**].

(a) Existence of multi-peak solutions

In this section, we consider the existence of multi-peak solutions of $(*)_\epsilon$. To state theorems, we set

$$\Psi_{\epsilon,\Lambda_i}(u) = \int_{\Lambda_i} \frac{1}{2}(\epsilon^2 |\nabla u|^2 + V(x)u^2) - \frac{1}{p+1} u_+^{p+1} dx. \tag{1.1}$$

and define

$$c_{\epsilon,\Lambda_i} = \inf_{u\in H_0^1(\Lambda_i)\setminus\{0\}} \left(\frac{\int_{\Lambda_i} \epsilon^2|\nabla u|^2 + V(x)u^2\, dx}{||u_+||^2_{L^{p+1}(\Lambda_i)}} \right)^{\frac{p+1}{p-1}}. \tag{1.2}$$

Under the assumptions (V.1)–(V.3), we can observe

$$\lim_{\epsilon\to 0} \epsilon^{-N} c_{\epsilon,\Lambda_i} < \infty, \qquad \liminf_{\epsilon\to 0} \epsilon^{-\frac{2(p+1)}{p-1}} c_{\epsilon,\Lambda_i} \in (0,\infty].$$

We remark that $\left(\frac{1}{2}-\frac{1}{p+1}\right) c_{\epsilon,\Lambda_i}$ is the lowest non-trivial critical value of the functional which corresponds to the equation

$$-\epsilon^2\Delta u + V(x)u = u^p, \quad u>0 \quad \text{in } \Lambda_i, \qquad u\in H_0^1(\Lambda_i). \tag{1.3}$$

Our first theorem is

Theorem 1.1 ([S]). *Assume that (V.1)–(V.2) hold and $\Lambda_1,\cdots,\Lambda_k \subset \mathbf{R}^N$ satisfy (V.3). Then there exists $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$, there exists a positive solution $u_\epsilon(x)$ of $(*)_\epsilon$ which satisfies*

$$\Psi_{\epsilon,\Lambda_i}(u_\epsilon) = \left(\tfrac{1}{2}-\tfrac{1}{p+1}\right)c_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad (i=1,\cdots,k), \tag{1.4}$$

$$\Psi_{\epsilon,\mathbf{R}^N\setminus(\cup_{i=1}^k\Lambda_i)}(u_\epsilon) = o(\epsilon^{\frac{2(p+1)}{p-1}}). \tag{1.5}$$

Moreover for some constants $C,c>0$, $u_\epsilon(x)$ satisfies

$$u_\epsilon(x) \le C\exp\left(-\frac{c\ dist(x,\cup_{i=1}^k A_i)}{\epsilon}\right) \qquad \text{for } x\in\mathbf{R}^N\setminus\left(\cup_{i=1}^k\Lambda_i\right). \tag{1.6}$$

We remark that under the assumptions of Theorem 1.1, for any non-empty subset $\{i_1,\cdots,i_\ell\}\subset\{1,\cdots,k\}$, $\Lambda_{i_1},\cdots,\Lambda_{i_\ell}$ also satisfy (V.3). Thus we have

Corollary 1.2 ([S]). *There exists $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$, there exists a positive solution $u_\epsilon(x)$ of $(*)_\epsilon$ such that*

$$\Psi_{\epsilon,\Lambda_{i_j}}(u_\epsilon) = \left(\tfrac{1}{2}-\tfrac{1}{p+1}\right)c_{\epsilon,\Lambda_{i_j}} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad (j=1,\cdots,\ell), \tag{1.7}$$

$$\Psi_{\epsilon,\mathbf{R}^N\setminus(\cup_{j=1}^\ell\Lambda_{i_j})}(u_\epsilon) = o(\epsilon^{\frac{2(p+1)}{p-1}}). \tag{1.8}$$

Moreover for some constants $C,c>0$, $u_\epsilon(x)$ satisfies

$$u_\epsilon(x) \le C\exp\left(-\frac{c\ dist(x,\cup_{j=1}^\ell A_{i_j})}{\epsilon}\right) \qquad \text{for } x\in\mathbf{R}^N\setminus\left(\cup_{j=1}^\ell\Lambda_{i_j}\right).$$

Especially, for small $\epsilon > 0$, $(*)_\epsilon$ has at least $2^k - 1$ positive solutions.

(b) Asymptotic behavior of solutions

In this section, we consider the asymptotic behavior of solutions of $(*)_\epsilon$ obtained in Theorem 1.1 and Corollary 1.2. For some $i \in \{1, \cdots, k\}$, we assume that Λ_i satisfies one of the following assumptions:

(L.1) $\inf_{x \in \Lambda_i} V(x) = m_i > 0$

(L.2) $V(x)$ has a unique local minimum x_i in Λ_i and $V(x)$ is represented as

$$V(x) = P(x - x_i) + Q(x - x_i).$$

Here $P(x)$ is a m-homogeneous positive function for some $m > 0$, that is,

$$P(x) > 0 \quad \text{for all } x \in \mathbf{R}^N \backslash \{0\},$$
$$P(tx) = t^m P(x) \quad \text{for all } (t, x) \in [0, \infty) \times \mathbf{R}^N,$$

and $\lim_{|x| \to 0} |x|^{-m} Q(x) = 0$.

(L.3) $V(x)$ has a unique local minimum x_i in Λ_i and $V(x)$ is represented as

$$V(x) = \exp\{-r(x - x_i)^\ell - Q(x - x_i)\}.$$

Here $\ell > 0$ and $r(x) : \mathbf{R}^N \backslash \{0\} \to (0, \infty)$ is a positive continuous function such that

$$r(x) > 0 \quad \text{for all } x \in \mathbf{R}^N \backslash \{0\},$$
$$r(tx) = \frac{1}{t} r(x) \quad \text{for all } (t, x) \in (0, \infty) \times (\mathbf{R}^N \backslash \{0\}).$$

We also assume $\Omega \equiv \{x \in \mathbf{R}^N;\ r(x) > 1\}$ is strictly star-shaped with respect to 0 and $\lim_{|x| \to 0} |x|^\ell Q(x) = 0$.

(L.4) $\Omega \equiv int\{x \in \Lambda_i;\ V(x) = 0\}$ is a non-empty connected bounded set with smooth boundary.

In [**BW1**], (L.2) is called finite case, (L.3) is infinite case and (L.4) is flat case. Now we have

Theorem 1.3 (**[S]**). *Suppose that the assumptions of Theorem 1.1 is satisfied and let $u_\epsilon(x)$ be a positive solution obtained in Theorem 1.1. Assume also, for some $i \in \{1, \cdots, k\}$, Λ_i satisfies one of the assumptions (L.1)–(L.4). Set $g(\epsilon) > 0$, $\Omega_0 \subset \mathbf{R}^N$ and $V_0(x) \in C(\Omega_0, \mathbf{R}^N)$ as in the table 0.1 below.*

(i) c_{ϵ,Λ_i} *given in* (1.2) *satisfies*

$$\lim_{\epsilon\to 0}\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda_i} = b_{V_0,\Omega_0}.$$

Here, b_{V_0,Ω_0} *is defined by*

$$b_{V_0,\Omega_0} = \inf_{u\in H_0^1(\Omega_0)\setminus\{0\}} \left(\frac{\int_{\Omega_0} |\nabla u|^2 + V_0(x)u^2\, dx}{||u_+||^2_{L^{p+1}(\Omega_0)}}\right)^{\frac{p+1}{p-1}}. \tag{1.9}$$

(ii) After extracting a subsequence $\epsilon_n \to 0$*, there exists a sequence* $x_n \in \Lambda_i$ *and a least energy solution* $w_0(x)$ *of* $(L)_{V_0,\Omega_0}$ *such that the rescaled function*

$$w_{\epsilon_n}(y) = \left(\frac{g(\epsilon_n)}{\epsilon_n}\right)^{\frac{2}{p-1}} u_{\epsilon_n}(g(\epsilon_n)y + x_n) \tag{1.10}$$

converges to $w_0(x)$ *in the following sense*

$$x_n \to x_i \in A_i, \tag{1.11}$$

$$||w_{\epsilon_n} - w_0||_{H^1(O_{i,n})} \to 0. \tag{1.12}$$

Here we set $O_{i,n} = \{y \in \mathbf{R}^N;\ g(\epsilon_n)y + x_n \in \Lambda_i\}$.

	(L.1)	*(L.2)*	*(L.3)*	*(L.4)*
$g(\epsilon)$	ϵ	$\epsilon^{\frac{2}{m+2}}$	$(\log \epsilon^{-2})^{-\frac{1}{2}}$	1
$V_0(x)$	m	$P(x)$	0	0
Ω_0	$\mathbf{R}^N$	$\mathbf{R}^N$	Ω	Ω

table 0.1

Remark 1.4. Byeon and Oshita [**BO**] constructed multi-peak solutions by a Lyapunov-Schmidt reduction method; More precisely, they showed the existence of multi-peak solutions under the following situations:

- $V(x) \in C(\mathbf{R}^N, \mathbf{R})$ and $V(x)$ satisfies $V(x) \geq 0$ for all $x \in \mathbf{R}^N$ and (V.2)–(V.3).
- All Λ_i $(i = 1, \cdots, k)$ satisfy one of the assumptions (L.2)–(L.4) and assumptions of the non-degeneracy of least energy solutions of limit equations.

Under the additional conditions $V(x) \in C^4(\mathbf{R}^N, \mathbf{R})$ and $u \to |u|^{p-1}u \in C^4(\mathbf{R})$, they also showed the existence of multi-peak solutions $u_\epsilon(x)$ which join peaks concentrating local minima satisfying (L.2)–(L.4) and topologically non-trivial critical points x_i of $V(x)$ in Λ_i with $V(x_i) > 0$. We remark that they also study the situation $\lim_{|x|\to\infty} V(x) = \infty$.

(c) Other limit equations

In this section, we consider about more general assumptions than (L.2)–(L.3). Firstly, we give some examples.

Examples.

(i) Suppose that $N = 1$ and $\Lambda = (t_1, t_2)$ for $t_1 < 0 < t_2$. We also assume that for $a > b > 1$,

$$V(x) = \begin{cases} |x|^a & \text{if } x \in (t_1, 0], \\ |x|^b & \text{if } x \in (0, t_2). \end{cases}$$

Then, setting $g(\epsilon) = \epsilon^{\frac{2}{a+2}}$, $\Omega_0 = (0, \infty)$ and $V_0(x) = |x|^a$, after extracting subsequence $\epsilon_n \to 0$, there exists a least energy solution $w_0(x)$ of $(L)_{V_0,\Omega_0}$ such that the rescaled function $w_{\epsilon_n}(y)$, which defined as (1.10), converges to $w_0(x)$.

(ii) Suppose that $N = 2$ and $0 \in \Lambda$. We also assume that for $a > b > 1$,

$$V(x_1, x_2) = |x_1|^a + |x_2|^b \quad \text{for } (x_1, x_2) \in \Lambda.$$

Then, setting $g(\epsilon) = \epsilon^{\frac{2}{b+2}}$, $\Omega_0 = \mathbf{R}^2$ and $V_0(x_1, x_2) = |x_2|^b$, after extracting subsequence $\epsilon_n \to 0$, there exists a a least energy solution $w_0(x)$ of $(L)_{V_0,\Omega_0}$ such that the rescaled function $w_{\epsilon_n}(y)$, which defined as (1.10), converges to $w_0(x)$.

In what follows, we give a condition that there exist limit problems. This condition contains (L.2)–(L.3) and two examples above. We fix a $i_0 \in \{1, \cdots, k\}$ and we write Λ_{i_0} by Λ. Without loss of generality, we assume that $0 \in \Lambda$ and 0 is a local minimum of $V(x)$. We set

$$V_\epsilon(x) = \left(\frac{g(\epsilon)}{\epsilon}\right)^2 V(g(\epsilon)x),$$
$$\Lambda_{g(\epsilon)} = \{y \in \mathbf{R}^N;\ g(\epsilon)y \in \Lambda\}.$$

Now, we consider the following assumption.

(L*) 0 is a unique local minimum of $V(x)$ in Λ. Moreover, there exist $g(\epsilon) > 0$ such that $g(\epsilon) \to 0$ as $\epsilon \to 0$ and setting

$$V_0(x) = \lim_{\epsilon\to 0} V_\epsilon(x) \in [0, \infty].$$
$$\Omega_0 = int\{x \in \mathbf{R}^N \mid V_0(x) < \infty\},$$

it holds the following:

(i) For any compact set $D \subset\subset \Omega_0$,

$$\lim_{\epsilon\to 0} \sup_{x\in D} |V_\epsilon(x) - V_0(x)| = 0. \tag{1.13}$$

(ii) For any closed set $E \supset\supset \Omega_0$,

$$\lim_{\epsilon\to 0} \inf_{x\in\Lambda_{g(\epsilon)}\setminus E} V_\epsilon(x) = \infty. \tag{1.14}$$

(iii) If $\Omega_0 = \mathbf{R}^N$, then for any $\delta \in (0,1)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that for $\epsilon \in (0, \epsilon_0)$,

$$V_\epsilon(x) \geq (1-\delta)V_0(x) \quad \text{for all } x \in \Lambda_{g(\epsilon)}, \tag{1.15}$$

Moreover, for $k \in \mathbf{N}$ with $1 \leq k \leq N$, $V_0(x)$ satisfies

$$V_0(x_1, \cdots, x_k, x_{k+1}, \cdots, x_N) = V_0(x_1, \cdots, x_k, 0, \cdots, 0) \quad \text{for all } x \in \mathbf{R}^N \tag{1.16}$$

$$\lim_{|x_1|+\cdots+|x_k|\to\infty} V_0(x_1, \cdots, x_k, 0, \cdots, 0) = \infty. \tag{1.17}$$

If $\Omega_0 \neq \mathbf{R}^N$, then $\partial\Omega_0$ is smooth and $V_0(x)$ satisfies

$$\lim_{|x|\to\infty} V_0(x) = \infty \tag{1.18}$$

We remark that $(L)_{V_0,\Omega_0}$ has a least energy solution under the condition (1.16)–(1.17) or (1.18). Here we have

Theorem 1.5. *Suppose that the assumptions of Theorem 1.1 is satisfied and let $u_\epsilon(x)$ be a positive solution obtained in Theorem 1.1. Assume also Λ satisfies the assumptions (L*).*

(i) c_{ϵ,Λ_i} given in (1.2) satisfies

$$\lim_{\epsilon\to 0} \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda_i} = b_{V_0,\Omega_0}.$$

Here, b_{V_0,Ω_0} is defined by (1.9).

(ii) After extracting a subsequence $\epsilon_n \to 0$, there exists a sequence $x_n \in \Lambda_i$ and a least energy solution $w_0(x)$ of $(L)_{V_0,\Omega_0}$ such that the rescaled function

$$w_{\epsilon_n}(y) = \left(\frac{g(\epsilon_n)}{\epsilon_n}\right)^{\frac{2}{p-1}} u_{\epsilon_n}(g(\epsilon_n)y + x_n) \tag{1.19}$$

converges to $w_0(x)$ in the following sense

$$x_n \to 0, \tag{1.20}$$

$$||w_{\epsilon_n} - w_0||_{H^1(O_{i,n})} \to 0. \tag{1.21}$$

Here we set $O_{i,n} = \{y \in \mathbf{R}^N;\ g(\epsilon_n)y + x_n \in \Lambda_i\}$.

In next sections, we will give a outline of the proof of main theorems.

2. Variational formulation

In this section, we give our variational formulation of $(*)_\epsilon$, which used in [**S**]. We will reduce to a variational problem defined on infinite dimensional torus $\Sigma_{\epsilon,\Lambda_1} \times \cdots \times \Sigma_{\epsilon,\Lambda_k}$.

(a) Preliminary

We use the following notation: for an open subset $D \subset \mathbf{R}^N$,

$$< u, v >_{\epsilon,D} = \int_D \epsilon^2 \nabla u \cdot \nabla v + V(x)uv\ dx \quad \text{for } u, v \in H^1(D),$$

$$||u||^2_{\epsilon,D} = < u, u >_{\epsilon,D} \quad \text{for } u \in H^1(D),$$

$$||f||^*_{\epsilon,D} = \sup_{u \in H^1(D), ||u||_{\epsilon,D} \le 1} |f(u)| \quad \text{for } f \in (H^1(D))^*.$$

For an open subset $D \subset \mathbf{R}^N$ and $W(x) \in C(D, \mathbf{R})$, we also set

$$E_{W,D} = \left\{v \in H^1(D);\ \textstyle\int_D W(x)v^2\ dx < \infty\right\},$$

$$< u, v >_{W,D} = \int_D \nabla u \cdot \nabla v + W(x)uv\ dx \quad \text{for } u, v \in E_{W,D},$$

$$||u||^2_{W,D} = < u, u >_{W,D} \quad \text{for } u \in E_{W,D}.$$

In what follows, we assume that Λ_i has smooth boundary. We set $\Lambda_* = \bigcup_{i=1}^k \Lambda_i$. By the following proposition, for subsets $D = \Lambda_1, \cdots, \Lambda_k$ or $\mathbf{R}^N \setminus \Lambda_*$, norm $||\cdot||_{\epsilon,D}$ is equivalent to $||\cdot||_{H^1(D)}$.

Proposition 2.1. *There exists $C_1 > 0$ independent of $\epsilon \in (0,1)$ such that for subsets $D = \Lambda_1, \cdots, \Lambda_k$ or $\mathbf{R}^N \setminus \Lambda_*$,*

$$||u||^2_{L^2(D)} \le \frac{C_1}{\epsilon^2}||u||^2_{\epsilon,D} \quad \textit{for } u \in H^1(D). \tag{2.1}$$

Proof. It follows from the Poincáre inequality. (See Proposition 1.1 of [**S**].) ∎

From Proposition 2.1, we can easily show the following lemma.

Lemma 2.2. *There exists $\nu_0 > 0$ independent of $\epsilon \in (0,1)$ such that for subsets $D = \Lambda_1, \cdots, \Lambda_k$ or $\mathbf{R}^N \setminus \Lambda_*$,*

$$\frac{1}{2}||u||_{\epsilon,D}^2 \leq ||u||_{\epsilon,D}^2 - 2\nu_0\epsilon^2||u||_{L^2(D)}^2 \quad \text{for } u \in H^1(D). \tag{2.2}$$

Proof. Setting $\nu_0 = \frac{1}{4C_1}$ for the constant C_1 appeared in (2.1), we get inequality (2.2). ∎

In what follows, we choose Λ_i' such that

$$A_i \subset\subset \Lambda_i' \subset\subset \Lambda_i,$$

and set $\Lambda_*' = \bigcup_{i=1}^k \Lambda_i'$. We also use the following lemma.

Lemma 2.3. *There exists a bounded linear operator $P : H^1(\Lambda_* \setminus \Lambda_*') \to H^1(\mathbf{R}^N \setminus \Lambda_*')$ such that for some $C_2 > 0$ independent of $\epsilon \in (0,1)$,*

$$\begin{aligned} &(Pu)(x) = u(x) \quad \text{for } x \in \Lambda_* \setminus \Lambda_*' \text{ and } u \in H^1(\Lambda_* \setminus \Lambda_*'), \\ &||Pu||_{\epsilon,\mathbf{R}^N \setminus \Lambda_*'} \leq C_2||u||_{\epsilon,\Lambda_*\setminus\Lambda_*'} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*'). \end{aligned} \tag{2.3}$$

Proof. By a standard way, there exists a bounded linear operator $P : H^1(\Lambda_* \setminus \Lambda_*') \to H^1(\mathbf{R}^N \setminus \Lambda_*')$ such that for some $C > 0$,

$$\begin{aligned} &(Pu)(x) = u(x) \quad \text{for } x \in \Lambda_* \setminus \Lambda_*' \text{ and } u \in H^1(\Lambda_* \setminus \Lambda_*'), \\ &||Pu||_{L^2(\mathbf{R}^N \setminus \Lambda_*')} \leq C||u||_{L^2(\Lambda_*\setminus\Lambda_*')} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*'), \\ &||Pu||_{H^1(\mathbf{R}^N \setminus \Lambda_*')} \leq C||u||_{H^1(\Lambda_*\setminus\Lambda_*')} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda_*'). \end{aligned}$$

Thus, noting $\inf_{x\in\Lambda_*\setminus\Lambda_*'} V(x) > 0$, we have

$$\begin{aligned} ||Pu||_{\epsilon,\mathbf{R}^N\setminus\Lambda_*'}^2 &\leq \epsilon^2||Pu||_{H^1(\mathbf{R}^N\setminus\Lambda_*')}^2 + \left(\sup_{x\in\mathbf{R}^N} V(x)\right)||Pu||_{L^2(\mathbf{R}^N\setminus\Lambda_*')}^2 \\ &\leq C^2\epsilon^2||u||_{H^1(\Lambda_*\setminus\Lambda_*')}^2 + C^2\left(\sup_{x\in\mathbf{R}^N} V(x)\right)||u||_{L^2(\Lambda_*\setminus\Lambda_*')}^2 \\ &\leq C^2\int_{\Lambda_*\setminus\Lambda_*'} \epsilon^2|\nabla u|^2\,dx + C^2\frac{\epsilon^2 + \sup_{x\in\mathbf{R}^N} V(x)}{\inf_{x\in\Lambda_*\setminus\Lambda_*'} V(x)}\int_{\Lambda_*\setminus\Lambda_*'} V(x)u^2\,dx \\ &\leq C^2\left(1 + \frac{1+\sup_{x\in\mathbf{R}^N} V(x)}{\inf_{x\in\Lambda_*\setminus\Lambda_*'} V(x)}\right)||u||_{\epsilon,\Lambda_*\setminus\Lambda_*'}. \end{aligned}$$

This is nothing but (2.3). ∎

(b) Functional setting

Firstly, to find critical points of $\Psi_\epsilon(u)$, we modify the nonlinearity $|u|^{p-1}u$. We use a local mountain pass approach introduced by del Pino and Felmer [**DF**]. We choose $f_\epsilon \in C^1([0,\infty),\mathbf{R})$ such that for some $0<\ell_1<\ell_2$ and $\alpha-2>\frac{2(p+1)}{p-1}-N>0$

$$f_\epsilon(\xi)=\begin{cases}\xi^p & \text{for } \xi\in(0,\epsilon^{\frac{\alpha}{p-1}}\ell_1),\\ \epsilon^\alpha\nu_0\xi & \text{for } \xi\in(\epsilon^{\frac{\alpha}{p-1}}\ell_2,\infty),\end{cases}\qquad 0\le f_\epsilon'(\xi)\le 2\epsilon^\alpha\nu_0 \quad \text{for } \xi\in[0,\infty). \tag{2.4}$$

Here ν_0 is given in Lemma 2.2. We set

$$g_\epsilon(x,\xi)=\begin{cases}\xi^p & \text{if } x\in\Lambda_* \text{ and } \xi\ge 0,\\ f_\epsilon(\xi) & \text{if } x\notin\Lambda_* \text{ and } \xi\ge 0,\\ 0 & \text{if } \xi\le 0,\end{cases} \tag{2.5}$$

$$G_\epsilon(x,\xi)=\int_0^\xi g_\epsilon(x,\tau)\,d\tau,\quad F_\epsilon(\xi)=\int_0^\xi f_\epsilon(\tau)\,d\tau.$$

Now, we define a functional $\Phi_\epsilon(u)$ by

$$\Phi_\epsilon(u)=\frac{1}{2}||u||^2_{\epsilon,\mathbf{R}^N}-\int_{\mathbf{R}^N}G_\epsilon(x,u)\,dx\in C^2(H^1(\mathbf{R}^N),\mathbf{R}). \tag{2.6}$$

We remark that $\Phi_\epsilon(u)$ satisfies (PS)-condition. (cf.[**DT**]) We also note that if $u_\epsilon(x)$ is a critical point of $\Phi_\epsilon(u)$ satisfying

$$0\le u_\epsilon(x)\le \epsilon^{\frac{\alpha}{p-1}}\ell_1 \qquad \text{for } x\in\mathbf{R}^N\setminus\Lambda_*, \tag{2.7}$$

then $u_\epsilon(x)$ is a critical point of $\Psi_\epsilon(u)$. Thus, in what follows, we will find critical points of $\Phi_\epsilon(u)$ which satisfy (2.7).

Next, we reduce our problem to a problem on $H^1(\Lambda_*)$. For given $u\in H^1(\Lambda_*)$, we consider the following minimizing problem:

$$I_\epsilon(u)=\inf_{\varphi\in H_0^1(\mathbf{R}^N\setminus\Lambda_*)}\Phi_\epsilon(Pu+\varphi). \tag{2.8}$$

Remark 2.4. Letting $\varphi_\epsilon(u)$ be a minimizer of (2.8), then $Pu+\varphi_\epsilon(u)$ is a minimizer of $\inf_{v\in E_u}\Phi_\epsilon(v)$, where $E_u=\{v\in H^1(\mathbf{R}^N)\ ;\ v=u \text{ on } \Lambda_*\}$. Thus $w(x)=(Pu+\varphi_\epsilon(u))|_{\mathbf{R}^N\setminus\Lambda_*}(x)$ satisfies the following boundary value problem:

$$-\epsilon^2\Delta w+V(x)w=f_\epsilon(w)\quad \text{in } \mathbf{R}^N\setminus\Lambda_*,\qquad w=u \quad \text{on } \partial\Lambda_*. \tag{2.9}$$

For the minimizing problem (2.8), we have the following proposition.

Proposition 2.5. *For any $u \in H^1(\Lambda_*)$ and $\epsilon \in (0,1)$, the minimizing problem (2.8) has a unique minimizer $\varphi_\epsilon(u)$ which satisfies*

(i) $u \mapsto \varphi_\epsilon(u) : H^1(\Lambda_*) \to H_0^1(\mathbf{R}^N \setminus \Lambda_*)$ *is of class* C^1.

(ii) $I_\epsilon(u) = \Phi_\epsilon(Pu + \varphi_\epsilon(u)) : H^1(\Lambda_*) \to \mathbf{R}$ *is of class* C^2.

(iii) $u \in H^1(\Lambda_*)$ *is a critical point of* $I_\epsilon(u)$ *if and only if* $Pu + \varphi_\epsilon(u)$ *is a critical point of* $\Phi_\epsilon(u)$.

(iv) $I_\epsilon(u)$ *satisfies (PS)-condition.*

Proof. We outline the proof. (See Proposition 1.6 of [**S**].) We consider the functional $J(\varphi) = \Phi_\epsilon(Pu + \varphi) : H_0^1(\mathbf{R}^N \setminus \Lambda_*) \to \mathbf{R}$. Then $J(\varphi)$ is strongly convex and coercive, that is, $J(\varphi)$ satisfies

$$J''(\varphi)(h,h) \geq \frac{1}{2}||h||^2_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} \quad \text{for any } \varphi, h \in H_0^1(\mathbf{R}^N \setminus \Lambda_*). \tag{2.10}$$

From (2.10), (2.8) has a unique minimizer $\varphi_\epsilon(u)$. (i) follows from the implicit function theorem. (ii)–(iv) follows from the fact that

$$\begin{aligned} &\Phi'_\epsilon(Pu + \varphi_\epsilon(u))h = 0 \quad \text{for all } h \in H_0^1(\mathbf{R}^N \setminus \Lambda_*), \\ &\varphi'_\epsilon(u)\zeta \in H_0^1(\mathbf{R}^N \setminus \Lambda_*) \quad \text{for all } \zeta \in H^1(\Lambda_*), \end{aligned} \tag{2.11}$$

that is,

$$I'_\epsilon(u)\zeta = \Phi'_\epsilon(Pu + \varphi_\epsilon(u))(P\zeta) \quad \text{for all } \zeta \in H^1(\Lambda_*).$$ ∎

Setting $Q_\epsilon(u) = (Pu + \varphi_\epsilon(u))|_{\mathbf{R}^N\setminus\Lambda_*}$, by Remark 2.4, $Q_\epsilon(u)$ satisfies (2.9). In what follows, we identify $H^1(\Lambda_*)$ and $H^1(\Lambda_1)\oplus\cdots\oplus H^1(\Lambda_k)$, Moreover, for functions $u_i \in H^1(\Lambda_i)$ $(i = 1, \cdots, k)$, we write $u = (u_1, \cdots, u_k)$ if $u \in H^1(\Lambda_*)$ satisfies $u_i = u|_{\Lambda_i}$ $(i = 1, \cdots, k)$. When $u = (u_1, \cdots, u_k)$, we also write

$$\begin{aligned} &I_\epsilon(u_1, \cdots, u_k) \quad \text{for} \quad I_\epsilon(u), \\ &Q_\epsilon(u_1, \cdots, u_k) \quad \text{for} \quad Q_\epsilon(u). \end{aligned} \tag{2.12}$$

Moreover, we can write

$$\begin{aligned} I_\epsilon(u_1, \cdots, u_k) = &\sum_{i=1}^k \left(\frac{1}{2}||u_i||^2_{\epsilon,\Lambda_i} - \frac{1}{p+1}||u_{i+}||^{p+1}_{L^{p+1}(\Lambda_i)} \right) \\ &+ \frac{1}{2}||Q_\epsilon(u_1, \cdots, u_k)||^2_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} - \int_{\mathbf{R}^N\setminus\Lambda_*} F_\epsilon(Q_\epsilon(u_1, \cdots, u_k))\, dx. \end{aligned}$$

For $h_\epsilon(x) \in L^\infty(\mathbf{R}^N \setminus \Lambda_*)$ satisfying $||h_\epsilon||_{L^\infty(\mathbf{R}^N \setminus \Lambda_*)} \leq 2\epsilon^\alpha \nu_0$, we consider the following linear boundary value problem:

$$-\epsilon^2 \Delta v + V(x)v = h_\epsilon(x)v \quad \text{in } \mathbf{R}^N \setminus \Lambda_*, \quad v = u \quad \text{on } \Lambda_* \setminus \Lambda'_*. \tag{2.13}$$

In a similar way to Proposition 2.5, we can easily observe (2.13) has a unique solution. We denote the unique solution of (2.13) by $Q_{h_\epsilon,\epsilon}(u_1, \cdots, u_k)(x)$. Then we can write as $Q_\epsilon(u_1, \cdots, u_k)(x) = Q_{f_\epsilon(Q_\epsilon(u_1,\cdots,u_k))/Q_\epsilon(u_1,\cdots,u_k),\epsilon}(u_1, \cdots, u_k)(x)$. Here, we define the auxiliary functionals

$$\begin{aligned} I_{\epsilon,\Lambda_i}(u_i) =& \frac{1}{2}||u_i||^2_{\epsilon,\Lambda_i} - \frac{1}{p+1}||u_{i+}||^{p+1}_{L^{p+1}(\Lambda_i)} + \frac{1}{2}||Q_{0,\epsilon}(0, \cdots, u_i, \cdots, 0)||^2_{\epsilon,\mathbf{R}^N \setminus \Lambda_*} \\ & \in C^2(H^1(\Lambda_i), \mathbf{R}) \quad (i = 1, \cdots, k). \end{aligned}$$

The following proposition gives estimates of difference between $I_\epsilon(u_1, \cdots, u_k)$ and auxiliary functional $\sum_{i=1}^k I_{\epsilon,\Lambda_i}(u_i)$ and it plays an important role in our proof.

Proposition 2.6. *There exists $C_3 > 0$ such that for $\epsilon \in (0,1)$ and $(u_1, \cdots, u_k) \in H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k)$,*

$$|I_\epsilon(u_1, \cdots, u_k) - \textstyle\sum_{i=1}^k I_{\epsilon,\Lambda_i}(u_i)| \leq \epsilon^{\alpha-2} C_3 \sum_{i=1}^k ||u_i||^2_{\epsilon,\Lambda_i}, \tag{2.14}$$

To prove Proposition 2.6, we show the following lemma.

Lemma 2.7. *There exists $C_4 > 0$ such that for $\epsilon \in (0,1)$ and $(u_1, \cdots, u_k) \in H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k)$,*

$$||Q_{h_\epsilon,\epsilon}(u_1, \cdots, u_k)||_{\epsilon,\mathbf{R}^N \setminus \Lambda_*} \leq C_4 ||u||_{\epsilon,\Lambda_* \setminus \Lambda'_*}, \tag{2.15}$$

$$||Q_\epsilon(u_1, \cdots, u_k)||_{\epsilon,\mathbf{R}^N \setminus \Lambda_*} \leq C_4 ||u||_{\epsilon,\Lambda_* \setminus \Lambda'_*}, \tag{2.16}$$

$$||Q_\epsilon(u_1, \cdots, u_k) - Q_{0,\epsilon}(u_1, \cdots, u_k)||_{\epsilon,\mathbf{R}^N \setminus \Lambda_*} \leq \epsilon^{\alpha-2} C_4 ||u||_{\epsilon,\Lambda_* \setminus \Lambda'_*}. \tag{2.17}$$

Proof. First of all, we set $v(x) = Q_{0,\epsilon}(u_1, \cdots, u_k)(x)$ and $w(x) = Q_\epsilon(u_1, \cdots, u_k)(x)$. From Remark 2.4, we remark that w is a unique minimizer of the minimizing problem $K_\epsilon(w) = \inf_{u \in E_u} K_\epsilon(u)$, where $K_\epsilon(u) = \int_{\mathbf{R}^N \setminus \Lambda_*} \frac{1}{2}(\epsilon^2 |\nabla u| + V(x)u^2) - F_\epsilon(u)\, dx$. Thus by Lemma 2.2 and Lemma 2.3, we have

$$\frac{1}{4}||w||^2_{\epsilon,\mathbf{R}^N \setminus \Lambda_*} \leq K_\epsilon(w) \leq K_\epsilon(Pu) \leq ||Pu||^2_{\epsilon,\mathbf{R}^N \setminus \Lambda_*} \leq C_2^2 ||u||^2_{\epsilon,\Lambda_* \setminus \Lambda'_*}. \tag{2.18}$$

Thus (2.16) follows from (2.18). We can show (2.15) in a similar way. Next we show (2.17). By variational characterizations for $w(x)$ and $v(x)$ $w - v$ satisfies

$$< w - v, \varphi >_{\epsilon,\mathbf{R}^N \setminus \Lambda_*} = \int_{\mathbf{R}^N \setminus \Lambda_*} f_\epsilon(w)\varphi\, dx \quad \text{for all } \varphi \in H_0^1(\mathbf{R}^N \setminus \Lambda_*). \tag{2.19}$$

Since $w - v \in H_0^1(\mathbf{R}^N \setminus \Lambda_*)$, setting $\varphi = w - v$ in (2.19), we have

$$\begin{aligned}||w - v||^2_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} &= \int_{\mathbf{R}^N\setminus\Lambda_*} f_\epsilon(w)(w-v)\, dx \\ &\le \epsilon^\alpha \nu_0 ||w||_{L^2(\mathbf{R}^N\setminus\Lambda_*)}||w-v||_{L^2(\mathbf{R}^N\setminus\Lambda_*)} \\ &\le \epsilon^{\alpha-2}\nu_0 C_1 ||w||_{\epsilon,\mathbf{R}^N\setminus\Lambda_*}||w-v||_{\epsilon,\mathbf{R}^N\setminus\Lambda_*}. \end{aligned} \tag{2.20}$$

Here we used Proposition 2.1. By (2.20), we get

$$||w - v||_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} \le \epsilon^{\alpha-2}\nu_0 C_1 ||w||_{\epsilon,\mathbf{R}^N\setminus\Lambda_*}. \tag{2.21}$$

From (2.16) and (2.21), for some $C_4 > 0$ which is independent of $u = (u_1, \cdots, u_k)$ and $\epsilon \in (0,1)$, we obtain

$$||w - v||_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} \le \epsilon^{\alpha-2} C_4 ||u||_{\epsilon,\Lambda_*\setminus\Lambda'_*}.$$

∎

Proof of Proposition 2.6. From (2.12), we have

$$\begin{aligned}&I_\epsilon(u_1,\cdots,u_k) - \sum_{i=1}^k I_{\epsilon,\Lambda_i}(u_i) \\ &= \frac{1}{2}\Big(||Q_\epsilon(u_1,\cdots,u_k)||^2_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} - \sum_{i=1}^k ||Q_{0,\epsilon}(0,\cdots,u_i,\cdots,0)||^2_{\epsilon,\mathbf{R}^N\setminus\Lambda_*}\Big) \\ &\quad - \int_{\mathbf{R}^N\setminus\Lambda_*} F_\epsilon(Q_\epsilon(u_1,\cdots,u_k))\, dx \\ &= \frac{1}{2}(I) + (II).\end{aligned}$$

Since $Q_{0,\epsilon}(u_1,\cdots,u_k)$ is a solution of the linear equation, we have $Q_{0,\epsilon}(u_1,\cdots,u_k) = \sum_{i=1}^k Q_{0,\epsilon}(0,\cdots,u_i,\cdots,0)$. Thus,

$$\begin{aligned}(I) &= ||Q_\epsilon(u_1,\cdots,u_k)||^2_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} - ||Q_{0,\epsilon}(u_1,\cdots,u_k)||^2_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} \\ &\quad + 2\sum_{1\le i<j\le k} < Q_{0,\epsilon}(0,\cdots,u_i,\cdots,0), Q_{0,\epsilon}(0,\cdots,u_j,\cdots,0) >_{\epsilon,\mathbf{R}^N\setminus\Lambda_*}. \end{aligned}\tag{2.22}$$

Now we remark that, for $i \ne j$, the following estimate holds:

$$| < Q_{0,\epsilon}(0,\cdots,u_i,\cdots,0), Q_{0,\epsilon}(0,\cdots,u_j,\cdots,0) >_{\epsilon,\mathbf{R}^N\setminus\Lambda_*} | \le Ce^{-\frac{a}{\epsilon}}||u_i||_{\epsilon,\Lambda_i}||u_j||_{\epsilon,\Lambda_j}, \tag{2.23}$$

where constants $C, a > 0$ are independent of $(u_1,\cdots,u_k) \in H^1(\Lambda_1)\oplus\cdots\oplus H^1(\Lambda_k)$ and $\epsilon \in (0,1)$. It is showed in [**ST**]. The key of the proof is the subsolution estimate in [**Si**]. (Also see [**GT**].) From Lemma 2.7, (2.22) and (2.23), it follows that

$$|(I)| \le 2\epsilon^{\alpha-2}C_4^2||u||^2_{\epsilon,\Lambda_*} + 2Ce^{-\frac{a}{\epsilon}}2\sum_{1\le i<j\le k}||u_i||_{\epsilon,\Lambda_i}||u_j||_{\epsilon,\Lambda_j} \tag{2.24}$$

On the other hand, since $F(\xi) \le \frac{1}{2}\epsilon^\alpha \nu_0 \xi^2$, we have

$$\begin{aligned} |(II)| &\le \frac{1}{2}\epsilon^\alpha \nu_0 ||Q_\epsilon(u_1, \cdots, u_k)||^2_{L^2(\mathbf{R}^N \setminus \Lambda_*)} \\ &\le \frac{1}{2}\epsilon^{\alpha-2}\nu_0 C_1 ||Q_\epsilon(u_1, \cdots, u_k)||^2_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \\ &\le \frac{1}{2}\epsilon^{\alpha-2}\nu_0 C_1 C_4^2 ||u||^2_{\epsilon, \Lambda_*}. \end{aligned} \tag{2.25}$$

Here we use Proposition 2.1 and Lemma 2.7. We get (2.14) from (2.24) and (2.25). ∎

We use the following notation: for $u_i, v_i \in H^1(\Lambda_i)$ $(i = 1, \cdots, k)$,

$$\begin{aligned} &< u_i, v_i >_{\epsilon, \Lambda_i, \#} = < u_i, v_i >_{\epsilon, \Lambda_i} \\ &\qquad + < Q_{0,\epsilon}(0, \cdots, u_i, \cdots, 0), Q_{0,\epsilon}(0, \cdots, v_i, \cdots, 0) >_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \quad (i = 1, \cdots, k), \\ &||u_i||^2_{\epsilon, \Lambda_i, \#} = < u_i, u_i >_{\epsilon, \Lambda_i, \#} \quad (i = 1, \cdots, k). \end{aligned}$$

By Lemma 2.7, we easily get

$$||u_i||_{\epsilon, \Lambda_i} \le ||u_i||_{\epsilon, \Lambda_i, \#} \le (1 + C_4)||u_i||_{\epsilon, \Lambda_i} \quad \text{for all } u_i \in H^1(\Lambda_i).$$

Thus $|| \cdot ||_{\epsilon, \Lambda_i, \#}$ is equivalent to $|| \cdot ||_{\epsilon, \Lambda_i}$ for each i. With this notation, $I_{\epsilon, \Lambda_i}(u)$ can be written as

$$I_{\epsilon, \Lambda_i}(u_i) = \frac{1}{2}||u_i||^2_{\epsilon, \Lambda_i, \#} - \frac{1}{p+1}||u_{i+}||^{p+1}_{L^{p+1}(\Lambda_i)} : H^1(\Lambda_i) \to \mathbf{R}. \quad (i = 1, \cdots, k)$$

We can easily see that $I_{\epsilon, \Lambda_i}(u_i)$ $(i = 1, \cdots, k)$ has a mountain pass geometry and satisfies $(PS)_c$-condition for all $c \in \mathbf{R}$ in a standard way.

(c) Reduction to a problem on $\Sigma_{\epsilon, \Lambda_1} \times \cdots \times \Sigma_{\epsilon, \Lambda_k}$

In this section, we reduce our problem to a variational problem on an infinite dimensional torus $\Sigma_{\epsilon, \Lambda_1} \times \cdots \times \Sigma_{\epsilon, \Lambda_k}$, where

$$\Sigma_{\epsilon, \Lambda_i} = \{v_i \in H^1(\Lambda_i); \ ||v_i||^2_{\epsilon, \Lambda_i, \#} = d_{\epsilon, \Lambda_i}\} \quad (i = 1, \cdots, k). \tag{2.26}$$

Here we define d_{ϵ, Λ_i} by

$$d_{\epsilon, \Lambda_i} = \inf_{v_i \in H^1(\Lambda_i) \setminus \{0\}} \left(\frac{||v_i||_{\epsilon, \Lambda_i, \#}}{||v_{i+}||_{L^{p+1}(\Lambda_i)}} \right)^{\frac{2(p+1)}{p-1}} \quad (i = 1, \cdots, k). \tag{2.27}$$

Then d_{ϵ, Λ_i} satisfies the following decay estimates.

Lemma 2.8. *(i) If* $\inf_{x\in\Lambda_i} V(x) = m_i > 0$*, then* $\lim_{\epsilon\to 0}\epsilon^{-N} d_{\epsilon,\Lambda_i} = b_{m_i,\mathbf{R}^N} > 0$.
(ii) If $\inf_{x\in\Lambda_i} V(x) = 0$*, then* $\lim_{\epsilon\to 0}\epsilon^{-N} d_{\epsilon,\Lambda_i} = 0$ *and* $\liminf_{\epsilon\to 0}\epsilon^{-\frac{2(p+1)}{p-1}} d_{\epsilon,\Lambda_i} \in (0,\infty]$.

Proof. Lemma 2.8 can be shown by a similar way of Proposition 4.2 below. ∎

We consider the auxiliary problems constrained on sphere $\Sigma_{\epsilon,\Lambda_i}$:

$$J_{\epsilon,\Lambda_i}(v_i) = \sup_{t>0} I_{\epsilon,\Lambda_i}(tv_i) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{d_{\epsilon,\Lambda_i}}{||v_{i+}||^2_{L^{p+1}(\Lambda_i)}}\right)^{\frac{p+1}{p-1}} : \Sigma_{\epsilon,\Lambda_i} \to \mathbf{R}\ (i = 1,\cdots,k).$$

For $v_i \in \Sigma_{\epsilon,\Lambda_i}$, we can see that $t \mapsto I_{\epsilon,\Lambda_i}(tv_i) : [0,\infty) \to \mathbf{R}$ takes a global maximum at

$$t = t_{\epsilon,\Lambda_i}(v_i) = d_{\epsilon,\Lambda_i}^{\frac{1}{p-1}} ||v_{i+}||_{L^{p+1}(\Lambda_i)}^{-\frac{p+1}{p-1}} \tag{2.28}$$

and

$$J_{\epsilon,\Lambda_i}(v_i) = I_{\epsilon,\Lambda_i}(t_{\epsilon,\Lambda_i}(v_i)v_i) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{d_{\epsilon,\Lambda_i}}{||v_{i+}||^2_{L^{p+1}(\Lambda_i)}}\right)^{\frac{p+1}{p-1}}.$$

We choose $r > 0$ such that $1 - p(1+r)^{-(p-1)} < 0$ and set

$$N_{\epsilon,\Lambda_i} = \{v_i \in \Sigma_{\epsilon,\Lambda_i};\ ||v_{i+}||^{p+1}_{L^{p+1}(\Lambda_i)} \geq (1+r)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_i}\} \quad (i = 1,\cdots,k). \tag{2.29}$$

N_{ϵ,Λ_i} is a neighborhood of least energy critical points of $J_{\epsilon,\Lambda_i}(v_i)$. In fact, we can easily get the following lemma.

Lemma 2.9. *For any* $\epsilon \in (0,1)$,

$$\left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_i} = \inf_{v_i\in N_{\epsilon,\Lambda_i}} J_{\epsilon,\Lambda_i}(v_i) \quad (i = 1,\cdots,k),$$
$$(1+r)\left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_i} \leq \inf_{v_i\in\partial N_{\epsilon,\Lambda_i}} J_{\epsilon,\Lambda_i}(v_i) \quad (i = 1,\cdots,k).$$

Proof. By a direct computation, we can easily see that

$$J_{\epsilon,\Lambda_i}(v_i) \leq (1+r)\left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_i} \text{ if and only if } ||v_{i+}||^{p+1}_{L^{p+1}(\Lambda_i)} \geq (1+r)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_i}. \tag{2.30}$$

Thus Lemma 2.9 follows. ∎

For minimizing sequences of J_{ϵ,Λ_i}, we have the following estimates.

Lemma 2.10. *If a sequence $v_{i,\epsilon} \in \Sigma_{\epsilon,\Lambda_i}$ $(\epsilon \to 0)$ satisfies*

$$J_{\epsilon,\Lambda_i}(v_{i,\epsilon}) = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}), \tag{2.31}$$

then we have

$$||v_{i,\epsilon+}||^{p+1}_{L^{p+1}(\Lambda_i)} = d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}). \tag{2.32}$$

Proof. From (2.31) and (2.30), we have

$$||v_{i,\epsilon+}||^{p+1}_{L^{p+1}(\Lambda_i)} = \left(1 + \frac{o(\epsilon^{\frac{2(p+1)}{p-1}})}{d_{\epsilon,\Lambda_i}}\right)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_i}.$$

Using a Taylor expansion, for small $|r|$, we find $(1+r)^{-\frac{p-1}{2}} = 1 + O(r)$. Thus we get

$$||v_{i,\epsilon+}||^{p+1}_{L^{p+1}(\Lambda_i)} = \left(1 + \frac{o(\epsilon^{\frac{2(p+1)}{p-1}})}{d_{\epsilon,\Lambda_i}}\right) d_{\epsilon,\Lambda_i} = d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}). \qquad \blacksquare$$

We set a subset N_ϵ as follows:

$$N_\epsilon = \{(v_1, \cdots, v_k) \in \Sigma_{\epsilon,\Lambda_1} \times \cdots \times \Sigma_{\epsilon,\Lambda_k};\ ||v_{i+}||^{p+1}_{L^{p+1}(\Lambda_i)} \geq (1+r)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_i}\ (i = 1, \cdots, k)\}. \tag{2.33}$$

From Lemma 2.9, we can see $N_\epsilon \neq \emptyset$. We try to find a critical point of $J_\epsilon(v_1, \cdots, v_k)$: $N_\epsilon \to (0, \infty]$ which is defined by

$$J_\epsilon(v_1, \cdots, v_k) = \sup_{s_1, \cdots, s_k \geq 0} I_\epsilon(s_1 v_1, \cdots, s_k v_k). \tag{2.34}$$

For simplicity, in what follows, we use notation: $\mathbf{v} = (v_1, \cdots, v_k) \in N_\epsilon$, $\mathbf{s} = (s_1, \cdots, s_k) \in [0, \infty)^k$, $\mathbf{sv} = (s_1 v_1, \cdots, s_k v_k)$.

The following proposition is important.

Proposition 2.11. *There exists $\epsilon_1 \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon_1)$ we have*

(i) There exist constants $R_2 > R_1 > 0$ independent of ϵ such that for any $\mathbf{v} \in N_\epsilon$, $\mathbf{s} \mapsto I_\epsilon(\mathbf{sv})$ has a unique maximizer $\mathbf{s}_\epsilon(\mathbf{v}) = (s_{1,\epsilon}(\mathbf{v}), \cdots, s_{k,\epsilon}(\mathbf{v}))$ in $[R_1, R_2]^k$.

(ii) $\mathbf{v} \mapsto \mathbf{s}_\epsilon(\mathbf{v}) : N_\epsilon \to \mathbf{R}^k$ is of class of C^1.

(iii) $J_\epsilon(\mathbf{v}) : N_\epsilon \to \mathbf{R}$ is of class of C^1.

(iv) $J_\epsilon(\mathbf{v}) : N_\epsilon \to \mathbf{R}$ satisfies (PS)-condition.

(v) If $\mathbf{v} \in N_\epsilon$ is a critical point of $J_\epsilon(\mathbf{v})$, then $\mathbf{s}_\epsilon(\mathbf{v})\mathbf{v} \in H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k)$ is a critical point of $I_\epsilon(u_1, \cdots, u_k)$.

(vi) For all $\mathbf{v} = (v_1, \cdots, v_k) \in N_\epsilon$, *we have*

$$J_\epsilon(\mathbf{v}) = \sum_{i=1}^k J_{\epsilon,\Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}}). \tag{2.35}$$

where $o(\epsilon^{\frac{2(p+1)}{p-1}})$ *is uniformly for* $\mathbf{v} \in N_\epsilon$.

Proof. See Proposition 1.13 and Proposition 1.14 of [**S**]. ∎

3. Outline of the proof of Theorem 1.1.

In this section, we will show Theorem 1.1. We define

$$c_\epsilon = \inf_{\mathbf{v}\in N_\epsilon} J_\epsilon(\mathbf{v}). \tag{3.1}$$

Then we show the following proposition.

Proposition 3.1. *For small* $\epsilon \in (0, \epsilon_1)$, *we have*
(i) $c_\epsilon = \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i=1}^k d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}})$.
(ii) $\inf_{\mathbf{v}\in N_\epsilon} J_\epsilon(\mathbf{v}) < \inf_{\mathbf{v}\in \partial N_\epsilon} J_\epsilon(\mathbf{v})$.
(iii) c_ϵ *is a critical value of* $J_\epsilon(\mathbf{v})$. *Moreover corresponding critical points lie in* N_ϵ.

Proof. From (2.35), we recall

$$J_\epsilon(\mathbf{v}) = \sum_{i=1}^k J_{\epsilon,\Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}}) \qquad \text{for all } \mathbf{v} \in N_\epsilon,$$

where $o(\epsilon^{\frac{2(p+1)}{p-1}})$ is uniformly for $\mathbf{v} \in N_\epsilon$. Thus from Lemma 2.9, it follows

$$\begin{aligned} c_\epsilon = \inf_{\mathbf{v}\in N_\epsilon} J_\epsilon(\mathbf{v}) &= \sum_{i=1}^k \inf_{v_i \in N_{\epsilon,\Lambda_i}} J_{\epsilon,\Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}}) \\ &= \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) \sum_{i=1}^k d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}). \end{aligned} \tag{3.2}$$

Next we show (ii). We note $\partial N_\epsilon = \bigcup_{i=1}^k (N_{\epsilon,\Lambda_1} \times \cdots \times \partial N_{\epsilon,\Lambda_i} \times \cdots \times N_{\epsilon,\Lambda_k})$. Again, from Lemma 2.9, we get for each j

$$\begin{aligned} \inf_{\mathbf{v}\in N_{\epsilon,\Lambda_1}\times\cdots\times\partial N_{\epsilon,\Lambda_j}\times\cdots\times N_{\epsilon,\Lambda_k}} J_\epsilon(\mathbf{v}) &= \sum_{i\neq j} \inf_{v_i\in N_{\epsilon,\Lambda_i}} J_{\epsilon,\Lambda_i}(v_i) + \inf_{v_j \in \partial N_{\epsilon,\Lambda_j}} J_{\epsilon,\Lambda_j}(v_j) + o(\epsilon^{\frac{2(p+1)}{p-1}}) \\ &\geq \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) \sum_{i\neq j} d_{\epsilon,\Lambda_i} + (1+r)\left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_j} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \\ &= c_\epsilon + r\left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_j} + o(\epsilon^{\frac{2(p+1)}{p-1}}). \end{aligned} \tag{3.3}$$

(ii) follows from (3.2)–(3.3). Since $J_\epsilon(\mathbf{v})$ satisfies (PS)-condition, we can see c_ϵ is a critical value of $J_\epsilon(\mathbf{v})$ in a standard way. ∎

Corollary 3.2. *Minimizer* $\mathbf{v}_\epsilon = (v_{1,\epsilon}, \cdots, v_{k,\epsilon})$ *of* (3.1) *satisfies*

$$J_{\epsilon,\Lambda_i}(v_{i,\epsilon}) = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad (i = 1, \cdots, k).$$

Proof. From (2.35), we have

$$\begin{aligned}\left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_i} \le J_{\epsilon,\Lambda_i}(v_{i,\epsilon}) &= J_\epsilon(\mathbf{v}_\epsilon) - \sum_{\ell\neq i} J_{\epsilon,\Lambda_\ell}(v_{\ell,\epsilon}) + o(\epsilon^{\frac{2(p+1)}{p-1}}) \\ &\le c_\epsilon - \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) \sum_{\ell\neq i} d_{\epsilon,\Lambda_\ell} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \\ &= \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) d_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}).\end{aligned}$$

Thus we get Corollary 3.2. ∎

From the a minimizer of (3.1), we get a critical point of $\Phi_\epsilon(u)$ by the following:

Proposition 3.3. *Let* $\mathbf{v}_\epsilon = (v_{1,\epsilon}, \cdots, v_{k,\epsilon})$ *be a minimizer of* (3.1). *Then*

$$u_\epsilon(x) = \begin{cases} s_{i,\epsilon}(\mathbf{v}_\epsilon) v_{i,\epsilon}(x) & \text{for } x \in \Lambda_i \quad (i = 1, \cdots, k), \\ Q_\epsilon(s_{1,\epsilon}(\mathbf{v}_\epsilon) v_{1,\epsilon}, \cdots, s_{k,\epsilon}(\mathbf{v}_\epsilon) v_{k,\epsilon})(x) & \text{for } x \in \mathbf{R}^N \setminus \Lambda_*. \end{cases} \tag{3.4}$$

is a critical point of $\Phi_\epsilon(u)$. *Here* $(s_{1,\epsilon}(\mathbf{v}), \cdots, s_{k,\epsilon}(\mathbf{v}))$ *are given in Proposition 2.11.*

Proof. It follows from Proposition 2.5, Proposition 2.11 and Proposition 3.1. ∎

Here we outline the proof of Theorem 1.1

Outline of the proof of Theorem 1.1. We can show that $u_\epsilon(x)$ defined by (3.4) has an exponential decay on $\mathbf{R}^N \setminus \Lambda'_*$ as $\epsilon \to 0$. (See [**S**].) Thus $u_\epsilon(x)$ satisfies (2.7) and $u_\epsilon(x)$ is a critical point of original functional $\Psi_\epsilon(u)$. Moreover, we have the following detailed estimate of $u_\epsilon(x)$ (See [**S**].):

$$||u_{\epsilon+}||^{p+1}_{L^{p+1}(\Lambda_i)} = c_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}), \tag{3.5}$$

$$||u_\epsilon||^2_{\epsilon,\Lambda_i} = c_{\epsilon,\Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}). \tag{3.6}$$

Thus (1.4)–(1.5) follow from (3.5)–(3.6). We get (1.6) by a standard way. The proof of Theorem 1.1 is completed. ∎

4. Asymptotic profile of solutions $u_\epsilon(x)$.

In this section, we will prove Theorem 1.5. By a similar way, we can prove Theorem 1.3. (See Section 3 of [**S**].) First of all, we note the following.

Remark 4.1. When assumption (L*) holds, there exists a constant $C_5 > 0$ independent of $\epsilon \in (0,1)$ such that

$$||w||_{H^1(\Lambda_{g(\epsilon)})} \leq C_5 ||w||_{V_\epsilon, \Lambda_{g(\epsilon)}} \quad \text{for all } w \in H^1(\Lambda_{g(\epsilon)}). \tag{4.1}$$

In fact we can see that, for some $\ell > 0$ and $\delta > 0$ independent of $\epsilon \in (0,1)$, $V_\epsilon(y)$ satisfies

$$V_\epsilon(y) \geq \begin{cases} 0 & \text{for } y \in (-\ell,\ell)^k \times \mathbf{R}^{N-k} \subset \mathbf{R}^N, \\ \delta & \text{elsewhere.} \end{cases}$$

Thus (4.1) can be shown as Proposition 2.1. (See Lemma 1.2 of [**S**].)

In the following arguments, $d_{\epsilon,\Lambda}$ defined in (2.27) and b_{V_0,Ω_0} defined in (1.9) will play important roles.

Proposition 4.2. *b_{V_0,Ω_0} is achieved by some $w(x) \in H_0^1(\Omega_0)$ and*

$$\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} d_{\epsilon,\Lambda} \to b_{V_0,\Omega_0} \quad \text{as } \epsilon \to 0. \tag{4.2}$$

Proof. From Lemma 2.11 of [**S**], it suffice to show

$$\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} \to b_{V_0,\Omega_0} \quad \text{as } \epsilon \to 0. \tag{4.3}$$

Firstly we show

$$\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} \leq b_{V_0,\Omega_0} + o(1). \tag{4.4}$$

Suppose that $w(x) \in H_0^1(\Omega)$ achieves $b_{V_0,\Omega}$. We choose a function $\varphi \in C_0^1(\mathbf{R}^N, [0,1])$ such that

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in \Lambda' \\ 0 & \text{for } x \in \mathbf{R}^N \setminus \Lambda \end{cases} \qquad |\nabla\varphi(x)| \leq C \quad \text{for } x \in \mathbf{R}^N,$$

and set

$$v_\epsilon(x) = \varphi(x) \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2}{p-1}} w\left(\frac{x}{g(\epsilon)}\right), \qquad \psi_\epsilon(x) = \varphi(g(\epsilon)x).$$

Then by direct computations, we have

$$\begin{aligned} ||v_\epsilon||_{\epsilon,\Lambda}^2 &= \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N \left[\int_{\Lambda_{g(\epsilon)}} \psi_\epsilon(x)^2 \{|\nabla w(x)|^2 + V_\epsilon(x) w(x)^2\}\, dx \right. \\ &\quad \left. + \int_{\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)}} |\nabla\psi_\epsilon(x)|^2 w(x)^2 + 2\nabla\psi_\epsilon(x) \cdot \nabla w(x) \psi_\epsilon(x) w(x)\, dx \right] \\ &= \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N \left((I) + (II)\right). \end{aligned}$$

Here we have

$$|(II)| \le C^2||w||^2_{L^2(\Lambda_{g(\epsilon)}\setminus\Lambda'_{g(\epsilon)})} + 2C||\nabla w||_{L^2(\Lambda_{g(\epsilon)}\setminus\Lambda'_{g(\epsilon)})}||w||_{L^2(\Lambda_{g(\epsilon)}\setminus\Lambda'_{g(\epsilon)})} \to 0$$

as $\epsilon \to 0$. Thus, from (1.13), we find

$$||v_\epsilon||^2_{\epsilon,\Lambda} = \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N(||w||^2_{V_0,\Omega_0} + o(1)). \tag{4.5}$$

By a similar way, we get

$$||v_{\epsilon+}||^{p+1}_{L^{p+1}(\Lambda)} = \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N(||w_+||^{p+1}_{L^{p+1}(\Omega_0)} + o(1)). \tag{4.6}$$

Thus from definition of $c_{\epsilon,\Lambda}$ and (4.5)–(4.6),

$$\begin{aligned}\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N}c_{\epsilon,\Lambda} &\le \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N}\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{||v_\epsilon||_{\epsilon,\Lambda}}{||v_{\epsilon+}||_{L^{p+1}(\Lambda)}}\right)^{\frac{2(p+1)}{p-1}} \\ &= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{||w||_{V_0,\Omega_0}+o(1)}{||w_+||_{L^{p+1}(\Omega_0)}+o(1)}\right)^{\frac{2(p+1)}{p-1}} \\ &= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{||w||_{V_0,\Omega_0}}{||w_+||_{L^{p+1}(\Omega_0)}}\right)^{\frac{2(p+1)}{p-1}} + o(1) \\ &= b_{V_0,\Omega_0} + o(1). \end{aligned} \tag{4.7}$$

Next, we show

$$b_{V_0,\Omega_0} \le \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N}c_{\epsilon,\Lambda} + o(1). \tag{4.8}$$

Let $v_\epsilon(x) \in H^1_0(\Lambda)$ attains $c_{\epsilon,\Lambda}$. We may assume $||v_\epsilon||^2_{\epsilon,\mathbf{R}^N} = \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N$. Then, from (4.4), we see that

$$\lim_{\epsilon\to 0}\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N}||v_{\epsilon+}||_{L^{p+1}(\Lambda)} > 0. \tag{4.9}$$

We set

$$w_\epsilon(x) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2}{p-1}} v_\epsilon(g(\epsilon)x).$$

Then $||w_\epsilon||_{V_\epsilon,\mathbf{R}^N} = 1$ and $\lim_{\epsilon\to 0}||w_\epsilon||_{L^{p+1}(\mathbf{R}^N)} > 0$. We need to divide in two cases: $\Omega_0 = \mathbf{R}^N$ or $\Omega_0 \ne \mathbf{R}^N$. Firstly, we consider the case $\Omega = \mathbf{R}^N$. By direct computations, we have

$$||w_\epsilon||^2_{V_0,\mathbf{R}^N} = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N}\int_\Lambda \epsilon^2|\nabla v_\epsilon|^2 + \left(\frac{\epsilon}{g(\epsilon)}\right)^2 V_0\left(\frac{x}{g(\epsilon)}\right)v_\epsilon^2(x)\,dx.$$

Thus, from (1.15), for any $\delta \in (0,1)$, we obtain

$$||w_\epsilon||^2_{(1-\delta)V_0,\mathbf{R}^N} \le \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N}||v_\epsilon||^2_{\epsilon,\mathbf{R}^N} + o(1). \tag{4.10}$$

On the other hand, we have

$$||w_\epsilon||^{p+1}_{L^{p+1}(\mathbf{R}^N)} = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N}||v_\epsilon||^{p+1}_{L^{p+1}(\Lambda)}. \tag{4.11}$$

From (4.10)–(4.11) we get

$$b_{(1-\delta)V_0,\mathbf{R}^N} \le \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon,\Lambda} + o(1).$$

(4.8) follows from the fact $b_{(1-\delta)V_0,\mathbf{R}^N} \to b_{V_0,\mathbf{R}^N}$ as $\delta \to 0$. Next, we consider the case $\Omega_0 \ne \mathbf{R}^N$. Since $||w_\epsilon||_{V_\epsilon,\mathbf{R}^N} = 1$, we can choose a subsequense $\epsilon_n \to 0$ and $w_0(x) \in H^1(\mathbf{R}^N)$ such that

$$w_{\epsilon_n}(x) \to w_0(x) \quad \text{weakly in } H^1(\mathbf{R}^N) \text{ and strongly in } L^{p+1}_{loc}(\mathbf{R}^N) \qquad (4.12).$$

From (1.18) and (4.12), we see $w_{\epsilon_n}(x) \to w_0(x)$ strongly in $L^{p+1}(\mathbf{R}^N)$. Moreover, from (1.14), for any closed set $E \supset\supset \Omega_0$ we find

$$||w_{\epsilon_n}||^2_{L^2(\Lambda_{g(\epsilon_n)}\setminus E)} \le \frac{1}{\inf_{x\in\Lambda_{g(\epsilon_n)}\setminus E} V_\epsilon(x)} \int_{\Lambda_{g(\epsilon_n)}\setminus E} V_\epsilon(x) w_{\epsilon_n}(x)^2\,dx \to 0.$$

In particular, we find $w_0 \in H^1_0(\Omega_0)$ and $w_{0+} \not\equiv 0$. Thus we have

$$\begin{aligned} b_{V_0,\Omega_0} &\le \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{||w_0||_{V_0,\Omega_0}}{||w_{0+}||_{L^{p+1}(\Omega_0)}}\right)^{\frac{2(p+1)}{p-1}} \\ &\le \liminf_{\epsilon_n\to 0}\left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{||w_{\epsilon_n}||_{\epsilon_n,\Lambda_{g(\epsilon_n)}}}{||w_{\epsilon_n+}||_{L^{p+1}(\Lambda_{g(\epsilon_n)})}}\right)^{\frac{2(p+1)}{p-1}} \\ &= \liminf_{\epsilon_n\to 0}\left(\frac{g(\epsilon_n)}{\epsilon_n}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon_n)^{-N}\left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{||v_{\epsilon_n}||_{\epsilon_n,\Lambda}}{||v_{\epsilon_n+}||_{L^{p+1}(\Lambda)}}\right)^{\frac{2(p+1)}{p-1}} \\ &= \liminf_{\epsilon_n\to 0}\left(\frac{g(\epsilon_n)}{\epsilon_n}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon_n)^{-N} c_{\epsilon_n,\Lambda}. \end{aligned} \tag{4.13}$$

Since (4.13) is not depend on subsequense, (4.8) holds. From (4.4) and (4.8), we get (4.4) and complete the proof of Proposition 4.2. ∎

To prove Theorem 1.5, we define a rescaled functional

$$\widetilde{\Phi}_{\Lambda_{g(\epsilon)}}(w) = \frac{1}{2}||w||^2_{V_\epsilon,\Lambda_{g(\epsilon)}} - \frac{1}{p+1}||w_+||^{p+1}_{L^{p+1}(\Lambda_{g(\epsilon)})} : H^1(\Lambda_{g(\epsilon)}) \to \mathbf{R}.$$

We also define a functional corresponding to the limit problem $(L)_{V_0,\Omega_0}$ by

$$I_{V_0,\Omega_0}(w) = \frac{1}{2}||w||^2_{V_0,\Omega_0} - \frac{1}{p+1}||w_+||^{p+1}_{L^{p+1}(\Omega_0)} : E_{V_0,\Omega_0} \cap H^1_0(\Omega_0) \to \mathbf{R}.$$

Now we have the following proposition.

Proposition 4.3. *Suppose* $(w_\epsilon(y)) \subset H^1(\Lambda_{g(\epsilon)})$ *satisfies*

$$||w_\epsilon||^{p+1}_{L^{p+1}(\Lambda_{g(\epsilon)})} = b_{V_0,\Omega_0} + o(1), \tag{4.14}$$

$$||w_\epsilon||^2_{V_\epsilon,\Lambda_{g(\epsilon)}} \leq b_{V_0,\Omega_0} + o(1), \tag{4.15}$$

$$\widetilde{\Phi}'_{\Lambda_{g(\epsilon)}}(w_\epsilon)\varphi = 0 \quad \text{for all } \varphi \in H^1_0(\Lambda_{g(\epsilon)}). \tag{4.16}$$

If (L) holds, then after extracting a subsequence* $\epsilon_n \to 0$, *there exist* $x_n \in \Lambda_{\epsilon_n}$ *and a least energy solution* $w_0 \in H^1_0(\Omega_0)$ *of* $(L)_{V_0,\Omega_0}$ *such that*

$$\epsilon_n x_n \to 0,$$
$$||w_{\epsilon_n} - w_0(\cdot - x_n)||_{H^1(\Lambda_{\epsilon_n})} \to 0 \quad \text{as } \epsilon_n \to 0,$$
$$I_{V_0,\Omega_0}(w_0) = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right)b_{V_0,\Omega_0}, \quad I'_{V_0,\Omega_0}(w_0) = 0.$$

To prove Proposition 4.3, we will use the following lemma.

Lemma 4.4. *Suppose* $w \in H^1_0(\Omega_0) \setminus \{0\}$ *satisfies* $I'_{V_0,\Omega_0}(w)w \leq 0$. *Then*

$$||w||^2_{V_0,\Omega_0} \geq b_{V_0,\Omega_0}.$$

Proof. $I'_{V_0,\Omega_0}(w)w \leq 0$ implies $||w||^2_{V_0,\Omega_0} \leq ||w_+||^{p+1}_{L^{p+1}(\Omega_0)}$. Thus, from the definition of b_{V_0,Ω_0}, we have

$$b_{V_0,\Omega_0} \leq \left(\frac{||w||_{V_0,\Omega_0}}{||w_+||_{L^{p+1}(\Omega_0)}}\right)^{\frac{2(p+1)}{p-1}} \leq ||w||^2_{V_0,\Omega_0}. \quad \blacksquare$$

Proof of Proposition 4.3. For the case $\Omega_0 = \mathbf{R}^N$, we show the proposition. For the case $\Omega_0 \neq \mathbf{R}^N$, we can show it by a similar way. We use concentration compactness argument. From (4.14)–(4.16) and (1.15)–(1.17), we can easily see that there exists $x_\epsilon = (0, \cdots, 0, x_{\epsilon,k}, \cdots, x_{\epsilon,N}) \in \mathbf{R}^N$ such that for large $R > 0$, $\liminf_{\epsilon\to 0} ||w_\epsilon||_{L^{p+1}(B_R(x_\epsilon))} > 0$.

From (4.15)–(4.16), there exist a subsequence $\epsilon_n \to 0$ and $w_0 \in H^1(\mathbf{R}^N)$ such that for any bounded set $D \subset \mathbf{R}^N$,

$$v_{\epsilon_n}(x) = w_{\epsilon_n}(x + x_{\epsilon_n}) \to w_0(x) \quad \text{weakly in } H^1(D) \text{ and strongly in } L^{p+1}(D). \tag{4.17}$$

For any $\varphi \in C_0^1(\Omega_0)$ we have

$$\int_{\Lambda_{g(\epsilon_n)}+x_{\epsilon_n}} \nabla v_{\epsilon_n} \cdot \nabla\varphi + V_{\epsilon_n}(x + x_{\epsilon_n})v_{\epsilon_n}\varphi\, dx - \int_{\Lambda_{g(\epsilon_n)}+x_{\epsilon_n}} v_{\epsilon_n+}^p \varphi\, dx = 0.$$

Since $\operatorname{supp}\varphi$ is compact, from (1.13), (1.15)–(1.17), $w_0(x)$ satisfies

$$\int_{\mathbf{R}^N} \nabla w_0 \cdot \nabla\varphi + V_0(x)w_0\varphi\, dx - \int_{\mathbf{R}^N} w_{0+}^p \varphi\, dx \le 0,$$

that is, $I'_{V_0,\Omega_0}(w_0)w_0 \le 0$. From Lemma 4.4 and (4.15), it follows that

$$b_{V_0,\mathbf{R}^N} \le ||w_0||^2_{V_0,\mathbf{R}^N} \le \liminf_{\epsilon_n\to 0} ||v_{\epsilon_n}||^2_{V_{\epsilon_n},\Lambda_{g(\epsilon_n)}+x_{\epsilon_n}} \le b_{V_0,\mathbf{R}^N}. \tag{4.18}$$

By (4.17) and (4.18), we can see

$$\lim_{\epsilon_n\to 0} ||v_{\epsilon_n} - w_0||^2_{V_{\epsilon_n},\Lambda_{g(\epsilon_n)}+x_{\epsilon_n}} = 0.$$

From (4.1), we also obtain $\lim_{\epsilon_n\to 0} ||v_{\epsilon_n} - w_0||_{H^1(\Lambda_{g(\epsilon_n)}+x_{\epsilon_n})} = 0$. Therefore we have $I'_{V_0,\Omega_0}(w_0) = 0$ and

$$\widetilde{\Phi}_{\Lambda_{g(\epsilon_n)}}(w_{\epsilon_n}) \to I_{V_0,\Omega_0}(w_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right) b_{V_0,\Omega_0}.$$

We complete the proof of Proposition 4.3. ∎

End of proof of Theorem 1.5. (i) of Theorem 1.5 follows from (4.3). Let $u_\epsilon(x)$ be a critical point of $\Psi_\epsilon(u)$ obtained in Theorem 1.1. We set $w_\epsilon(x) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2}{p-1}} u_\epsilon(g(\epsilon)x)$. Then from Proposition 4.2 and (3.5)–(3.6), $w_\epsilon|_{\Lambda_{g(\epsilon)}}(x)$ satisfies (4.14)–(4.16). This implies (ii) of Theorem 1.5. Thus we complete the proof of Theorem 1.5. ∎

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