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RADIAL AND NONRADIAL STEADY-STATES WITH CLUSTERING LAYERS IN ALLEN CAHN EQUATION

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1. INTRODUCTION

This is a joint work with Yihong Du (University of New England, Australia).

Consider the Allen-Cahn equation

$$-\epsilon^2\Delta u = u(u-a(|x|))(1-u) \text{ in } \Omega,\ \partial_\nu u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega = B_1$ denotes the unit ball in $\mathbf{R}^N$ $(N \geq 2)$, centered at the origin, $\nu = \nu(x)$ denotes the unit outer normal at $x \in \partial\Omega$, $\epsilon > 0$ is a small constant and $a(r)$ is a C^1 function satisfying $0 < a(r) < 1$ for $r \in [0,1]$. Therefore $a(|x|)$ is Lipschitz continuous in $\overline{B}_1$ and C^1 in $\overline{B}_1 \setminus \{0\}$.

Problem (1.1) arises from several applied fields and has been extensively investigated in the last two decades. In the one dimensional case, it is known that when $\epsilon > 0$ is small, there are solutions with sharp layers near those values of r such that $a(r) = 1/2$, and with sharp spikes near certain local extremum points of $a(r)$; these solutions are generally unstable, and their Morse indices can be calculated according to the number of layers and spikes they have (see [ACH] and [UNY] for further details). The stable solutions for the one dimensional case were earlier investigated in detail in [AMPP]. Relations between Morse indices and location of layers are first studied in [N]. It is shown in [N] that Morse indices of clustering layered solutions are completely determined by number of layers and a location of layers. Further related results for the one dimensional case can be found in [ABF], [HS], [NT] and the references therein. Much less is known for the higher dimensional case.

In [DY1] unstable solutions of (1.1) over the unit ball was studied, and it was shown that (1.1) has unstable radially symmetric solutions $u_\epsilon(r)$ with one or several sharp layers near a point $r_0 \in (0,1)$ where $a(r_0) = 1/2$ and $a'(r_0) \neq 0$. This result is similar to those in the one dimensional case. However, there exist fundamentally different properties of $u_\epsilon(r)$ between the one dimensional and high dimensional cases; its Morse index is one of these properties.

This article is based on [DN], where we treat the case that u_ϵ is a single layered unstable solution. In the one dimensional case, it is known that such a solution has bounded Morse index. In higher dimension, it is expected that the Morse index of u_ϵ goes to infinity as

$\epsilon \to 0$. In this paper, we will give some accurate estimates of the small eigenvalues of the linearized eigenvalue problem of (1.1) at u_ϵ, and obtain a rather sharp asymptotic formula for the Morse index of u_ϵ, which we denote by m^ϵ:

$$\lim_{\epsilon\to 0} m^\epsilon \epsilon^{(N-1)/2} = \mu^*,$$

where μ^* is a positive constant which can be calculated (see Theorem 3.9 for more details).

Our estimates for the small eigenvalues associated to u_ϵ have many other applications. In [DN2], we will show that Morse indices of these radial solutions of clustering layers are order of $\epsilon^{-\frac{N}{2}}$. Moreover in [KN], we will find nonradial solutions bifurcating from radial solutions.

Let us now describe u_ϵ more accurately. For convenience of notation, we often write

$$f(r,u) = u(u-a(r))(1-u); \;\; f(u) = u(u-1/2)(1-u).$$

Clearly

$$f'(0) = f'(1) = -1/2, \; \int_0^1 f(u)du = 0.$$

It is well known that the problem

$$-u'' = f(u), \; u' > 0 \text{ in } \mathbf{R}^1, \; u(0) = 1/2, \; u(-\infty) = 0, \; u(\infty) = 1 \tag{1.2}$$

has a unique solution $u = \phi(t)$, and it satisfies $\phi(t) + \phi(-t) = 1$ and

$$\begin{cases} \lim_{t\to\infty} e^{t/\sqrt{2}}[1-\phi(t)] = c_0, \; \lim_{t\to-\infty} e^{-t/\sqrt{2}}\phi(t) = c_0, \\ \lim_{t\to\pm\infty} e^{|t|/\sqrt{2}}\phi'(t) = c_0/\sqrt{2}, \; \lim_{t\to\pm\infty} e^{|t|/\sqrt{2}}\phi''(t) = \mp c_0/2, \end{cases} \tag{1.3}$$

where c_0 is a positive constant.

Moreover, since $(f'(\phi(t)) + 1/2) \to 0$ exponentially as $|t| \to \infty$, by standard theory on Schrödinger operators (see [LL]) the eigenvalue problem

$$-\psi'' = f'(\phi(t))\psi + \lambda\psi \text{ in } \mathbf{R}^1, \; \psi \in H^1(\mathbf{R}^1)$$

has a smallest eigenvalue λ_1, it corresponds to a positive eigenfunction, which is unique up to a multiplicative constant, and any other eigenvalue $\lambda < 1/2$ (if exists) is isolated and corresponds to eigenfunctions which change sign. It follows that 0 is the smallest eigenvalue with corresponding eigenfunctions $\psi(t) = \alpha\phi'(t)$, $\alpha \in \mathbf{R}^1$. The other eigenvalues (and real part of the spectrum) are positive and bounded away from 0.

Problem (1.1) has many radially symmetric solutions. The following result was proved by Dancer and Yan in [DY1].

Theorem A *Suppose that* $r_0 \in (0,1)$ *satisfies* $a(r_0) = 1/2$ *and* $a'(r_0) \neq 0$. *Then for any integer* $k > 0$, *there exists* $\epsilon_0 > 0$ *such that for* $\epsilon \in (0, \epsilon_0)$, *(1.1) has a solution of the form*

(i) $u_\epsilon = \overline{\psi}_{\epsilon,1} + \Sigma_{i=2}^k w_{\epsilon,i} + \omega_\epsilon$ *if* $a'(r_0) < 0$,

(ii) $u_\epsilon = \Sigma_{i=1}^{k-1} w_{\epsilon,i} + \psi_{\epsilon,k} + \omega_\epsilon$ *if* $a'(r_0) > 0$,
where ω_ϵ *is a "higher order term" satisfying*

$$\int_0^1 [\epsilon^2 \omega_\epsilon'(r)^2 + \omega_\epsilon(r)^2] r^{N-1} dr = o(\epsilon),$$

$w_{\epsilon,i} = \psi_{\epsilon,i} + \overline{\psi}_{\epsilon,i} - 1$ *has two sharp layers near* $r_i = r_{\epsilon,i}$ *and* $\overline{r}_i = \overline{r}_{\epsilon,i}$, *where for some constants* $\tau, M > 0$ *independent of* ϵ,

$$r_0 - M\epsilon \ln(1/\epsilon) \le r_{\epsilon,1} < \overline{r}_{\epsilon,1} < \cdots < r_{\epsilon,k} < \overline{r}_{\epsilon,k} \le r_0 + M\epsilon \ln(1/\epsilon),$$

$$r_{\epsilon,i} - \overline{r}_{\epsilon,i-1} \ge \tau\epsilon \ln(1/\epsilon),\ \overline{r}_{\epsilon,i} - r_{\epsilon,i} \ge \tau\epsilon \ln(1/\epsilon),$$

$$\psi_{\epsilon,i}(r) = \Psi_{\epsilon,r_i}(r),\ \overline{\psi}_{\epsilon,i}(r) = 1 - \Psi_{\epsilon,\overline{r}_i}(r),$$

and for $r_* \in (0,1)$, Ψ_{ϵ,r_*} *is a* C^2 *function satisfying*

$$\Psi_{\epsilon,r_*}(r) = \begin{cases} 0 & \textit{for } r \in [0, r_* - (R+1)\epsilon \ln(1/\epsilon)], \\ \phi(\frac{r-r_*}{\epsilon}) & \textit{for } r \in [r_* - R\epsilon \ln(1/\epsilon), r_* + R\epsilon \ln(1/\epsilon)], \\ 1 & \textit{for } r \in [r_* + (R+1)\epsilon \ln(1/\epsilon), 1], \end{cases}$$

with $R > 0$ *a large constant such that*

$$|\Psi_{\epsilon,r_*}^{(j)}(r)| = O(\epsilon^2) \textit{ for } j = 0,1,2,\ r \in [0, r_* - R\epsilon \ln(1/\epsilon)],$$

$$|[\Psi_{\epsilon,r_*}(r) - 1]^{(j)}| = O(\epsilon^2) \textit{ for } j = 0,1,2,\ r \in [r_* + R\epsilon \ln(1/\epsilon), 1].$$

Remark 1.1. The solutions in Theorem A are different from the minimizer (and hence stable) solutions of (1.1) obtained in [DY2]. By Theorems 1.3 and 1.4 of [DY2], it is easy to obtain the following result: If $a(r_0) = 1/2$ and $a'(r_0) < 0$, then (1.1) has a solution of the form

$$u_\epsilon = \psi_{\epsilon,1} + \omega_\epsilon;$$

if $a(r_0) = 1/2$ and $a'(r_0) > 0$, then (1.1) has a solution of the form

$$u_\epsilon = \overline{\psi}_{\epsilon,1} + \omega_\epsilon.$$

Estimate of small eigenvalues is an important topic in the stability analysis of patterned solutions in reaction diffusion systems, see, e.g. [NS] and the references therein. When the spatial domain is a ball, sharp estimates of the small eigenvalues are usually achievable, see, for instance [RW], where a system of elliptic equations are considered and the estimates are based on formal expansions of the eigenvalues and eigenfunctions in powers of ϵ. Our method here is significantly different. In a future paper, we will study the small eigenvalues of the linearized problem of (1.1) at a solution u_ϵ which has clustering layers near some $r_0 \in (0,1)$ as described in Theorem A above.

The rest of this paper is arranged as follows. In section 2, we give a good asymptotic approximation for the first eigenvalue of the linearized problem of (1.1) at u_ϵ. In section 3, we make use of polar coordinates and spherical harmonics to estimate the other small eigenvalues, and hence obtain an asymptotic expression for the Morse index of u_ϵ as $\epsilon \to 0$.

2. ESTIMATES OF THE FIRST EIGENVALUE FOR A SINGLE LAYERED SOLUTION

In this section, we provide some sharp estimates for the first eigenvalue of the linearized eigenvalue problem of (1.1) at a single layered unstable solution obtained from Theorem A. For definiteness, we assume that

$$r_0 \in (0,1),\ a(r_0) = 1/2,\ a'(r_0) > 0.$$

Then by Theorem A (ii), for all small $\epsilon > 0$, (1.1) has a solution of the form

$$u_\epsilon(r) = \psi_{\epsilon,1}(r) + \omega_\epsilon(r),$$

where

$$\psi_{\epsilon,1}(r) = \begin{cases} 0 & \text{for } r \in [0, r_1 - (R+1)\epsilon \ln(1/\epsilon)], \\ \phi(\frac{r-r_1}{\epsilon}) & \text{for } r \in [r_1 - R\epsilon\ln(1/\epsilon), r_1 + R\epsilon \ln(1/\epsilon)], \\ 1 & \text{for } r \in [r_1 + (R+1)\epsilon\ln(1/\epsilon), 1], \end{cases} \tag{2.1}$$

with $r_1 = r_1^\epsilon \in [r_0 - M\epsilon\ln(1/\epsilon), r_0 + M\epsilon\ln(1/\epsilon)]$ for some constants $R, M > 0$ independent of ϵ. Moreover, for $j = 0, 1, 2$,

$$\begin{cases} |\psi_{\epsilon,1}^{(j)}(r)| = O(\epsilon^2) & \text{for } r \in [0, r_1 - R\epsilon\ln(1/\epsilon)], \\ |[\psi_{\epsilon,1}(r) - 1]^{(j)}| = O(\epsilon^2) & \text{for } r \in [r_1 + R\epsilon\ln(1/\epsilon), 1]. \end{cases} \tag{2.2}$$

Furthermore, by standard elliptic estimates (as remarked in [DY1, Remark 4.2]),

$$\|\omega_\epsilon\|_\infty = o(1). \tag{2.3}$$

(The argument in Remark 4.2 of [DY1] has to be modified slightly though, since their rescaling of u does not quite yield (4.22) there.)

Lemma 2.1. *$\|\omega_\epsilon'\|_\infty = o(\epsilon^{-1})$, and hence, for all small $\epsilon > 0$, $u_\epsilon(r) = 1/2$ has a unique solution $r = r_\epsilon$, and $r_\epsilon = r_1^\epsilon + o(\epsilon)$.*

Proof. For any given function $v(r)$, $r \in [0,1]$, let us define

$$\tilde{v}(r) = v(\epsilon r),\ r \in [0, 1/\epsilon].$$

Then clearly

$$\begin{cases} -\tilde{u}_\epsilon'' - \frac{N-1}{r}\tilde{u}_\epsilon' = f(\epsilon r, \tilde{u}_\epsilon(r)),\ r \in (0, 1/\epsilon), \\ \tilde{u}_\epsilon'(0) = \tilde{u}_\epsilon'(1/\epsilon) = 0. \end{cases} \tag{2.4}$$

(Recall that $f(r,u) = u(u - a(r))(1-u)$.)

By (2.1), (2.2), (2.3) and the fact that

$$|r_1 - r_0| = O(\epsilon \ln(1/\epsilon)),\ f(r_0, u) = f(u),\ -\phi'' = f(\phi),$$

we easily see that

$$\begin{aligned} -\tilde{\psi}''_{\epsilon,1} - \frac{N-1}{r}\tilde{\psi}'_{\epsilon,1} &= f(r_0, \tilde{\psi}_{\epsilon,1}) - \frac{N-1}{r}\tilde{\psi}'_{\epsilon,1} + O(\epsilon^2) \\ &= f(r_1, \tilde{\psi}_{\epsilon,1}) - \frac{N-1}{r}\tilde{\psi}'_{\epsilon,1} + O(\epsilon \ln(1/\epsilon)) \end{aligned}$$

uniformly for $r \in (0, 1/\epsilon]$. Therefore, from $\tilde{\omega}_\epsilon = \tilde{u}_\epsilon - \tilde{\psi}_{\epsilon,1}$ we deduce

$$\begin{aligned} -\tilde{\omega}''_\epsilon - \frac{N-1}{r}\tilde{\omega}'_\epsilon &= f(\epsilon r, \tilde{u}_\epsilon) - f(r_1, \tilde{\psi}_{\epsilon,1}) + \frac{N-1}{r}\tilde{\psi}'_{\epsilon,1} + O(\epsilon \ln(1/\epsilon)) \\ &= f(\epsilon r, \tilde{\psi}_{\epsilon,1}) - f(r_1, \tilde{\psi}_{\epsilon,1}) + \frac{N-1}{r}\tilde{\psi}'_{\epsilon,1} + o(1). \end{aligned}$$

By (2.1) we find that $-\frac{N-1}{r}\tilde{\psi}'_{\epsilon,1} = 0$ if $|r - \frac{r_1}{\epsilon}| \geq (R+1)\ln(1/\epsilon)$. If $|r - \frac{r_1}{\epsilon}| \leq (R+1)\ln(1/\epsilon)$, then $| - \frac{N-1}{r}\tilde{\psi}'_{\epsilon,1}| = O(1/r) = O(\epsilon)$. Hence we have

$$|\frac{N-1}{r}\tilde{\psi}'_{\epsilon,1}| = O(\epsilon)$$

uniformly for all r.

If $|r - r_1/\epsilon| \geq R\ln(1/\epsilon)$, then by (2.1) and (2.2), $|\tilde{\psi}_{\epsilon,1}(r)| = O(\epsilon^2)$ or $|\tilde{\psi}_{\epsilon,1}(r) - 1| = O(\epsilon^2)$, and hence

$$|f(r_1, \tilde{\psi}_{\epsilon,1})|, |f(\epsilon r, \tilde{\psi}_{\epsilon,1})| = O(|\tilde{\psi}_{\epsilon,1}(r)||\tilde{\psi}_{\epsilon,1}(r) - 1|) = O(\epsilon^2).$$

If $|r - r_1/\epsilon| \leq R\ln(1/\epsilon)$, then $\tilde{\psi}_{\epsilon,1}(r) = \phi(r - r_1/\epsilon)$ and $|\epsilon r - r_1| \leq R\epsilon \ln(1/\epsilon) \to 0$ as $\epsilon \to 0$. Hence

$$\begin{aligned} & f(\epsilon r, \tilde{\psi}_{\epsilon,1}) - f(r_1, \tilde{\psi}_{\epsilon,1}) + \frac{N-1}{r}\tilde{\psi}'_{\epsilon,1} \\ = \ & f(\epsilon r, \phi(r - r_1/\epsilon)) - f(r_1, \phi(r - r_1/\epsilon)) + O(\epsilon) = o(1) \end{aligned}$$

uniformly for $|r - r_1/\epsilon| \leq R\ln(1/\epsilon)$. Thus we always have

$$|f(\epsilon r, \tilde{\psi}_{\epsilon,1}) - f(r_1, \tilde{\psi}_{\epsilon,1})| = o(1)$$

uniformly for $r \in [0, 1/\epsilon]$ as $\epsilon \to 0$.

Now from

$$-\tilde{\omega}''_\epsilon - \frac{N-1}{r}\tilde{\omega}'_\epsilon = o(1),\ \tilde{\omega}'_\epsilon(-r_1/\epsilon) = \tilde{\omega}'_\epsilon((1 - r_1)/\epsilon) = 0,$$

or equivalently

$$-\Delta\tilde{\omega}_\epsilon = o(1) \text{ in } B_{1/\epsilon},\ \partial_\nu \tilde{\omega}_\epsilon = 0 \text{ on } \partial B_{1/\epsilon},$$

and (2.3), we deduce by applying standard elliptic estimates (on bounded sets contained in $B_{1/\epsilon}$) that $\tilde{\omega}_\epsilon(r)$, $\tilde{\omega}'_\epsilon(r) \to 0$ uniformly for $r \in [0, 1/\epsilon]$ as $\epsilon \to 0$. Therefore $\epsilon\omega'_\epsilon(r) \to 0$ as $\epsilon \to 0$ uniformly for $r \in [0, 1]$.

By (2.1), (2.2) and (2.3), we find that

$$\tilde{u}_\epsilon(r) = o(1) \text{ uniformly for } r \in [0, (r_1/\epsilon) - R\ln(1/\epsilon)],$$

$$\tilde{u}_\epsilon(r) = 1 + o(1) \text{ uniformly for } r \in [(r_1/\epsilon) + R\ln(1/\epsilon), 1/\epsilon],$$

$$\tilde{u}_\epsilon(r) = \phi(r - r_1/\epsilon) + \tilde{\omega}_\epsilon(r) \text{ for } r \in [(r_1/\epsilon) - R\ln(1/\epsilon), (r_1/\epsilon) + R\ln(1/\epsilon)].$$

Suppose $\tilde{u}_\epsilon(\tilde{r}_\epsilon + r_1/\epsilon) = 1/2$. Then necessarily $\tilde{r}_\epsilon \in [-R\ln(1/\epsilon), R\ln(1/\epsilon)]$. Since $\tilde{\omega}_\epsilon(r), \tilde{\omega}_\epsilon'(r) = o(1)$ uniformly in $[0, 1/\epsilon]$, and $\phi(0) = 1/2, \phi'(r) > 0$, it follows from the implicit function theorem that $\tilde{r}_\epsilon$ is unique and $\tilde{r}_\epsilon = o(1)$. Denote $r_\epsilon = r_1 + \epsilon\tilde{r}_\epsilon$. Then $u_\epsilon(r_\epsilon) = 1/2$ and $r_\epsilon = r_1 + o(\epsilon)$. Moreover, $r = r_\epsilon$ is the unique solution of $u_\epsilon(r) = 1/2$ for all small $\epsilon > 0$. $\square$

Let λ_1^ϵ be the first eigenvalue of the linearized eigenvalue problem of (1.1) at u_ϵ, that is,

$$-\epsilon^2\psi'' - \epsilon^2\frac{N-1}{r}\psi' = f_u(r, u_\epsilon)\psi + \lambda_1^\epsilon\psi \text{ in } (0,1),\ \psi'(0) = \psi'(1) = 0 \tag{2.5}$$

for some $\psi > 0$, $\|\psi\|_\infty = 1$. We first have the following rough estimate.

Lemma 2.2. *There exist constants $C > 0$ and $C_\epsilon > -C$ such that $\lim_{\epsilon\to 0} C_\epsilon = 0$ and $-C \le \lambda_1^\epsilon \le C_\epsilon$ for all small $\epsilon > 0$.*

Proof. By the variational characterization of the first eigenvalue,

$$\lambda_1^\epsilon = \inf_{v \in H^1(B_1)\setminus\{0\}} \int_{B_1} [\epsilon^2|\nabla v|^2 - f_u(|x|, u_\epsilon)v^2]dx \Big/ \int_{B_1} v^2 dx. \tag{2.6}$$

A simple comparison argument shows that $0 < u_\epsilon < 1$. Since

$$f_u(|x|, u_\epsilon) \le C := \max_{r,t \in [0,1]} f_u(r,t),$$

we deduce from (2.6) that $\lambda_1^\epsilon \ge -C$.

Next we use $\psi_{\epsilon,1}'(|x|)$ as a test function to obtain an upper bound for λ_1^ϵ. Define

$$C_\epsilon := \int_{B_1} [\epsilon^2|\nabla v_0|^2 - f_u(|x|, u_\epsilon)v_0^2]dx \Big/ \int_{B_1} v_0^2 dx, \text{ where } v_0(x) = \psi_{\epsilon,1}'(|x|).$$

Clearly $\lambda_1^\epsilon \le C_\epsilon$. It remains to show that $C_\epsilon \to 0$ as $\epsilon \to 0$.

We have

$$\begin{aligned}
\int_{B_1} \psi'_{\epsilon,1}(|x|)^2 dx &= \int_0^1 \psi'_{\epsilon,1}(r)^2 r^{N-1} dr \\
&= O(\epsilon^4) + \int_{r_1 - R\epsilon\ln(1/\epsilon)}^{r_1 + R\epsilon\ln(1/\epsilon)} \epsilon^{-2}\phi'((r-r_1)/\epsilon)^2 r^{N-1} dr \\
&= O(\epsilon^4) + \int_{-R\ln(1/\epsilon)}^{R\ln(1/\epsilon)} \epsilon^{-1}\phi'(s)^2 (r_1 + \epsilon s)^{N-1} ds \\
&= O(\epsilon^4) + \epsilon^{-1}[r_0^{N-1} + o(1)]\Big[\int_{-\infty}^{\infty} \phi'(s)^2 ds + o(1)\Big] \\
&= \epsilon^{-1}[r_0^{N-1} + o(1)] \int_{-\infty}^{\infty} \phi'(s)^2 ds.
\end{aligned}$$

$$\begin{aligned}
\int_{B_1} \epsilon^2 |\nabla \psi'_{\epsilon,1}(|x|)|^2 dx &= \int_0^1 \epsilon^2 \psi''_{\epsilon,1}(r)^2 r^{N-1} dr \\
&= O(\epsilon^6) + \int_{r_1 - R\epsilon\ln(1/\epsilon)}^{r_1 + R\epsilon\ln(1/\epsilon)} \epsilon^{-2}\phi''((r-r_1)/\epsilon)^2 r^{N-1} dr \\
&= O(\epsilon^6) + \int_{-R\ln(1/\epsilon)}^{R\ln(1/\epsilon)} \epsilon^{-1}\phi''(s)^2 (r_1 + \epsilon s)^{N-1} ds \\
&= O(\epsilon^6) + \epsilon^{-1}[r_0^{N-1} + o(1)]\Big[\int_{-\infty}^{\infty} \phi''(s)^2 ds + o(1)\Big] \\
&= \epsilon^{-1}[r_0^{N-1} + o(1)] \int_{-\infty}^{\infty} \phi''(s)^2 ds.
\end{aligned}$$

$$\begin{aligned}
&\int_{B_1} f_u(|x|, u_\epsilon(|x|)) \psi'_{\epsilon,1}(|x|)^2 dx \\
&= \int_0^1 f_u(r, u_\epsilon(r)) \psi'_{\epsilon,1}(r)^2 r^{N-1} dr \\
&= O(\epsilon^4) + \int_{r_1 - R\epsilon\ln(1/\epsilon)}^{r_1 + R\epsilon\ln(1/\epsilon)} [f_u(r_0, \psi_{\epsilon,1}(r)) + o(1)]\epsilon^{-2}\phi'((r-r_1)/\epsilon)^2 r^{N-1} dr \\
&= O(\epsilon^4) + \int_{-R\ln(1/\epsilon)}^{R\ln(1/\epsilon)} [f_u(r_0, \phi(s)) + o(1)]\epsilon^{-1}\phi'(s)^2 (r_1 + \epsilon s)^{N-1} ds \\
&= O(\epsilon^4) + \epsilon^{-1}[r_0^{N-1} + o(1)]\Big[\int_{-\infty}^{\infty} f_u(r_0, \phi(s))\phi'(s)^2 ds + o(1)\Big] \\
&= \epsilon^{-1}[r_0^{N-1} + o(1)] \int_{-\infty}^{\infty} \phi''(s)^2 ds,
\end{aligned}$$

where we have used, in the last step, $f(r_0, u) = f(u)$, $-\phi'' = f(\phi)$, $-(\phi')'' = f'(\phi)\phi'$ and (1.2).

The above estimates clearly imply $C_\epsilon = o(1)$. ☐

Since the first eigenvalue λ_1^ϵ is simple, there is a unique function ψ satisfying (2.5) and $\psi > 0$, $||\psi||_\infty = 1$. Let us denote it by ψ_ϵ.

Let ϵ_n be a sequence of constants decreasing to 0 and denote $\psi_n = \psi_{\epsilon_n}$. Then there exists $r_n \in [0,1]$ such that $\psi_n(r_n) = 1 = ||\psi_n||_\infty$.

Lemma 2.3. $\lim_{n\to\infty} r_n = r_0$.

Proof. Choose $x_n \in B_1 = B_1(0)$ such that $|x_n| = r_n$. Then from

$$-\epsilon^2\Delta\psi_n = f_u(|x|, u_{\epsilon_n})\psi_n + \lambda_1^{\epsilon_n}\psi_n$$

we deduce

$$f_u(r_n, u_{\epsilon_n}(r_n)) + \lambda_1^{\epsilon_n} \geq 0 \tag{2.7}$$

if $r_n \in [0,1)$. If $r_n = 1$, then making use of the boundary condition $\psi_n'(1) = 0$ we deduce $\psi_n''(1) \leq 0$ and hence (2.7) holds for this case as well. We claim that (2.7) implies

$$|r_n - r_1^{\epsilon_n}| \leq M\epsilon_n \text{ for some } M > 0 \text{ and all } n. \tag{2.8}$$

Otherwise, by passing to a subsequence, we may assume that

$$\frac{|r_n - r_1^{\epsilon_n}|}{\epsilon_n} \to \infty. \tag{2.9}$$

By (2.3), $u_{\epsilon_n}(r_n) = \psi_{\epsilon_n,1}(r_n) + o(1)$. Thus in view of (2.1), (2.9) implies

$$u_{\epsilon_n}(r_n)[1 - u_{\epsilon_n}(r_n)] \to 0.$$

From the formula for $f_u(r_n, u_{\epsilon_n}(r_n))$ we easily see that it is close to $-a(r_n)$ when $u_{\epsilon_n}(r_n)$ is close to 0, and it is close to $a(r_n) - 1$ when $u_{\epsilon_n}(r_n)$ is close to 1. Therefore

$$\overline{\lim}_{n\to\infty} f_u(r_n, u_{\epsilon_n}(r_n)) \leq \sigma_0 < 0,$$

where

$$\sigma_0 = \max\{-\min_{r\in[0,1]} a(r), \max_{r\in[0,1]} a(r) - 1\}.$$

Making use of Lemma 2.2, we now deduce

$$\overline{\lim}_{n\to\infty}[f_u(r_n, u_{\epsilon_n}(r_n)) + \lambda_1^{\epsilon_n}] \leq \sigma_0 < 0,$$

which contradicts (2.7). This proves (2.8). Since $r_1^{\epsilon_n} = r_0 + O(\epsilon_n \ln \epsilon_n^{-1})$, we infer from (2.8) that $|r_n - r_0| = O(\epsilon_n \ln \epsilon_n^{-1}) = o(1)$. ☐

Lemma 2.4. *If we define* $\overline{\psi}_n(r) = \psi_n(r_n + \epsilon_n r)$, *then* $\overline{\psi}_n \to \phi'/\phi'(0)$ *in* $C^1_{loc}(\mathbf{R}^1)$. *Moreover,*

$$\lim_{n\to\infty} \lambda_1^{\epsilon_n} = 0, \quad \lim_{n\to\infty}(r_n - r_1^{\epsilon_n})/\epsilon_n = 0.$$

Proof. By the definition of $\overline{\psi}_n$, we have

$$\begin{cases} -\overline{\psi}_n'' - \epsilon_n \frac{N-1}{r_n+\epsilon_n r}\overline{\psi}_n' = f_u(r_n+\epsilon_n r, u_{\epsilon_n}(r_n+\epsilon_n r))\overline{\psi}_n + \lambda_1^{\epsilon_n}\overline{\psi}_n \text{ in } (-\frac{r_n}{\epsilon_n}, \frac{1-r_n}{\epsilon_n}), \\ \overline{\psi}_n'(-\frac{r_n}{\epsilon_n}) = \overline{\psi}_n'(\frac{1-r_n}{\epsilon_n}) = 0. \end{cases} \tag{2.10}$$

Due to Lemma 2.2, we may assume that $\lambda_1^{\epsilon_n} \to \tilde{\lambda}_1$ as $n \to \infty$. By passing to a subsequence, we have three possibilities:

$$\text{(i) } \lim_{n\to\infty} \frac{r_n - r_1^{\epsilon_n}}{\epsilon_n} = \infty, \quad \text{(ii) } \lim_{n\to\infty} \frac{r_n - r_1^{\epsilon_n}}{\epsilon_n} = -\infty, \quad \text{(iii) } \lim_{n\to\infty} \frac{r_n - r_1^{\epsilon_n}}{\epsilon_n} = c \in \mathbf{R}^1.$$

In case (i), $u_{\epsilon_n}(r_n+\epsilon_n r) \to 1$ uniformly for r in bounded sets of $\mathbf{R}^1$, and therefore, applying standard interior elliptic estimates to (2.10) and the Sobolev imbedding theorems, we can find a subsequence of $\{\overline{\psi}_n\}$ such that $\overline{\psi}_n \to \overline{\psi}$ in $C^1_{loc}(\mathbf{R}^1)$. From (2.10) we find that $\overline{\psi}$ satisfies

$$-\overline{\psi}'' = -(1/2)\overline{\psi} + \tilde{\lambda}_1\overline{\psi} \text{ in } \mathbf{R}^1, \ \overline{\psi}(0) = 1, \ 0 \le \overline{\psi} \le 1.$$

Since $\tilde{\lambda}_1 \le 0$ and $\overline{\psi}(0) = 1$, $\overline{\psi}'(0) = 0$, we can solve for $\overline{\psi}(r)$ to obtain a unique unbounded solution, which contradicts the fact that $0 \le \overline{\psi}(r) \le 1$.

Similarly, case (ii) leads to a contradiction. Therefore only case (iii) is possible. In such a case, firstly we can use (2.1) and (2.3) to see that $\overline{u}_n(r) := u_{\epsilon_n}(r_n+\epsilon_n r) \to \phi(r+c)$ uniformly in r. Moreover, as above, by passing to a subsequence, $\overline{\psi}_n \to \overline{\psi}$ in $C^1_{loc}(\mathbf{R}^1)$. It then follows from (2.10) that

$$-\overline{\psi}'' = f_u(r_0, \phi(r+c))\overline{\psi} + \tilde{\lambda}_1\overline{\psi} \text{ in } \mathbf{R}^1, \ \overline{\psi}(0) = 1, \ 0 \le \overline{\psi} \le 1. \tag{2.11}$$

Since $a(r_0) = 1/2$, $f_u(r_0, \phi(r+c)) = f'(\phi(r+c))$, and hence (2.11) can be rewritten as

$$-\overline{\psi}'' = f'(\phi(r+c))\overline{\psi} + \tilde{\lambda}_1\overline{\psi} \text{ in } \mathbf{R}^1, \ \overline{\psi}(0) = 1, \ 0 \le \overline{\psi} \le 1. \tag{2.12}$$

Since $\tilde{\lambda}_1 \in [-C, 0]$ and $f'(\phi(r+c)) \to -1/2$ as $|r| \to \infty$, and $0 \le \overline{\psi}(r) \le 1$, an elementary analysis of (2.12) shows that $\overline{\psi}(r) \to 0$ exponentially as $|r| \to \infty$. (This also follows from a simple application of Lemma 2.5 below.) Therefore we must have $\tilde{\lambda}_1 = 0$ and $\overline{\psi}(r) = \alpha\phi'(r+c)$ for some $\alpha > 0$, since the only possible solution $\psi \in H^1(\mathbf{R}^1)$ of the problem

$$-\psi'' = f'(\phi(r+c))\psi + \lambda\psi \text{ in } \mathbf{R}^1, \ \psi(0) = 1, \ 0 \le \psi \le 1$$

is $\lambda = 0$ and $\psi(r) = \alpha\phi'(r+c)$ for some $\alpha > 0$.

We show next that $c = 0$. From the properties of $\phi(r)$ we see that $\max_{\mathbf{R}^1} \overline{\psi}(r) = \overline{\psi}(-c) = \alpha\phi'(0)$ and $\overline{\psi}(r) < \overline{\psi}(-c)$ for $r \ne c$. But we already have $\overline{\psi}(0) = 1 = \max_{\mathbf{R}^1} \overline{\psi}(r)$. Therefore we necessarily have $c = 0$, that is, $(r_n - r_1^{\epsilon_n})/\epsilon_n \to 0$ as $n \to \infty$. Since now $\overline{\psi}$ is uniquely determined, namely, $\overline{\psi}(r) = \phi'(r)/\phi'(0)$, we must have $\overline{\psi}_n \to \overline{\psi}$ in $C^1_{loc}(\mathbf{R}^1)$ for the entire original sequence. Similarly $(r_n - r_1^{\epsilon_n})/\epsilon_n \to 0$ for the entire original sequence. ☐

In order to prove our main result of this section, and also for later applications, we introduce another lemma whose proof is ommited.

Lemma 2.5. *Suppose that* v_n *satisfies*

$$v_n'' + \delta_n(t)v_n' = \alpha_n(t)v_n + f_n(t), \ |v_n(t)| \le M_0 \text{ in } [0, T_n], \tag{2.13}$$

where

$$\lim_{n\to\infty} T_n = \infty, \ |\delta_n(t)| \le M_1, \ |f_n(t)| \le M_2 e^{-\beta_0 t}, \ \alpha_n(t) \ge \alpha_0 \text{ in } [0, T_n], \tag{2.14}$$

and α_0, β_0, M_0, M_1, M_2 *are positive constants independent of* n, $\delta_n(t)$, $\alpha_n(t)$ *and* $f_n(t)$ *are continuous functions on* $[0, T_n]$. *Then for any given* $\xi \in (0,1)$ *we can find* $\epsilon_0 \in (0, \beta_0]$ *and* $C_0 > 0$ *such that*

$$|v_n(t)|, \ |v_n'(t)|, \ |v_n''(t)| \le C_0 e^{-\epsilon_0 t} \text{ for all } t \in [0, \xi T_n] \text{ and all large } n. \tag{2.15}$$

Theorem 2.6. $\lambda_1^\epsilon = \mu_0\epsilon + o(\epsilon)$, *where*

$$\mu_0 = -\frac{a'(r_0)}{6\int_{-\infty}^{\infty} \phi'(r)^2 dr}. \tag{2.16}$$

Proof. It suffices to show that $\lambda_1^{\epsilon_n}/\epsilon_n \to \mu_0$ for any decreasing sequence ϵ_n which converges to 0. Let $\{\epsilon_n\}$ be such a sequence and let r_n, $\psi_n(r)$ and $\overline{\psi}_n(r)$ be defined as in Lemmas 2.3 and 2.4 above. Then $\overline{\psi}_n$ satisfies (2.10) as before. In what follows, it is convenient for us to write

$$f(r,u) = f(u) + [\frac{1}{2} - a(r)](u - u^2), \ f_u(r,u) = f'(u) + [\frac{1}{2} - a(r)](1 - 2u).$$

Let us also denote $\overline{u}_n(r) = u_{\epsilon_n}(r_n + \epsilon_n r)$. Then from (2.10) we obtain

$$\begin{cases} -\overline{\psi}_n'' - \epsilon_n \frac{N-1}{r_n+\epsilon_n r}\overline{\psi}_n' = f_u(r_n + \epsilon_n r, \overline{u}_n(r))\overline{\psi}_n + \lambda_1^{\epsilon_n}\overline{\psi}_n \text{ in } (-\frac{r_n}{\epsilon_n}, \frac{1-r_n}{\epsilon_n}), \\ \overline{\psi}_n'(-\frac{r_n}{\epsilon_n}) = \overline{\psi}_n'(\frac{1-r_n}{\epsilon_n}) = 0, \ \overline{\psi}_n(0) = 1, \ 0 < \overline{\psi}_n(r) \le 1. \end{cases} \tag{2.17}$$

From the equation for u_{ϵ_n}, we obtain

$$\begin{cases} -\overline{u}_n'' - \epsilon_n \frac{N-1}{r_n+\epsilon_n r}\overline{u}_n' = f(\overline{u}_n) + [\frac{1}{2} - a(r_n + \epsilon_n r)](\overline{u}_n - \overline{u}_n^2), \\ \overline{u}_n'(-\frac{r_n}{\epsilon_n}) = \overline{u}_n'(\frac{1-r_n}{\epsilon_n}) = 0. \end{cases} \tag{2.18}$$

Differentiating (2.23) with respect to r we obtain, for $v_n(r) := \overline{u}_n'(r)$,

$$\begin{cases} -v_n'' - \epsilon_n \frac{N-1}{r_n+\epsilon_n r} v_n' + \epsilon_n^2 \frac{N-1}{(r_n+\epsilon_n r)^2} v_n = & f'(\overline{u}_n)v_n - \epsilon_n a'(r_n + \epsilon_n r)(\overline{u}_n - \overline{u}_n^2) \\ & +[\frac{1}{2} - a(r_n + \epsilon_n r)](1 - 2\overline{u}_n)v_n. \end{cases} \tag{2.19}$$

We show next that for all large $T_0, T > 0$ the following estimates hold:

$$|\overline{u}_n^{(i)}(r)| = O(e^{-2r/T}) \text{ uniformly for } r \in [-2T\ln(1/\epsilon_n), -T_0], i = 0, 1, 2, \tag{2.20}$$

$$|[1 - \overline{u}_n(r)]^{(i)}| = O(e^{-2r/T}) \text{ uniformly for } r \in [T_0, 2T\ln(1/\epsilon_n)], i = 0, 1, 2, \tag{2.21}$$

$$|v_n^{(i)}(r)| = O(e^{-2r/T}) \text{ uniformly for } |r| \in [T_0, 2T\ln(1/\epsilon_n)], i = 0, 1, 2, \tag{2.22}$$

$$|\overline{\psi}_n^{(i)}(r)| = O(e^{-2r/T}) \text{ uniformly for } |r| \in [T_0, 2T\ln(1/\epsilon_n)], i = 0, 1, 2. \tag{2.23}$$

To show (2.25), we define $V_n(r) = \overline{u}_n(-r - T_0)$ for $r \in [0, T_n]$ with $T_n = \frac{r_n}{2\epsilon_n} - T_0$ and $T_0 > 0$ to be specified later. Then

$$V_n'' + \delta_n(r)V_n' = \alpha_n(r)V_n, \; 0 \le V_n \le 1 \text{ in } [0, T_n],$$

where for $r \in [0, T_n]$,

$$\delta_n(r) := \epsilon_n \frac{N-1}{r_n - \epsilon_n(r + T_0)} \to 0 \text{ uniformly in } r \text{ as } n \to \infty,$$

and since $\overline{u}_n(r)$ is close to 0 for large negative r,

$$\begin{aligned} \alpha_n(r) : &= -\left[\overline{u}_n(-r - T_0) - a(r_n - \epsilon_n(r + T_0))\right]\left[1 - \overline{u}_n(-r - T_0)\right] \\ &\ge (1/2) \min_{[0,1]} a > 0 \end{aligned}$$

if T_0 is chosen large enough. Therefore we can apply Lemma 2.5 to find $C_0, \epsilon_0 > 0$ such that

$$|V_n^{(i)}(r)| \le C_0 e^{-\epsilon_0 r} \; \forall r \in [0, (1/2)T_n], \; i = 0, 1, 2.$$

It follows that

$$|\overline{u}_n^{(i)}(s)| \le C_0 e^{-\epsilon_0(-s - T_0)} = C_1 e^{-\epsilon_0 |s|} \; \forall s \in [-(1/2)T_n - T_0, -T_0], \; i = 0, 1, 2.$$

Choose $T = 2/\epsilon_0$. Then for all large n,

$$[-2T\ln(1/\epsilon_n), -T_0] \subset [-(1/2)T_n - T_0, -T_0],$$

and hence

$$|\overline{u}_n^{(i)}(r)| \le C_1 e^{-\epsilon_0 |r|} \text{ for } r \in [-2T\ln(1/\epsilon_n), -T_0], \; i = 0, 1, 2.$$

This proves (2.25).

To prove (2.26), we consider $V_n(r) := 1 - \overline{u}_n(r + T_0)$. Then by (2.23) we obtain

$$V_n'' + \delta_n(r)V_n' = \alpha_n(r)V_n$$

with

$$\delta_n(r) = \epsilon_n \frac{N-1}{r_n + \epsilon_n(r + T_0)},$$

$$\alpha_n(r) = \overline{u}_n(r + T_0)[\overline{u}_n(r + T_0) - a(r_n + \epsilon_n(r + T_0))].$$

Then (2.26) follows from a similar argument to that used to prove (2.25) above.

Since $v_n = \overline{u}_n'$, (2.27) follows directly from (2.25) and (2.26) when $i = 0, 1$. For v_n'', we can use (2.24) and the estimates for u_n, v_n and v_n'.

Finally we can prove (2.28) by making use of (2.22) and Lemma 2.5 much as above.

Let us now denote $R_n = T\ln(1/\epsilon_n)$ and note that, by (2.3) and Lemma 2.1,

$$\overline{u}_n(r) \to \phi(r),\ v_n(r) = \overline{u}_n'(r) \to \phi'(r) \text{ uniformly for } r \in [-R_n, R_n] \text{ as } n \to \infty,$$

which imply, by (2.24),

$$v_n''(r) = \overline{u}_n''(r) \to f(\phi(r)) \text{ uniformly for } r \in [-R_n, R_n].$$

Moreover, by Lemma 2.4,

$$\overline{\psi}_n \to \phi'/\phi'(0) \text{ in } C^1_{loc}(\mathbf{R}^1).$$

We now use integration by parts and (2.24)–(2.28) to obtain

$$\begin{aligned}
&\int_{-R_n}^{R_n} \left[-\overline{\psi}_n'' - \epsilon_n \tfrac{N-1}{r_n+\epsilon_n r}\overline{\psi}_n' \right] v_n dr \\
&= \int_{-R_n}^{R_n} -v_n''\overline{\psi}_n dr + \int_{-R_n}^{R_n} \left(\epsilon_n \tfrac{N-1}{r_n+\epsilon_n r} v_n\right)'\overline{\psi}_n dr + O(\epsilon_n^2) \\
&= \int_{-R_n}^{R_n} \left[-v_n'' + \epsilon_n \tfrac{N-1}{r_n+\epsilon_n r} v_n' - \epsilon_n^2 \tfrac{N-1}{(r_n+\epsilon_n r)^2} v_n \right] \overline{\psi}_n dr + O(\epsilon_n^2) \\
&= \int_{-R_n}^{R_n} \Big[2\epsilon_n \tfrac{N-1}{r_n+\epsilon_n r} v_n' - 2\epsilon_n^2 \tfrac{N-1}{(r_n+\epsilon_n r)^2} v_n + f'(\overline{u}_n) v_n - \epsilon_n a'(r_n+\epsilon_n r)(\overline{u}_n - \overline{u}_n^2) \\
&\quad + \left[\tfrac{1}{2} - a(r_n+\epsilon_n r)\right](1-2\overline{u}_n)v_n \Big] \overline{\psi}_n dr + O(\epsilon_n^2).
\end{aligned} \tag{2.24}$$

On the other hand, by (2.22) we have

$$\begin{aligned}
&\int_{-R_n}^{R_n} \left[-\overline{\psi}_n'' - \epsilon_n \tfrac{N-1}{r_n+\epsilon_n r}\overline{\psi}_n' \right] v_n dr \\
&= \int_{-R_n}^{R_n} f'(\overline{u}_n)\overline{\psi}_n v_n dr + \int_{-R_n}^{R_n} \left[\tfrac{1}{2} - a(r_n+\epsilon_n r)\right](1-2\overline{u}_n) v_n \overline{\psi}_n dr \\
&\quad + \lambda_1^{\epsilon_n} \int_{-R_n}^{R_n} v_n \overline{\psi}_n dr.
\end{aligned} \tag{2.25}$$

Combining (2.29) and (2.30) we deduce

$$\begin{aligned}
&\int_{-R_n}^{R_n} \left[2\epsilon_n \tfrac{N-1}{r_n+\epsilon_n r} v_n' - 2\epsilon_n^2 \tfrac{N-1}{(r_n+\epsilon_n r)^2} v_n - \epsilon_n a'(r_n+\epsilon_n r)(\overline{u}_n - \overline{u}_n^2) \right] \overline{\psi}_n dr \\
&= \lambda_1^{\epsilon_n} \int_{-R_n}^{R_n} \overline{\psi}_n v_n dr + O(\epsilon_n^2).
\end{aligned} \tag{2.26}$$

We further have

$$\begin{aligned}
\int_{-R_n}^{R_n} 2\epsilon_n \tfrac{N-1}{r_n+\epsilon_n r} v_n' \overline{\psi}_n dr &= 2\epsilon_n \left(\tfrac{N-1}{r_0} + o(1) \right) \int_{-R_n}^{R_n} v_n' \overline{\psi}_n dr \\
&= 2\epsilon_n \left(\tfrac{N-1}{r_0} + o(1) \right) \left[\int_{-R_n}^{R_n} f(\phi(r)) \tfrac{\phi'(r)}{\phi'(0)} dr + o(1) \right] \\
&= 2\epsilon_n \left(\tfrac{N-1}{r_0} + o(1) \right) \left[\phi'(0)^{-1} \int_0^1 f(\phi) d\phi + o(1) \right] \\
&= o(\epsilon_n), \text{ since } \int_0^1 f(\phi)d\phi = 0,
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
\int_{-R_n}^{R_n} 2\epsilon_n^2 \tfrac{N-1}{(r_n+\epsilon_n r)^2} v_n \overline{\psi}_n dr &= 2\epsilon_n^2 \left(\tfrac{N-1}{r_0^2} + o(1) \right) \left[\int_{-R_n}^{R_n} \phi'(r)^2 \phi'(0)^{-1} dr + o(1) \right] \\
&= O(\epsilon_n^2),
\end{aligned} \tag{2.28}$$

$$\begin{aligned}&\textstyle\int_{-R_n}^{R_n} -\epsilon_n a'(r_n+\epsilon_n r)(\overline{u}_n-\overline{u}_n^2)\overline{\psi}_n dr\\ &= -\epsilon_n[a'(r_0)+o(1)]\textstyle\int_{-R_n}^{R_n}(\overline{u}_n-\overline{u}_n^2)\overline{\psi}_n dr\\ &= -\epsilon_n[a'(r_0)+o(1)]\Big[\textstyle\int_{-R_n}^{R_n}(\phi-\phi^2)\phi'\phi'(0)^{-1}dr+o(1)\Big]dr\\ &= -\epsilon_n[a'(r_0)+o(1)]\phi'(0)^{-1}\Big[\textstyle\int_0^1(\phi-\phi^2)d\phi+o(1)\Big]\\ &= -\epsilon_n[a'(r_0)+o(1)]\phi'(0)^{-1}[\tfrac{1}{6}+o(1)]\\ &= -\tfrac{1}{6}a'(r_0)\phi'(0)^{-1}\epsilon_n+o(\epsilon_n),\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\textstyle\int_{-R_n}^{R_n}\overline{\psi}_n v_n dr &= \textstyle\int_{-R_n}^{R_n}\phi'(r)^2\phi'(0)^{-1}dr+o(1)\\ &= \textstyle\int_{-\infty}^{\infty}\phi'(r)^2\phi'(0)^{-1}dr+o(1).\end{aligned} \tag{2.30}$$

Substituting (3.32)-(3.35) into (3.31), we obtain

$$-\frac{1}{6}a'(r_0)\phi'(0)^{-1}\epsilon_n+o(\epsilon_n)=\lambda_1^{\epsilon_n}\Big[\phi'(0)^{-1}\int_{-\infty}^{\infty}\phi'(r)^2dr+o(1)\Big].$$

Thus,

$$\lambda_1^{\epsilon_n}=-\frac{1}{6}a'(r_0)\epsilon_n\Big[\int_{-\infty}^{\infty}\phi'(r)^2dr\Big]^{-1}+o(\epsilon_n).$$

□

Remark 2.7. If u_ϵ is a stable solution of the form $u_\epsilon=\overline{\psi}_{\epsilon,1}+\omega_\epsilon$ as given in Remark 1.1, then we can similarly prove that

$$\lambda_1^\epsilon=|\mu_0|\epsilon+o(\epsilon), \tag{2.31}$$

with μ_0 determined by (2.21).

3. MORSE INDEX OF A SINGLE LAYERED UNSTABLE SOLUTION

Let u_ϵ be as in Section 2. We now consider the eigenvalue problem

$$-\epsilon^2\Delta\Phi=f_u(|x|,u_\epsilon)\Phi+\lambda\Phi \text{ in } B_1,\ \partial_\nu\Phi|_{\partial B_1}=0. \tag{3.1}$$

Here Φ is not assumed to be radially symmetric. It is well known that (3.1) has a sequence of different eigenvalues $\lambda_1^\epsilon<\lambda_2^\epsilon<...$, with λ_1^ϵ the principal eigenvalue whose corresponding eigenfunction ψ_ϵ can be chosen positive, and $\lambda_k^\epsilon\to\infty$ as $k\to\infty$. Moreover, ψ_ϵ is radially symmetric and therefore solves (2.5). Any other eigenvalue λ_k^ϵ corresponds to a finite number of linearly independent sign-changing eigenfunctions which span a finite dimensional space H_k^ϵ. Note that we have $H_1^\epsilon=span\{\psi_\epsilon\}$. Denote $m_k^\epsilon=dim(H_k^\epsilon)$, and suppose $\lambda_j^\epsilon<0$, $\lambda_{j+1}^\epsilon\geq 0$; then

$$m^\epsilon:=\Sigma_{i=1}^j m_k^\epsilon$$

is called the Morse index of u_ϵ. The Morse index gives the dimension of the unstable manifold of u_ϵ as a steady-state solution of the parabolic problem corresponding to (1.1). Therefore it is a measure of the stability of u_ϵ.

In order to estimate the Morse index, and more importantly, in order to construct solutions of (1.1) which are perturbations of u_ϵ with sharp spikes, we need to obtain good estimates to all the λ_k^ϵ which are close to 0 for small $\epsilon > 0$. To this end we make use of polar coordinates :

$$x = (r,\xi),\ r = |x|,\ \xi \in S^{N-1}$$

and the Laplace-Beltrami operator $\Delta_{S^{N-1}}$ on the unit sphere S^{N-1}. We have

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}.$$

It is well-known (see, e.g., [T]) that the eigenvalues of $-\Delta_{S^{N-1}}$ are $\sigma_k = k(k+N-2)$, $k = 0,1,2,\ldots$, and the corresponding eigenfunctions of σ_k span the space of homogeneous and harmonic polynomials of degree k, which we denote by $\mathcal{H}^k$. Moreover, the following orthogonal decomposition holds

$$L^2(S^{N-1}) = \bigoplus_{k\geq 0}\mathcal{H}^k.$$

Now suppose that $\Phi = \Phi(r,\xi)$ is an eigenfunction of (3.1) corresponding to some eigenvalue λ. Clearly Φ is C^2 in $\overline{B}_1$. Given $\Psi_k \in \mathcal{H}^k$ define

$$A_k(r) = \int_{S^{N-1}} \Phi(r,\xi)\Psi_k(\xi)d\sigma(\xi).$$

Then $A_k \in C^2((0,1]) \cap C([0,1])$, $A_k'(1) = 0$ and

$$\Phi(r,\xi) = \Sigma_{k\geq 0}A_k(r)\Psi_k(\xi).$$

Moreover,

$$-\epsilon^2 A_k'' - \epsilon^2\frac{N-1}{r}A_k' + \epsilon^2\frac{\sigma_k}{r^2}A_k = f_u(r,u_\epsilon)A_k + \lambda A_k,\ \forall k \geq 0. \tag{3.2}$$

Since $\Phi \not\equiv 0$, there exists $k \geq 0$ such that $A_k \not\equiv 0$. This suggests that we should examine closely the eigenvalues of the problem

$$-\epsilon^2 A'' - \epsilon^2\frac{N-1}{r}A' + \epsilon^2\frac{\sigma}{r^2}A = f_u(r,u_\epsilon)A + \lambda A,\ 0 < r < 1, \tag{3.3}$$

with $A \in C^2((0,1]) \cap C([0,1])$ satisfying $A'(1) = 0$, and $\sigma > 0$. We will show later that if (λ, A) solves (3.3) with $\sigma = \sigma_k$, and if $\Psi_k \in \mathcal{H}^k$, then (λ,Φ) with $\Phi = A(r)\Psi_k(\xi)$ solves (3.1). Hence λ is an eigenvalue of (3.1) if and only if it is an eigenvalue of (3.3) with $\sigma = \sigma_k$ for some $k \geq 0$.

We would like to point out that $A_k'(0) = 0$ does not always hold (which was mistakenly assumed in some references). To make this point clear, we provide some detailed discussions on the behavior of any possible solution $A(r)$ of (3.3) near the singular point $r = 0$ of the equation. (We suspect that the conclusions in Lemmas 3.1-3.3 below are well known but have failed to locate a proper reference. In [DN] we have shown the entire proof.)

Lemma 3.1. *If* $\epsilon, \sigma > 0$ *are fixed and* $A \in C^2((0,1]) \cap C([0,1])$ *is a nontrivial solution to (3.3) for some* $\lambda \in \mathbf{R}^1$*, then* $A(0) = 0$.

Proof. Using an indirect argument, we assume that $A(0) \neq 0$. Without loss of generality we may assume that $A(0) > 0$. Now we choose $\delta \in (0,1)$ small enough such that

$$A(r) \geq (1/2)A(0), \ \epsilon^2\sigma - r^2[f_u(r, u_\epsilon(r)) + \lambda] \geq (1/2)\epsilon^2\sigma, \ \forall r \in [0, \delta].$$

It then follows from (3.3) that

$$\epsilon^2(r^{N-1}A')' = r^{N-3}\Big(\epsilon^2\sigma - r^2[f_u(r, u_\epsilon) + \lambda]\Big)A(r) \geq \epsilon^2 c r^{N-3} > 0$$

for all $r \in (0, \delta]$ and $c = (1/4)\sigma A(0) > 0$. Therefore

$$(r^{N-1}A')' \geq c r^{N-3}, \ \forall r \in (0, \delta]. \tag{3.4}$$

It follows that $A(r)$ cannot have a local maximum in $(0, \delta)$, for if it has a local maximum at $r_* \in (0, \delta]$), then by (3.4),

$$c r_*^{N-3} \leq r_*^{N-1}A''(r_*) + (N-1)r_*^{N-2}A'(r_*) \leq 0.$$

This implies that we have either $A'(r) \geq 0$ in $(0, \delta)$ or there exists $\delta_1 \in (0, \delta)$ such that $A'(r) \leq 0$ in $(0, \delta_1]$.

Consider now the case $A'(r) \geq 0$ in $(0, \delta)$. From (3.4) we deduce, for $0 < s < r < \delta$,

$$r^{N-1}A'(r) \geq r^{N-1}A'(r) - s^{N-1}A'(s) \geq c\int_s^r t^{N-3}dt.$$

When $N = 2$ this already gives a contradiction if we let $s \to 0$. If $N \geq 3$ then letting $s \to 0$ we deduce

$$r^{N-1}A'(r) \geq \frac{c}{N-2}r^{N-2}.$$

Hence $A'(r) \geq c_1 r^{-1}$ and

$$A(r) - A(s) \geq c_1 \ln(r/s) \to \infty \text{ as } s \to 0.$$

Thus the first case leads to a contradiction.

Consider next the second case that $A'(r) \leq 0$ in $(0, \delta_1]$. Integrating (3.4) over $[r, \delta_1]$ we deduce

$$-r^{N-1}A'(r) \geq \delta_1^{N-1}A'(\delta_1) - r^{N-1}A'(r) \geq \begin{cases} c\ln(\delta_1/r) & \text{if } N = 2, \\ c\frac{\delta_1^{N-2} - r^{N-2}}{N-2} & \text{if } N \geq 3. \end{cases}$$

Hence for $r \in (0, \delta_1/2)$ we have

$$-r^{N-1}A'(r) \geq c_2 > 0.$$

It follows that

$$A(s) - A(r) \geq c_2 \int_s^r t^{1-N} dt \to \infty \text{ as } s \to 0,$$

again a contradiction. This finishes the proof. □

For the proof of the following lemma see [DN].

Lemma 3.2. *Under the conditions of Lemma 3.1, there exists $\delta > 0$ small such that $A(r)$ is strictly monotone in $[0, \delta]$. Moreover, if*

$$\gamma = \frac{1}{2}(2 - N + \sqrt{(N-2)^2 + 4\sigma}),$$

then for any pair (γ^-, γ^+) satisfying $0 < \gamma^- < \gamma < \gamma^+$, we can find $M, M^-, M^+ > 0$ such that

$$M^+ r^{\gamma^+} \leq |A(r)| \leq M^- r^{\gamma^-} \text{ and } |A'(r)| \leq M r^{\gamma^- - 1} \; \forall r \in (0, \delta].$$

Lemma 3.3. *Suppose that (λ, A) solves (3.3) with $\sigma = \sigma_k$, $k \geq 1$, where $A \in C^2((0,1]) \cap C([0,1])$, $A'(1) = 0$. Then for any $\Psi_k \in \mathcal{H}^k$, (λ, Φ) with $\Phi = A(r)\Psi_k(\xi))$ solves (3.1) in the classical sense.*

Proof. Clearly Φ satisfies (3.1) in the classical sense over $\overline{B}_1 \setminus \{0\}$. It remains to show that 0 is a removable singularity of Φ. From classical results on removable singularity for linear elliptic equations (see [P]) it follows from $\Phi \in C(\overline{B}_1)$ that 0 is a removable singularity of Φ in the distributional sense, that is Φ is a solution of (3.1) over B_1 in the sense of distribution. Since $\sigma_k \geq \sigma_1 = N - 1$, we find that γ defined in Lemma 3.2 satisfies $\gamma \geq 1$. Therefore by Lemma 3.2, for any $\gamma^- \in (0,1)$, there exists $M > 0$ such that

$$|A(r)| \leq M r^{\gamma^-}, \; |A'(r)| \leq M r^{\gamma^- - 1} \text{ for all small } r > 0.$$

Since

$$\nabla \Phi = A'(r)\Psi_k(\xi)\xi + \frac{1}{r} A(r) \nabla_{S^{N-1}} \Psi_k(\xi),$$

the above estimates for $A(r)$ near $r = 0$ imply that $\Phi \in W^{1,p}(B_1)$ for any $p > 1$. Therefore Φ is a weak solution of (3.1). It then follows from standard regularity theory for elliptic equations that Φ is a classical solution of (3.1) in $\overline{B}_1$. □

We next consider the existence problem for (3.3). For later applications, we consider a more general problem.

$$\begin{cases} -\epsilon^2 A'' - \epsilon^2 \frac{N-1}{r} A' + \epsilon^\alpha \frac{\sigma}{r^2} A = f_u(r, u_\epsilon) A + \lambda A, \; 0 < r < 1, \\ A'(1) = 0, \; A \in C^2((0,1]) \cap C([0,1]), \end{cases} \tag{3.5}$$

where $\sigma > 0$ and $\alpha \in [1, 2]$.

Lemma 3.4. *Given $\sigma^* > 0$, there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0]$ and $\sigma \in [0, \sigma^*]$, $\alpha \in [1,2]$, (3.11) has a solution pair (λ, A) with $A(r) > 0$ in $(0,1]$. Moreover, if (λ_*, A_*) is another solution pair of (3.11) with $A_*(r) > 0$ in $(0,1]$, then $\lambda_* = \lambda$ and $A_* = \alpha A$ for some $\alpha > 0$.*

Proof. For $\xi \in (0,1)$ and $\epsilon, \sigma > 0$ let us consider the auxiliary problem over $(\xi, 1)$,

$$-\epsilon^2 A'' - \epsilon^2 \frac{N-1}{r} A' + \epsilon^\alpha \frac{\sigma}{r^2} A = f_u(r, u_\epsilon) A + \lambda A, \quad A(\xi) = 0, \; A'(1) = 0. \tag{3.6}$$

This is a regular eigenvalue problem, and let us denote its first eigenvalue by $\lambda_1^{\epsilon,\xi}$. By its variational characterization one easily sees that $\lambda_1^{\epsilon,\xi}$ varies continuously with ξ and is strictly increasing in ξ. Fix $\xi_0 \in (0, r_0)$. Then for any $\xi \in (0, \xi_0]$, the proof of Lemma 2.2 can be applied to (3.12) to conclude that there exists $C > 0$ and C_ϵ satisfying $\lim_{\epsilon \to 0} C_\epsilon = 0$, both independent of $\xi \in (0, \xi_0]$ and $\sigma \in [0, \sigma^*]$ and $\alpha \in [1,2]$, such that $\lambda_1^{\epsilon,\xi} \in [-C, C_\epsilon]$ for all $\xi \in (0, \xi_0]$. As in the proof of Lemma 2.3, we can find some $\epsilon_0 > 0$ small so that for $r \in [0, \xi_0]$ and $\epsilon \in (0, \epsilon_0]$,

$$f_u(r, u_\epsilon(r)) + C_\epsilon \le \sigma_0 < 0$$

for some negative constant σ_0. Hence

$$f_u(r, u_\epsilon(r)) + \lambda_1^{\epsilon,\xi} \le \sigma_0 < 0, \; \forall r \in [0, \xi_0], \; \forall \xi \in (0, \xi_0], \; \forall \epsilon \in (0, \epsilon_0]. \tag{3.7}$$

Fix $\epsilon \in (0, \epsilon_0]$ and let A_ξ be the corresponding eigenfunction of $\lambda_1^{\epsilon,\xi}$ with the properties $A_\xi(r) > 0$ in $(\xi, 1)$ and $\|A_\xi\|_\infty = 1$. We claim that when $\xi < \xi_0$, $A_\xi(r)$ is strictly increasing for r in $[\xi, \xi_0]$. Otherwise, due to $A'_\xi(\xi) > 0$ (by the Hopf boundary lemma) $A_\xi(r)$ must have a local maximum at some $r_* \in (\xi, \xi_0)$. It follows that $A''_\xi(r_*) \le 0$ and $A'_\xi(r_*) = 0$. But then (3.12) evaluated at $r = r_*$ leads to a contradiction to (3.13). Using this property of $A_\xi(r)$ and (3.11) and standard elliptic estimates, we can find a sequence $\xi_n \to 0$ such that $A_{\xi_n} \to A_0$ in $C^1_{loc}((0,1])$, and A_0 satisfies $\|A_0\|_\infty = 1$, $A_0(r) \ge 0$ in $(0,1]$ and

$$-\epsilon^2 A_0'' - \epsilon^2 \frac{N-1}{r} A_0' + \epsilon^\alpha \frac{\sigma}{r^2} A_0 = f_u(r, u_\epsilon) A_0 + \lambda_1^{\epsilon,0} A_0 \text{ in } (0,1], \; A_0'(1) = 0,$$

where $\lambda_1^{\epsilon,0} = \lim_{\xi \to 0} \lambda_1^{\epsilon,\xi} \in [-C, C_\epsilon]$. Moreover, $A_0(r)$ is nondecreasing in $(0, \xi_0]$. Therefore we must have $A_0 \in C([0,1])$. Standard elliptic regularity theory shows that $A_0 \in C^2((0,1])$. We can now apply Lemmas 3.1 and 3.2 to describe the behavior of $A_0(r)$ for r near 0.

It remains to show the uniqueness. Suppose that (λ_*, A_*) and (λ, A) are two pairs of solutions of (3.11) as described in the statement of the lemma. Suppose that $\lambda \neq \lambda_*$. By Lemma 3.2, the behavior of $A_*(r)$ and $A(r)$ for r near 0 allows us to use integration by parts to obtain

$$\int_0^1 (r^{N-1}A')'A_* dr = \int_0^1 (r^{N-1}A_*')'A dr.$$

Therefore we can multiply the equation for A by A_* and integrate over $[0,1]$ to deduce

$$(\lambda - \lambda_*)\int_0^1 A(r)A_*(r)r^{N-1}dr = 0.$$

But this is impossible since $A(r), A_*(r) > 0$ in $(0,1]$. This contradiction proves that $\lambda = \lambda_*$. Then by uniqueness of initial value problems for ordinary differential equations we find that $A_*(r) \equiv \alpha A(r)$ with $\alpha = A_*(1)/A(1)$. $\square$

From now on, we fix $\sigma^* > |\mu_0|/r_0^2$, where μ_0 is given by (2.21), and let $\epsilon_0 > 0$ be determined by Lemma 3.4. Then for any $\sigma \in [0,\sigma^*]$, $\alpha \in [1,2]$ and $\epsilon \in (0,\epsilon_0]$, (3.11) has a unique solution pair $(\lambda, A) = (\lambda_1^{\epsilon,\sigma,\alpha}, A^{\epsilon,\sigma,\alpha})$ with $A(r) > 0$ in $(0,1]$ and $\|A\|_\infty = 1$.

Let $\{\epsilon_n\} \subset (0,\epsilon_0]$ be a decreasing sequence converging to 0, and denote

$$\lambda^n = \lambda_1^{\epsilon_n,\sigma,\alpha},\ A_n = A^{\epsilon_n,\sigma,\alpha}.$$

Then we can find $\hat{r}_n \in (0,1]$ such that $A_n(\hat{r}_n) = 1$. An examination of the proof of Lemma 2.3 shows that the arguments used there carry over to (3.11) and we have

Lemma 3.5. $\lim_{n\to\infty} \hat{r}_n = r_0$ *uniformly for* $\sigma \in [0,\sigma^*]$ *and* $\alpha \in [1,2]$.

We now define $\overline{A}_n(r) = A_n(\hat{r}_n + \epsilon_n r)$. Then it is easy to check that the proof of Lemma 2.4 can be easily modified to show the following result.

Lemma 3.6. $\overline{A}_n \to \phi'/\phi'(0)$ *in* $C^1_{loc}(\mathbf{R}^1)$ *uniformly for* $\sigma \in [0,\sigma^*]$ *and* $\alpha \in [1,2]$. *Moreover,*

$$\lim_{n\to\infty} \lambda^n = 0,\ \lim_{n\to\infty} (\hat{r}_n - r_1^{\epsilon_n})/\epsilon_n = 0,$$

uniformly for $\sigma \in [0,\sigma^*]$ *and* $\alpha \in [1,2]$.

Finally an examination of the proof of Theorem 2.6 shows that the arguments there can be applied to (3.11). Thus we have

Theorem 3.7. *As* $\epsilon \to 0$, *we have*

$$\lambda_1^{\epsilon,\sigma,\alpha} = \mu_0\epsilon + o(\epsilon)\ \textit{if}\ \alpha \in (1,2],$$

$$\lambda_1^{\epsilon,\sigma,\alpha} = (\mu_0 + \sigma r_0^{-2})\epsilon + o(\epsilon)\ \textit{if}\ \alpha = 1.$$

Lemma 3.8. *Suppose that* $\sigma \in [0,\sigma^*]$, $\alpha \in [1,2]$, $\epsilon \in (0,\epsilon_0]$, *and let* $(\lambda_\epsilon, A_\epsilon)$ *be a solution pair to (3.11) with* $\lambda_\epsilon \neq \lambda_1^{\epsilon,\sigma,\alpha}$ *and* $\|A_\epsilon\|_\infty = 1$. *Then there exists* $\epsilon_0^* \in (0,\epsilon_0]$ *and* $\lambda_0 > 0$, *both independent of* (σ,α), *such that* $\lambda_\epsilon \geq \lambda_0$ *if* $\epsilon \in (0,\epsilon_0^*]$.

Proof. As in the proof of Lemma 3.4, from $\lambda_\epsilon \neq \lambda_1^{\epsilon,\sigma,\alpha}$ we easily deduce that

$$\int_0^1 A^{\epsilon,\sigma,\alpha}(r)A_\epsilon(r)dr = 0. \tag{3.8}$$

Hence $A_\epsilon(r)$ must change sign. Let $r_\epsilon \in (0,1)$ be the largest zero of $A_\epsilon(r)$. Then multiply (3.11) with $(\lambda, A) = (\lambda_\epsilon, A_\epsilon)$ by $r^{N-1}A^{\epsilon,\sigma,\alpha}$, integrate over $(r_\epsilon, 1)$ and use integration by parts, we deduce

$$\lambda_\epsilon > \lambda_1^{\epsilon,\sigma,\alpha}.$$

If the conclusion of the lemma is not true, then we can find $\epsilon_n \to 0$, $\sigma_n \in [0,\sigma^*]$ and $\alpha_n \in [1,2]$ such that $\overline{\lim}_{n\to\infty}\lambda_{\epsilon_n} \leq 0$. Since $\lambda_1^{\epsilon_n,\sigma_n,\alpha_n} \to 0$ as $n \to \infty$, we necessarily have $\lambda_{\epsilon_n} \to 0$.

Let $r_n^* \in (0,1]$ satisfy $|A_{\epsilon_n}(r_n^*)| = 1$. Replacing A_{ϵ_n} by $-A_{\epsilon_n}$ when necessary, we can assume that $A_{\epsilon_n}(r_n^*) = 1$. We can now argue as in the proof of Lemma 2.3 to show that $r_n^* \to r_0$ as $n \to \infty$. Define

$$\tilde{A}_n(r) = A_{\epsilon_n}(r_n^* + \epsilon_n r).$$

Then

$$\begin{cases} -\tilde{A}_n'' - \epsilon_n \frac{N-1}{r_n^*+\epsilon_n r}\tilde{A}_n' + \epsilon_n^{\alpha_n}\frac{\sigma_n}{(r_n^*+\epsilon_n r)^2}\tilde{A}_n = & f_u(r_n^* + \epsilon_n r, u_{\epsilon_n}(r_n^* + \epsilon_n r))\tilde{A}_n \\ & \qquad +\lambda_{\epsilon_n}\tilde{A}_n \text{ in } \left(-\frac{r_n^*}{\epsilon_n}, \frac{1-r_n^*}{\epsilon_n}\right), \\ \tilde{A}_n(0) = 1,\ \tilde{A}_n'(0) = 0,\ |\tilde{A}_n(r)| \leq 1. & \end{cases} \tag{3.9}$$

As in the proof of Lemma 2.4, by passing to a subsequence, we have three possibilities:

$$\text{(i)} \lim_{n\to\infty}\frac{r_n^* - r_1^{\epsilon_n}}{\epsilon_n} = \infty,\quad \text{(ii)} \lim_{n\to\infty}\frac{r_n^* - r_1^{\epsilon_n}}{\epsilon_n} = -\infty,\quad \text{(iii)} \lim_{n\to\infty}\frac{r_n^* - r_1^{\epsilon_n}}{\epsilon_n} = c \in \mathbf{R}^1.$$

In case (i), $u_{\epsilon_n}(r_n^* + \epsilon_n r) \to 1$ uniformly for r in bounded sets of $\mathbf{R}^1$, and we can use standard elliptic estimates to (3.15) and Sobolev imbedding theorems to conclude that, subject to a subsequence, $\tilde{A}_n \to \tilde{A}$ in $C_{loc}^1(\mathbf{R}^1)$ and $\tilde{A}$ satisfies

$$-\tilde{A}'' = -(1/2)\tilde{A} \text{ in } \mathbf{R}^1,\ \tilde{A}(0) = 1,\ \tilde{A}'(0) = 0, \tag{3.10}$$

and $-1 \leq \tilde{A}(r) \leq 1$. However, the unique solution of (3.16) is

$$\tilde{A}(r) = \frac{1}{2}(e^{r/\sqrt{2}} + e^{-r/\sqrt{2}}),$$

which is unbounded in $\mathbf{R}^1$. This contradiction shows that case (i) cannot occur. Similarly, case (ii) cannot occur.

Therefore we necessarily have case (iii). In such a case, $u_{\epsilon_n}(r_n^* + \epsilon_n r) \to \phi(r + c)$ uniformly in $r \in \mathbf{R}^1$. As before, we can use elliptic estimates and Sobolev imbedding theorems to conclude that, subject to a subsequence, $\tilde{A}_n \to \tilde{A}$ in $C_{loc}^1(\mathbf{R}^1)$, and $\tilde{A}$ satisfies

$$-\tilde{A}'' = f_u(r_0, \phi(r + c))\tilde{A} \text{ in } \mathbf{R}^1,\ \tilde{A}(0) = 1,\ \tilde{A}'(0) = 0,\ |\tilde{A}(r)| \leq 1. \tag{3.11}$$

Since

$$f_u(r_0, \phi(r+c)) = f'(\phi(r+c)) \to -1/2 \text{ as } |r| \to \infty,$$

we can apply Lemma 2.5 to (3.17) to deduce that $|\tilde{A}(r)| \to 0$ exponentially as $|r| \to \infty$. Therefore we can conclude from (3.17) that $\tilde{A}(r) = \gamma\phi'(r+c)$ for some $\gamma \neq 0$. From $\tilde{A}(0) = 1$ and $\tilde{A}'(0) = 0$ we further deduce that $c = 0$ and $\gamma = \phi'(0)^{-1}$. Therefore $\tilde{A}(r) = \phi'(r)/\phi'(0)$.

Next we use (3.14) to deduce a contradiction. Denote

$$A_n^*(r) = A^{\epsilon_n, \sigma_n, \alpha_n}(r_n^* + \epsilon_n r).$$

Then (3.14) gives

$$\int_{-r_n^*/\epsilon_n}^{(1-r_n^*)/\epsilon_n} (r_n^* + \epsilon_n r)^{N-1} A_n^*(r)\tilde{A}_n(r)dr = 0.$$

Since we are in case (iii), $r_n^* - r_1^{\epsilon_n} = o(\epsilon_n)$, and we easily see from Lemma 3.6 that $A_n^* \to \phi'/\phi'(0)$ in $C^1_{loc}(\mathbf{R}^1)$. We will show next that there exists $C, \delta > 0$ such that for all large n,

$$|\tilde{A}_n(r)| \le Ce^{-\delta|r|}\,\forall r \in \Big[-\frac{r_n^*}{\epsilon_n}, \frac{1-r_n^*}{\epsilon_n}\Big]. \tag{3.12}$$

If (3.18) is proved, then for any fixed $R > 0$,

$$\begin{aligned} 0 &= \int_{-r_n^*/\epsilon_n}^{(1-r_n^*)/\epsilon_n} (r_n^* + \epsilon_n r)^{N-1} A_n^*(r)\tilde{A}_n(r)dr \\ &\ge \int_{-R}^{R} (r_n^* + \epsilon_n r)^{N-1} A_n^*(r)\tilde{A}_n(r)dr - \Big(\int_{-r_n^*/\epsilon_n}^{-R} + \int_R^{(1-r_n^*)/\epsilon_n}\Big)Ce^{-\delta|r|}dr \\ &\to \int_{-R}^{R} r_0^{N-1}\Big[\frac{\phi'(r)}{\phi'(0)}\Big]^2 dr - 2C\frac{e^{-\delta R}}{\delta}. \end{aligned}$$

Therefore,

$$0 \ge \int_{-R}^{R} r_0^{N-1}\Big[\frac{\phi'(r)}{\phi'(0)}\Big]^2 dr - 2C\frac{e^{-\delta R}}{\delta},\ \forall R > 0.$$

Clearly this is impossible if R is large enough.

It remains to prove (3.18). Since $r_n^* - r_1^{\epsilon_n} = o(\epsilon_n)$, we have

$$u_{\epsilon_n}(r_n^* + \epsilon_n r) \to \phi(r) \text{ uniformly in } r \in \mathbf{R}^1 \text{ as } n \to \infty.$$

Using this and the properties of $f_u(t,u)$ for u close to 0 and 1, we easily see that

$$f_u(r_n^* + \epsilon_n r, u_{\epsilon_n}(r_n^* + \epsilon_n r)) \to f_u(r_0, \phi(r)) = f'(\phi(r))$$

uniformly for r in bounded sets of $\mathbf{R}^1$ as $n \to \infty$, and for fixed $T_0 > 0$, there exists $\alpha_0 > 0$ such that for all large n, say $n \ge n_0$, and $|r| \ge T_0$,

$$\alpha_n(r) := -f_u(r_n^* + \epsilon_n r, u_{\epsilon_n}(r_n^* + \epsilon_n r)) - \lambda_{\epsilon_n} \ge \alpha_0. \tag{3.13}$$

Clearly, for fixed $T > 0$, as $n \to \infty$,

$$\delta_n(r) := \epsilon_n \frac{N-1}{r_n^* + \epsilon_n r} \to 0 \text{ uniformly for } |r| \leq 2T \ln \epsilon_n^{-1}.$$

Hence, by enlarging n_0 if necessary, we can assume that

$$|\delta_n(r)| \leq 1, \ \forall |r| \leq 2T \ln \epsilon_n^{-1}, \ \forall n \geq n_0.$$

Now for $r \in [-2T \ln \epsilon_n^{-1}, -T_0] \cup [T_0, 2T \ln \epsilon_n^{-1}]$, we have

$$\tilde{A}_n'' + \delta_n(r) \tilde{A}_n' = \Big[\epsilon_n^{\alpha_n} \frac{\sigma_n}{(r_n^* + \epsilon_n r)^2} + \alpha_n(r) \Big] \tilde{A}_n \text{ and } |\tilde{A}_n(r)| \leq 1.$$

Therefore we can apply Lemma 2.5 to deduce that

$$|\tilde{A}_n(r)| \leq C_0 e^{-\delta_0 |r|} \text{ for } |r| \in [T_0, T \ln \epsilon_n^{-1}] \tag{3.14}$$

and some $C_0, \delta_0 > 0$. In particular,

$$|A_{\epsilon_n}(r_n^* - T\epsilon_n \ln \epsilon_n^{-1})| \leq C_0 e^{-\delta_0 T \ln \epsilon_n^{-1}}.$$

To estimate $\tilde{A}_n(r)$ for $r \in [-r_n^*/\epsilon_n, -T \ln \epsilon_n^{-1}]$, we let

$$\hat{A}_n(r) = A_{\epsilon_n}(\epsilon_n r), \ R_n = r_n^*/\epsilon_n - T \ln \epsilon_n^{-1}.$$

Then $\hat{A}_n(|x|)$ satisfies

$$-\Delta \hat{A}_n + \epsilon_n^{\alpha_n} \frac{\sigma_n}{|\epsilon_n x|^2} \hat{A}_n + \alpha_n(\epsilon_n |x|) \hat{A}_n = 0 \text{ for } 0 < |x| \leq R_n,$$

with $\alpha_n(\epsilon_n |x|) \geq \alpha_0 > 0$ for $n \geq n_0$.

Let v_n be the unique solution of

$$-\Delta v_n + \alpha_0 v_n = 0 \text{ for } |x| < R_n, \ v_n = |\hat{A}_n(R_n)| \text{ for } |x| = R_n.$$

Then v_n is radially symmetric and it is well-known that

$$0 < v_n(r) \leq C_1 v_n(R_n) e^{-\delta_1 (R_n - r)}$$

for some $C_1 > 0$, $\delta_1 \in (0, \alpha_0)$ and all $r \in (0, R_n)$. We may assume that $\delta_1 \leq \delta_0$. Therefore,

$$-\Delta v_n + \epsilon_n^{\alpha_n} \frac{\sigma_n}{|\epsilon_n x|^2} v_n + \alpha_n(\epsilon_n |x|) v_n \geq 0 \text{ in } B_{R_n} \setminus \{0\}.$$

Since $\hat{A}_n(0) = 0$, we can apply the comparison principle over $B_{R_n} \setminus \{0\}$ to conclude that

$$|\hat{A}_n(r)| \leq v_n(r) \ \forall r \in (0, R_n], \ \forall n \geq n_0.$$

Therefore, for $n \geq n_0$ and $r \in (0, R_n]$,

$$|\hat{A}_n(r)| \leq C_1 |\hat{A}_n(R_n)| e^{-\delta_1 (R_n - r)} \leq C_1 C_0 e^{-\delta_0 T \ln \epsilon_n^{-1} - \delta_1 (R_n - r)}.$$

Denote $C_2 = C_1 C_0$ and we obtain

$$|A_{\epsilon_n}(r)| = |\hat{A}_n(\frac{r}{\epsilon_n})| \leq C_2 e^{-\delta_1 \frac{r_n^* - r}{\epsilon_n}} \ \forall r \in (0, R_n], \forall n \geq n_0.$$

It follows that

$$|\tilde{A}_n(r)| \le C_2 e^{-\delta_1|r|}\ \forall r \in [-r_n^*/\epsilon_n, -T\ln \epsilon_n^{-1}],\ \forall n \ge n_0.$$

Together with (3.20), we have proved

$$|\tilde{A}_n(r)| \le C_2 e^{-\delta_1|r|}\ \forall r \in [-r_n^*/\epsilon_n, -T_0],\ \forall n \ge n_0.$$

Denote

$$T_n = T\ln \epsilon_n^{-1},\ T_n^* = (1-r_n^*)/\epsilon_n.$$

Then from (3.20) we have

$$|\tilde{A}_n(r)| \le C_0 e^{-\delta_0 r}\ \forall r \in [T_0, T_n], \forall n \ge n_0.$$

We now estimate $|\tilde{A}_n(r)|$ for $r \in [T_n, T_n^*]$. From the equation for $\tilde{A}_n$ (see (3.15)) we can write

$$\tilde{A}_n'' + \delta_n(r)\tilde{A}_n' = \tilde{\alpha}_n(r)\tilde{A}_n,\ |\tilde{A}_n(r)| \le 1,\ \tilde{A}_n'(T_n^*) = 0,$$

where

$$\delta_n(r) = \epsilon_n \frac{N-1}{r_n^* + \epsilon_n r} \to 0 \text{ uniformly for } r \in [T_n, T_n^*] \text{ as } n \to \infty,$$

and by (3.19),

$$\tilde{\alpha}_n(r) \ge \alpha_n(r) \ge \alpha_0 > 0 \text{ for } r \in [T_n, T_n^*] \text{ and } n \ge n_0.$$

Therefore we may assume that

$$|\delta_n(r)| \le 1,\ \tilde{\alpha}_n(r) \ge \alpha_0\ \forall r \in [T_n, T_n^*], \forall n \ge n_0.$$

Choose $\beta \in (0, \delta_0]$ such that $\beta(\beta+1) \le \alpha_0$. Then define

$$w_n(r) = A_n e^{-\beta r} + B_n e^{\beta r}$$

with

$$A_n = \frac{|\tilde{A}(T_n)|}{e^{-\beta T_n} + e^{-\beta(2T_n^* - T_n)}},\ B_n = e^{-2\beta T_n^*} A_n.$$

It is easily checked that, for all large n,

$$w_n'' + \delta_n(r) w_n' \le \tilde{\alpha}_n(r) w_n \text{ in } [T_n, T_n^*],\ w_n(T_n) = |\tilde{A}_n(T_n)|,\ w_n'(T_n^*) = 0.$$

It then follows from the comparison principle that

$$|\tilde{A}_n(r)| \le w_n(r)\ \forall r \in [T_n, T_n^*].$$

Clearly

$$A_n \le |\tilde{A}_n(T_n)| e^{\beta T_n} \le C_0 e^{-\epsilon_0 T_n + \beta T_n} \le C_0,\ B_n \le C_0 e^{-2\beta T_n^*}.$$

Therefore

$$w_n(r) \le C_0 e^{-\beta r} + C_0 e^{-2\beta T_n^* + \beta r} \le 2C_0 e^{-\beta r}$$

for all large n and all $r \in [T_n, T_n^*]$. Thus, for all large n,

$$|\tilde{A}_n(r)| \leq 2C_0 e^{-\beta r} \text{ in } [T_n, T_n^*].$$

The estimates for $|\tilde{A}_n(r)|$ over $[-T_0, T_0]$ is trivial since $|\tilde{A}_n(r)| \leq 1$. □

From Lemma 3.3, Theorem 3.7 and Lemma 3.8, we find that the eigenvalues of (3.1) which are close to zero when $\epsilon > 0$ is small are $\lambda_1^{\epsilon,\sigma_k,2}$, $k = 0, 1, 2, \ldots.$ Moreover, from Theorem 3.7, for any given small $\delta > 0$, if $\sigma_k \leq r_0^2(|\mu_0| - \delta)\epsilon^{-1}$, then

$$\lambda_1^{\epsilon,\sigma_k,2} \leq \lambda_1^{\epsilon,r_0^2(|\mu_0|-\delta)\epsilon^{-1},2} = \lambda_1^{\epsilon,r_0^2(|\mu_0|-\delta),1} = -\delta\epsilon + o(\epsilon) < 0 \tag{3.15}$$

for all small $\epsilon > 0$, and if $\sigma_k \geq r_0^2(|\mu_0| + \delta)\epsilon^{-1}$, then

$$\lambda_1^{\epsilon,\sigma_k,2} \geq \lambda_1^{\epsilon,r_0^2(|\mu_0|-\delta)\epsilon^{-1},2} = \lambda_1^{\epsilon,r_0^2(|\mu_0|+\delta),1} = \delta\epsilon + o(\epsilon) > 0 \tag{3.16}$$

for all small $\epsilon > 0$. Here we have used the following property of $\lambda_1^{\epsilon,\sigma,\alpha}$:

$$\sigma \geq \sigma' \text{ implies } \lambda_1^{\epsilon,\sigma,\alpha} \geq \lambda_1^{\epsilon,\sigma',\alpha},$$

which follows from the proof of Lemma 3.4 and the corresponding property of the first eigenvalue of (3.12).

Let

$$N(\lambda) := \Sigma_{k \leq \lambda} dim(\mathcal{H}^k).$$

Then by the well-known asymptotic estimate for eigenvalues (see Theorem 3.1 in [T]),

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{(N-1)/2}} = \frac{|S^{N-1}|}{\Gamma(\frac{N+1}{2})(4\pi)^{(N-1)/2}}. \tag{3.17}$$

We are now ready to give an asymptotic estimate for the Morse index m^ϵ of u_ϵ as $\epsilon \to 0$.

Theorem 3.9.

$$\lim_{\epsilon \to 0} \frac{m^\epsilon}{\epsilon^{-(N-1)/2}} = \left(\frac{r_0^2|\mu_0|}{4\pi}\right)^{(N-1)/2} \frac{|S^{N-1}|}{\Gamma(\frac{N+1}{2})}.$$

Proof. From (3.21) and (3.22) we see that

$$m^\epsilon = N\Big(r_0^2|\mu_0|\epsilon^{-1} + o(\epsilon^{-1})\Big).$$

The conclusion then follows from (3.23). □

Remark 3.10. Our results remain the same if B_1 is replaced by a general ball $B_R := \{x \in R^N : |x| < R\}$ or by an annulus $A_{R_0,R} := \{x \in R^N : R_0 < |x| < R\}$. In the case of B_R, we simply change $0 < r < 1$ to $0 < r < R$ everywhere. Note that this does not affect our proofs, and more importantly, this does not change our asymptotic formulas for the eigenvalues (the parameters in our formulas are independent of the value of R). In the case of $A_{R_0,R}$, the situation is simpler. For example, Lemmas 3.1-3.4 become trivial, since the singularity at $r = 0$ disappears in the equation. On the other hand, all our arguments carry over easily; we simply replace $0 < r < 1$ by $R_0 < r < R$ and $A'(R_0) = 0$.

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