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Kyoto University
Decay Rates of the Derivatives of the Solutions of the Heat Equations and Related Topics

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1 Introduction

In this paper, we consider the initial-boundary value problem of the heat equation in the exterior domain of a ball,

\[
\begin{cases}
\frac{\partial}{\partial t} u = \Delta u - V(|x|)u & \text{in } \Omega_L \times (0, \infty), \\
\mu u + (1 - \mu) \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial \Omega_L \times (0, \infty), \\
u(\cdot, 0) = \phi(\cdot) \in L^p(\Omega_L),
\end{cases}
\]

where $0 \leq \mu \leq 1$, $p \geq 1$, $\Omega_L = \{x \in \mathbb{R}^N : |x| > L\}$, $N \geq 2$, $L > 0$, and $\nu$ is the outer unit normal vector to $\partial \Omega_L$. Throughout this paper, we assume that $V = V(|x|)$ satisfies the following condition $(V_\omega^\ell)$ for some $\omega \geq 0$ and $\ell \in \mathbb{N}$:

\[
(V_\omega^\ell) \begin{cases}
(0) & V = V(|x|) \in C^\ell(\mathbb{R}^N), \ V \geq 0 \text{ in } \mathbb{R}^N, \\
(i) & \lim_{r \to \infty} r^2 V(r) = \omega, \\
(ii) & \int_L^\infty r \left| V(r) - \frac{\omega}{r^2} \right| \, dr < \infty, \\
(iii) & \sup_{r \geq L} r^{2+j} \left( \frac{d^j}{dr^j} V \right) (r) < \infty, \quad j = 1, \ldots, \ell.
\end{cases}
\]
The purpose of this paper is to study the decay rates of the derivatives of the solution of (1.1) under the condition \((V_{\omega}^{l})\), as \(t \to \infty\).

Now, we introduce some notations. For any set \(A\) and \(B\), let \(f = f(\lambda, \nu)\) and \(g = g(\lambda, \mu)\) be maps from \(A \times B\) to \((0, \infty)\). Then we say

\[ f(\lambda, \mu) \preceq g(\lambda, \mu) \text{ for all } \lambda \in A \]

if, for any \(\mu \in B\), there exists a positive constant \(C\) such that \(f(\lambda, \mu) \leq C g(\lambda, \mu)\) for all \(\lambda \in A\). Furthermore, we say

\[ f(\lambda, \mu) \asymp g(\lambda, \mu) \text{ for all } \lambda \in A \]

if \(f(\lambda, \mu) \preceq g(\lambda, \mu)\) and \(g(\lambda, \mu) \preceq f(\lambda, \mu)\) for all \(\lambda \in A\). We put

\[
N_{0} = \mathbb{N} \cup \{0\}, \quad N_{0}^{N} = \{(n_{1}, \ldots, n_{N}) : n_{i} \in N_{0}, i = 1, \ldots, N\}.
\]

Furthermore, for any \(j = (j_{1}, \ldots, j_{N}) \in N_{0}^{N}\), we write \(|j| = \sum_{i=1}^{N} j_{i}\) and \(\nabla_{x}^{j} = \partial^{j_{1}} \partial_{1}^{j_{1}} \cdots \partial_{N}^{j_{N}}\).

To state historical remarks, let \(\Omega\) be an unbounded domain in \(\mathbb{R}^{N}\). Then, under the suitable assumptions on \(\Omega\) and \(V\), for any \(j \in N_{0}^{N}\), the solution \(u\) of (1.1) in the domain \(\Omega\) satisfies

\[
|||\nabla_{x}^{j}u(\cdot, t)|||_{L^{\infty}(\Omega)} \preceq t^{-\frac{N}{2p}} ||\phi||_{L^{p}(\Omega)}
\]

for all sufficiently large \(t\). (See Theorem 10.1 of Chapters 3 and 4 in [6].)

On the other hand, for the case when \(\Omega = \mathbb{R}^{N}\) (or \(\Omega = \mathbb{R}_{+}^{N}\)) and \(V \equiv 0\), the explicit representation of the fundamental solution of the heat equation implies that, for any \(j \in N_{0}^{N}\),

\[
|||\nabla_{x}^{j}u(\cdot, t)|||_{L^{\infty}(\mathbb{R}^{N})} \preceq t^{-\frac{N}{2p} - \frac{|j|}{2}} ||\phi||_{L^{p}(\mathbb{R}^{N})}
\]

for all \(t > 0\). Furthermore, for the case when \(\Omega\) is a convex domain in \(\mathbb{R}^{N}\) and \(V \equiv 0\), Li and Yau [7] studied the behavior of the nonnegative solution of (1.1) with \(\mu = 0\), and obtained the inequality

\[
\frac{|\nabla_{x}u|^{2}}{u^{2}} - \frac{\partial_{t}u}{u} \leq \frac{1}{t}, \quad (x, t) \in \Omega \times (0, \infty).
\]

Then, by the standard arguments in the parabolic equations, we see that, for any \(j \in N_{0}^{N}\) with \(|j| \leq 1\), the inequality (1.3) holds for all \(t > 0\).

On the other hand, Grigor'yan and Saloff-Coste [2] studied the asymptotic behavior of the Green function \(G_{\mu}^{V} = G_{\mu}^{V}(x, y, t)\) of (1.1) for the case
when $\Omega$ is the exterior domain of a compact set, $\mu = 1$, and $V \equiv 0$. They proved that, for any fixed $x, y \in \Omega$,
\[
G_1^V(x, y, t) \asymp t^{-\frac{N}{2}}
\]
for all sufficiently large $t$ if $N \geq 3$. This together with the mean value theorem, the Dirichlet boundary condition, and (1.2) implies that
\[
\|(\nabla_x G_1^V)(\cdot, \cdot, t)\|_{L^\infty(\Omega \times \Omega)} \asymp t^{-\frac{N}{2}}
\]
for all sufficiently large $t$. So we see that the solution of (1.1) with $\mu = 1$ does not necessarily satisfy the inequality (1.3) even for the case $|j| = 1$. The first author of this paper studied the asymptotic behavior of the solution of the heat equation under the Neumann boundary condition in the exterior domain of a ball in [3]. His results imply that, for the case $\mu = 0$ and $V \equiv 0$ on $\Omega_L$, the inequality (1.3) does not necessarily hold for the case $|j| = 2$. Recently, Shibata and Shimizu [8] studied the decay properties of the Stokes semigroup in the exterior domain of a compact set, under the Neumann boundary condition. Their results are applicable to the heat equation, and we see that the inequality (1.3) holds for the case when $N \geq 3$, $\Omega$ is the exterior domain of a compact set, $V \equiv 0$ on $\Omega$, and $\mu = 0$. Our motivation is how the decay rate is affected in the presence of $V$ under various boundary conditions.

Let $u_\mu^V = u_\mu^V(x, t : \phi)$ be a solution of the initial-boundary value problem (1.1) in the exterior domain $\Omega_L$. For any $p \geq 1$ and $t > 0$, put
\[
\||(\nabla_x^j u_\mu^V)(\cdot, t)\|_{L^\infty(\Omega_L)} = \sup\{||\nabla_x^j u_\mu^V(\cdot, t : \emptyset)||_{L^\infty(\Omega_L)} : ||\phi||_{L^p(\Omega_L)} = 1\},
\]
where $j \in \mathbb{N}_0^N$.

Let $\Delta_{\mathbb{S}^{N-1}}$ be the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$ and $\\{\omega_k\}_{k=0}^{\infty}$ the eigenvalues of
\[
-\Delta_{\mathbb{S}^{N-1}} Q = \omega Q \quad \text{on} \quad \mathbb{S}^{N-1}, \quad Q \in L^2(\mathbb{S}^{N-1}),
\]
that is,
\[
\omega_k = k(N + k - 2), \quad k \in \mathbb{N}_0.
\]
Furthermore, let $\{Q_{k,i}\}_{i=1}^{l_k}$ and $l_k$ be the orthonormal system and the dimension of the eigenspace corresponding to $\omega_k$, respectively. Let $U_{\mu,L}^V(r)$ be a solution of the initial value problem for the ordinary differential equation,
\[
(OV) \quad \left\{
\begin{array}{ll}
\partial_r^2 U + \frac{N-1}{r} \partial_r U - V(r) U = 0 & \text{in } (L, \infty), \\
(\partial_r U)(L) = \mu, & U(L) = 1 - \mu,
\end{array}
\right.
\]
where $0 \leq \mu \leq 1$. Put

$$g(t : \omega) = (1 + t)^{-\frac{\alpha(\omega)}{2}}. \tag{1.7}$$

Here $\alpha = \alpha(\omega)$ is a nonnegative root of the equation $\alpha(\alpha + N - 2) = \omega$, that is,

$$\alpha(\omega) = \frac{-(N - 2) + \sqrt{(N - 2)^2 + 4\omega}}{2}. \tag{1.8}$$

Then, under the condition $(V^{1}_\omega)$, we see that

$$g(t : \omega) \asymp [U_{\mu,L}^{V}(t^{1/2})]^{-1}$$

for all sufficiently large $t$ (see Proposition 2.1).

Now, we give the main results of this paper for the case $N \geq 3$.

**Theorem 1.1** Let $N \geq 3$ and consider the initial-boundary value problem (1.1) under the condition $(V^{\ell}_\omega)$ with $\omega \geq 0$ and $\ell \in \mathbb{N}$. Let $p \geq 1$. Assume either

$$\mu \neq \frac{2n'}{2n' + L} \quad \text{or} \quad V(r) \neq \frac{\omega_{2n'}}{r^2} \quad \text{on} \quad [L, \infty) \tag{1.9}$$

for any $n' \in \mathbb{N}_0$ with $2n' \leq \ell + 1$. Then, for any $j \in \mathbb{N}^N_0$ with $|j| \leq \ell + 1$,

$$\|\nabla^j_x G_{\mu}^{V}(t)\|_{p \rightarrow \infty} \preceq t^{-\frac{N}{2p} - \frac{|j|}{2}} \quad \text{if} \quad |j| \leq \alpha(\omega), \tag{1.10}$$

$$\|\nabla^j_x G_{\mu}^{V}(t)\|_{p \rightarrow \infty} \preceq t^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}} \quad \text{if} \quad |j| > \alpha(\omega) \tag{1.11}$$

for all sufficiently large $t$.

If, for some $n' \in \mathbb{N}_0$, the equalities hold in (1.9), we have another decay property.

**Theorem 1.2** Let $N \geq 3$ and consider the initial-boundary value problem (1.1). Assume that there exists a natural number $n'$ such that

$$n = 2n', \quad V(r) \equiv \frac{\omega_{n}}{r^2} \quad \text{on} \quad [L, \infty), \quad \mu = \frac{n}{n + L}. \tag{1.12}$$

Let $p \geq 1$. Then, for any $j \in \mathbb{N}^N_0$,

$$\|\nabla^j_x G_{\mu}^{V}(t)\|_{p \rightarrow \infty} \preceq t^{-\frac{N}{2p} - \frac{|j|}{2}} \quad \text{if} \quad |j| \leq n, \tag{1.13}$$

$$\|\nabla^j_x G_{\mu}^{V}(t)\|_{p \rightarrow \infty} \preceq t^{-\frac{N}{2p} - \frac{\alpha(\omega) + \omega_{1}}{2}} \quad \text{if} \quad |j| > n \tag{1.14}$$

for all sufficiently large $t$. 

Here we remark that, under the condition $(1.12)$, $V$ satisfies the condition $(V^\ell)$ for all $\ell \in \mathbb{N}$ and $\alpha(\omega) = \alpha(\omega_n) = n$. Furthermore, as a corollary of Theorems 1.1 and 1.2, we have

**Corollary 1.1** Let $N \geq 3$ and $u^V_\mu = u^V_\mu(x, t : \phi)$ be a solution of the initial-boundary value problem (1.1) with $\phi \in L^p(\Omega_L)$, under the condition $(V^\ell)$ with $\omega \geq 0$ and $\ell \in \mathbb{N}$. Let $p \geq 1$ and $j \in \mathbb{N}_0^N$ with $|j| \leq \ell + 1$. Then there exist positive constants $C$ and $T$ such that

$$
\| (\nabla_x^j u^V_\mu)(\cdot, t : \phi) \|_{L^\infty(\Omega_L)} \leq Ct^{-\frac{N}{2p} - \frac{|j|}{2}} \| \phi \|_{L^p(\Omega_L)}
$$

for all $t \geq T$ and all $\phi \in L^p(\Omega_L)$ if and only if, either $\omega \geq \omega_{|j|}$ or

$$
|j| = 1, \quad V(r) \equiv 0 \text{ on } [L, \infty), \quad \mu = 0.
$$

According to Corollary 1.1, we may say that results in [7] and [8] are exceptional cases.

For the decay rates of the derivatives of the solution for case $N = 2$, similar results and peculiar results are both obtained although we will not give any proofs to the results for $N = 2$.

We first consider the cases either

(1.15) \hspace{1cm} N = 2 \quad \text{and} \quad \omega > 0

or

(1.16) \hspace{1cm} N = 2, \quad \mu = 0, \quad \text{and} \quad V \equiv 0 \text{ on } [L, \infty).

**Theorem 1.3** Assume either (1.15) or (1.16). Then Theorems 1.1 and 1.2 hold true.

Next, we consider the cases either

(1.17) \hspace{1cm} (N, \omega) = (2, 0) \quad \text{and} \quad \mu > 0

or

(1.18) \hspace{1cm} (N, \omega, \mu) = (2, 0, 0) \quad \text{and} \quad V \not\equiv 0 \text{ on } [L, \infty).

Then we see that

$$
U^0_{\mu,L}(r) = 1 - \mu + \mu \log \left( \frac{r}{L} \right).
$$
THEOREM 1.4 Let $N = 2$ and consider the initial-boundary value problem (1.1) under the condition $(\tilde{V}_{\omega}^{\ell})$ with $\omega = 0$ and $\ell \in \mathbb{N}$. Let $p \geq 1$, and $R > L$. Assume either (1.17) or (1.18). Then, for any $j \in \mathbb{N}_{0}^{N}$ with $|j| \leq \ell + 1$,

$$
\|\nabla_{x}^{j}g_{\mu}^{V}(t)\|_{p} \lesssim t^{-\frac{1}{p}-\frac{|j|}{2}}
$$

for all sufficiently large $t$.

In Section 2, we give fundamental lemmas and propositions without proofs. For their proofs, readers consult Sections 2 and 3 of [5]. Section 3 is devoted to the large time behavior of a radial solution to (1.1) with a radial initial value and its derivatives. Upper estimates for proofs of Theorems 1.1 and 1.2 are given in Section 4 and their proofs are provided in Section 5. As concluding remarks, some related topics are stated in Section 6.

2 Preliminaries

In this section, we give preliminary lemmas, whose proofs can be seen in Section 2 of [5], in order to study the decay rates of the derivatives of the solution (1.1) for the case $N \geq 3$.

For any $\mu \in [0, 1]$, $R \geq L$, and $\omega \geq 0$, let $U_{\mu,R}^{\omega}$ be the solution of

$$(O_{\omega}) \quad \left\{ \begin{array}{l}
\partial_{r}^{2}U + \frac{N-1}{r}\partial_{r}U - \frac{\omega}{r^{2}}U = 0 \quad \text{in} \quad (R, \infty), \\
(\partial_{r}U)(R) = \mu, \quad U(R) = 1 - \mu.
\end{array} \right.$$

Put

$$
U_{+}^{\omega}(r) = \left( \frac{r}{L} \right)^{\alpha(\omega)}, \quad U_{-}^{\omega}(r) = \left( \frac{r}{L} \right)^{-\beta(\omega)},
$$

where $\beta(\omega) = N - 2 + \alpha(\omega)$. Then the functions $U_{+}^{\omega}(r)$ and $U_{-}^{\omega}(r)$ are solutions of the ordinary differential equation

$$
\partial_{r}^{2}U + \frac{N-1}{r}\partial_{r}U - \frac{\omega}{r^{2}}U = 0 \quad \text{in} \quad (0, \infty),
$$

and $U_{+}^{\omega}(r) \neq U_{-}^{\omega}(r)$ on $(0, \infty)$. So, by the uniqueness of the solution of $(O_{\omega})$, there exist constants $c_{1}$ and $c_{2}$ such that

$$
U_{\mu,R}^{\omega}(r) = c_{1}U_{+}^{\omega}(r) + c_{2}U_{-}^{\omega}(r), \quad r \geq R.
$$
Therefore, by $U_{\mu,R}^\omega(R) = 1 - \mu$ and $\partial_r U_{\mu,R}^\omega(R) = \mu$, we obtain

\begin{equation}
U_{\mu,R}^\omega(r) = \frac{\alpha - \mu \alpha - R \mu}{\alpha + \beta} \left( \frac{r}{R} \right)^{-\beta} + \frac{R \mu - \beta \mu + \beta}{\alpha + \beta} \left( \frac{r}{R} \right)^{\alpha}
\end{equation}

where $\alpha = \alpha(\omega)$ and $\beta = \beta(\omega)$. In what follows, we put

$U_{\mu,R}^{\omega,k}(r) = U_{\mu,R}^{\omega+k}(r)$, $U_+^{\omega,k}(r) = U_+^{\omega+k}(r)$, $U_-^{\omega,k}(r) = U_-^{\omega+k}(r)$,

for simplicity. Then we have the following lemma on $U_{\mu,R}^\omega$.

**Lemma 2.1** Let $L \leq R < S$ and $a, b \geq 0$. Assume $N \geq 3$. Then

\begin{equation}
U_{\mu,R}^{a,k}(r) \asymp U_{\mu,R}^{b,k}(r)
\end{equation}

for all $r \in [R, S]$, $\mu \in [0, 1]$, and $k \in \mathbb{N}_0$,

\begin{equation}
U_{\mu,R}^{a,k}(r) \asymp \left[ \frac{\mu}{k+1} + 1 - \mu \right] \left( \frac{r}{R} \right)^{\alpha(a+\omega_k)}
\end{equation}

for all $r \geq S$, $\mu \in [0, 1]$, and $k \in \mathbb{N}_0$, and

\begin{equation}
U_{0,R}^{a,k}(r) \asymp U_+^{a,k}(r)
\end{equation}

for all $r \geq R$ and $k \in \mathbb{N}_0$. Furthermore

\begin{equation}
0 \leq \frac{d}{dr} U_{\mu,R}^{a,k}(r) \leq \mu + (k + 1)(1 - \mu) \left( \frac{r}{R} \right)^{\alpha(a+\omega_k)-1},
\end{equation}

\begin{equation}
0 < U_{\mu,R}^{a,k}(r) \leq \left[ \frac{\mu}{k+1} + 1 - \mu \right] \left( \frac{r}{R} \right)^{\alpha(a+\omega_k)},
\end{equation}

for all $r > R$, $0 \leq \mu \leq 1$, and $k \in \mathbb{N}_0$.

Next we recall the following two lemmas on the decay rate of the solutions of the initial-boundary value problem (1.1) under the condition $(V_\omega^i)$.

**Lemma 2.2** Let $u_{\mu}^V$ be a solution of (1.1) under the condition $(V_\omega^1)$ with $\omega \geq 0$. Let $1 \leq p \leq q \leq \infty$ and $i = 1, 2, \ldots$. Then there exists a positive constant $C$, independent of $V$, such that

\begin{equation}
\|u_{\mu}^V(\cdot, t)\|_{L^q(\Omega_L)} \leq Ct^{-\frac{N}{2}(1-\frac{1}{q})} \|\phi\|_{L^p(\Omega_L)}
\end{equation}

for all $t > 0$. 

LEMMA 2.3 Let $u_{\mu}^{V}$ be a solution of (1.1) under the condition $(V_{\omega}^{\ell})$ with $\omega \geq 0$ and $\ell \geq 1$. Then, for any $\epsilon \in (0, 1)$ and $p \geq 1$, there exists a positive constant $C$ such that

\begin{equation}
|\langle \mathcal{P}^{j} \nabla^{i} u_{\mu}^{V} \rangle(x, t)| \leq Ct^{-\frac{N}{2p} - \frac{|j|}{2} - i} ||\phi||_{L^{p}(\Omega_{L})},
\end{equation}

for all $(x, t) \in \Omega_{L} \times (0, \infty)$ with $|x| \geq \epsilon t^{1/2} > L + 2$ and all $i \in \mathbb{N}_{0}$ and $j \in \mathbb{N}_{0}^{N}$ with $2i + |j| \leq \ell + 1$.

Next, we study the behavior of the solution $U_{\mu,L}^{V}(r)$ of $(O_{V})$ under the assumption $(V_{\omega}^{\ell})$. Put

$$V_{k}(r) = V(r) + \frac{\omega_{k}}{r^{2}}, \quad k \in \mathbb{N}_{0}.$$ 

In what follows, for $k \in \mathbb{N}_{0}$ and $\lambda \in \mathbb{R}$, we put

$$\alpha_{k} = \alpha(\omega + \omega_{k}), \quad \beta_{k} = N - 2 + \alpha_{k}, \quad h_{\lambda}(r) = V(r) - \frac{\lambda}{r^{2}}$$

for simplicity. We first prove the following lemma.

LEMMA 2.4 Let $R \geq L$, $a \geq 0$, and $k \in \mathbb{N}_{0}$. For any $g \in C([R, \infty))$, put

$$H_{R}^{a,k}[g](r) = U_{-}^{a,k}(r) \int_{R}^{r} s^{1-N}[U_{-}^{a,k}(s)]^{-2} \left( \int_{R}^{s} \tau^{N-1} U_{-}^{a,k}(\tau) g(\tau) d\tau \right) ds.$$

Then

(i) $H_{R}^{a,k}[g](r)$ is a solution of the ordinary differential equation

$$U'' + \frac{N - 1}{r} U' - \frac{a + \omega_{k}}{r^{2}} U = g \quad \text{in} \quad (R, \infty),$$

with $U(R) = U'(R) = 0$. In particular,

$$U_{\mu,R}^{V_{k}}(r) = U_{\mu,R}^{a,l}(r) + H_{R}^{a,l}[h_{\omega_{l} + a - \omega_{k}} U_{\mu,R}^{V_{k}}](r)$$

for all $r \geq R$, $k \in \mathbb{N}_{0}$, and $l = 0, \ldots, k$.

(ii) If $g(r) \geq 0$ on $[R, R_{1}]$ with $R_{1} > R$, then

\begin{equation}
H_{R}^{a,k}[g](r) \geq 0, \quad H_{R}^{a,k}[g](r) \geq 0, \quad R \leq r \leq R_{1}.
\end{equation}

(iii) Assume that there exists a positive constant $A$ such that

\begin{equation}
|g(r)| \leq A|h_{\lambda}(r)| U_{\mu,R}(r), \quad r \geq R.
\end{equation}
Then there exist positive constants $C_1$ and $C_2$, independent of $R$ and $k$, such that

\begin{equation}
|H_{R}^{a,k}[g](r)| \leq C_1 Ar^{-1}U_{\mu,R}^{a,k}(r) \int_{R}^{r} \tau|h_{a}(\tau)|d\tau,
\end{equation}

for all $r \geq R$.

In view of Lemma 2.4, we have the following proposition on the behavior of $U_{\mu,L}^{V_{k}}(r)$ as $r \to \infty$, by using the function $U_{\mu,L}^{a,k}(r) = U_{\mu,L}^{a(\omega+\omega_{k})}(r)$.

**PROPOSITION 2.1** Assume $(V_{\omega}^{1})$ with $\omega \geq 0$ and $N \geq 3$. Then

\begin{equation}
0 \leq (\partial_{r}U_{\mu,L}^{V_{k}})(r) \leq (k+1) \left( \frac{r}{L} \right)^{\alpha_{k}-1}
\end{equation}

for all $r > L$, $0 \leq \mu \leq 1$, and $k \in \mathbb{N}_{0}$. Furthermore

\begin{align}
U_{\mu,L}^{V_{k}}(r) & \preceq U_{\mu,L}^{\omega,k}(r), \quad 0 \leq \mu \leq 1, \\
U_{0,L}^{V_{k}}(r) & \preceq U_{+}^{\omega,k}(r)
\end{align}

for all $r \geq L$ and $k \in \mathbb{N}_{0}$. In particular,

\begin{equation}
U_{V_{k}}^{V_{k}}(r) \simeq \left[ \frac{\mu}{k+1} + 1 - \mu \right] U_{+}^{\omega,k}
\end{equation}

for all sufficiently large $r$, $0 \leq \mu \leq 1$, and $k \in \mathbb{N}_{0}$.

Furthermore, by Proposition 2.1, we have the following proposition.

**PROPOSITION 2.2** Assume $(V_{\omega}^{1})$ with $\omega \geq 0$ and $N \geq 3$. For any $g \in C([L, \infty))$, put

\begin{equation}
F_{L}^{V}[g](r) = U_{0,L}^{V}(r) \int_{L}^{r} s^{1-N}[U_{0,L}^{V}(s)]^{-2} \left( \int_{L}^{s} \tau^{N-1}U_{0,L}^{V}(\tau)g(\tau) d\tau \right) ds.
\end{equation}

Then, for any $k \in \mathbb{N}_{0}$, $F_{L}^{V_{k}}[g](r)$ is a solution of

\begin{equation}
\begin{cases}
U'' + \frac{N-1}{r}U' - V_{k}(r)U = g \quad \text{in} \quad (L, \infty), \\
U(L) = U'(L) = 0.
\end{cases}
\end{equation}
If there exist constants $A > 0$ such that
\[ |g(r)| \leq AU_{0,L}^{V_{k}}(r), \quad r \geq L, \]
then there exists a positive constant $C$, independent of $k$, such that
\[
|F_{L}^{V_{k}}[g](r)| \leq CA(k+1)^{-1}r^{2}U_{0,L}^{V_{k}}(r),
\]
for all $r \geq L$.

Next, we consider the case (1.12).

**Proposition 2.3** Assume $(V_{\omega}^\ell)$ with $\omega \geq 0$ and $\ell \in \mathbb{N}$. Furthermore assume that there exists a multi-index $J \in \mathbb{N}_{0}^{N}$ with $|J| = n+1 \leq \ell+2$ such that
\[
(\nabla_{x}^{j}U_{\mu,L}^{V})(|x|) \neq 0 \text{ in } \Omega_{L}, \text{ for all } j \in \mathbb{N}_{0}^{N} \text{ with } |j| \leq n,
\]
\[
(\nabla_{x}^{j}U_{\mu,L}^{V})(|x|) \equiv 0 \text{ in } \Omega_{L}.
\]
Then there exists a nonnegative integer $n'$ such that (1.12),
\[
U_{\mu,L}^{V}(|x|) = \frac{1-\mu}{L^{n}}(x_{1}^{2} + \cdots + x_{N}^{2})^{n'} = \frac{1-\mu}{L^{n}}|x|^{n'}, \quad x \in \Omega_{L},
\]
and
\[
(\nabla_{x}^{j}U_{\mu,L}^{V})(|x|) \equiv 0 \text{ in } \Omega_{L}
\]
hold for all $j \in \mathbb{N}_{0}^{N}$ with $|j| \geq n+1$.

3 Derivatives of the solutions of $(P_{\mu}^{k})$

In this section, we consider the radial solution $v$ of the initial-boundary value problem
\[
(P_{\mu}^{k}) \quad \left\{
\begin{array}{l}
\partial_{t}v = \Delta v - V_{k}(|x|)v \quad \text{in } \Omega_{L} \times (0, \infty), \\
\mu v - (1-\mu)\partial_{r}v = 0 \quad \text{on } \partial \Omega_{L} \times (0, \infty), \\
v(\cdot, 0) = \psi(\cdot) \in L^{p}(\Omega_{L}),
\end{array}
\right.
\]
where $0 \leq \mu \leq 1$, $p \geq 1$, $k \in \mathbb{N}_{0}$, and $\psi$ is a radial function in $\Omega_{L}$. For any positive $\epsilon$ and $T$, put
\[
D_{\epsilon}(T) = \left\{(x,t) \in \Omega_{L} \times (T, \infty) : |x| < \epsilon(1+t)^{1/2}\right\},
\]
\[
\Gamma_{\epsilon}(T) = \left\{(x,t) \in \Omega_{L} \times (T, \infty) : |x| = \epsilon(1+t)^{1/2}\right\}
\]
\[
\cup \left\{(x,T) : x \in \Omega_{L}, |x| \leq \epsilon(1+T)^{1/2}\right\}.
\]
We will construct a super-solution of \((P_{\mu}^{k})\) in \(D_{\epsilon}(T)\) for some positive constants \(\epsilon\) and \(T\), and give some estimates on the derivatives of the solution \(v_{\mu}^{k}\) of \((P_{\mu}^{k})\) in \(D_{\epsilon}(T)\). In what follows, under the assumption \((V_{\omega}^{\ell})\), we put
\[
U_{k}(r) = U_{0,L}^{V_{k}}(r), \quad g_{k}(t) = g(t : \omega + \omega_{k})
\]
for simplicity. We first construct a super-solution of \((P_{\mu}^{k})\).

**Lemma 3.1** Assume \(N \geq 3\) and \((V_{\omega}^{\ell})\) with \(\omega \geq 0\) and \(k \in \mathbb{N}_{0}\). Let \(\gamma > 0\). Then there exist positive constants \(T\), \(\epsilon\), and \(C\), which are independent of \(k\), and a function \(W = W(x, t)\) in \(\Omega_{L} \times (0, \infty)\) such that

\[
\partial_{t}W \geq \Delta W - V_{k}(|x|)W \quad \text{in} \quad D_{\epsilon}(T),
\]

\[
\mu W(x, t) + (1 - \mu)\frac{\partial}{\partial \nu}W(x, t) \geq 0 \quad \text{on} \quad \partial\Omega_{L} \times (T, \infty),
\]

\[
W(x, t) \geq C^{-\alpha_{k}}(1+t)^{-\gamma} \quad \text{on} \quad \Gamma_{\epsilon}(T),
\]

\[
0 < W(x, t) \leq (1+t)^{-\gamma}g_{k}(t)U_{k}(|x|) \quad \text{in} \quad D_{\epsilon}(T).
\]

**Proof.** Let \(A\) and \(\epsilon\) be constants to be chosen later such that \(A > 0\) and \(0 < \epsilon < 1\). Let \(T_{\epsilon}\) be a positive constant such that \(\epsilon(1+T_{\epsilon})^{1/2} = L+1\). Put
\[
W(x, t) = (1+t)^{-\gamma}g_{k}(t) \left[ U_{k}(|x|) - A(1+k)(1+t)^{-1}F_{L}^{V_{k}}[U_{k}](|x|) \right]
\]
for all \((x, t) \in \Omega_{L} \times (T_{\epsilon}, \infty)\). Then, there exists a constant \(C_1 = C_1(\gamma)\) such that

\[
\partial_{t}W \geq [-\gamma(1+t)^{-\gamma-1}g_{k}(t) + (1+t)^{-\gamma}g'_{k}(t)]U_{k}(|x|) \geq -C_{1}(1+k)(1+t)^{-\gamma-1}g_{k}(t)U_{k}(|x|)
\]
and by (2.19), we have
\[
\Delta W - V_{k}(|x|)W = -A(1+k)(1+t)^{-\gamma-1}g_{k}(t)U_{k}(|x|)
\]
in \(\Omega_{L} \times (T_{\epsilon}, \infty)\). Let \(A = C_{1}\). Then, by (3.5) and (3.6), we have
\[
\partial_{t}W \geq \Delta W - V_{k}(|x|)W \quad \text{in} \quad \Omega_{L} \times (T_{\epsilon}, \infty).
\]

On the other hand, by Proposition 2.2, there exists a positive constant \(C_{2}\), independent of \(\epsilon\), such that
\[
0 \leq A(1+k)(1+t)^{-1}F_{L}^{V_{k}}[U_{k}](|x|) \leq C_{2}A\epsilon U_{k}(|x|)
\]
for all \((x, t) \in D_{\epsilon}(T_{\epsilon})\). Let \(0 < \epsilon \leq \min\{1, 1/2C_{2}A\}\). Then we have
\[
(3.9) \quad \frac{1}{2}g_{k}(t)U_{k}(|x|) \leq (1 + t)^{\gamma}W(x, t) \leq g_{k}(t)U_{k}(|x|)
\]
for all \((x, t) \in D_{\epsilon}(T_{\epsilon})\). Then, by the definition of \(W\), we have
\[
(3.10) \quad \mu W + (1 - \mu)\frac{\partial}{\partial \nu}W = \mu W \geq 0 \quad \text{on} \quad \partial \Omega_{L} \times (0, \infty).
\]
By Proposition 2.1 and (1.7), we see that
\[
(3.11) \quad U_{k}(\epsilon(1 + t)^{1/2}) \leq (1 + t)^{\gamma}W(x, t) \leq U_{k}(\epsilon(1 + t)^{1/2})
\]
for all \(t \geq T_{\epsilon}\) and \(k \in \mathbb{N}_{0}\). By (3.9) and (3.11), there exists a positive constant \(C_{3}\) such that
\[
(3.12) \quad (1 + t)^{\gamma}W(x, t) \geq \frac{1}{2}g_{k}(t)U_{k}(\epsilon(1 + t)^{1/2}) \geq C_{3}^{-1}(k + 1)^{-1}\left(\frac{\epsilon}{L}\right)^{\alpha_{k}}
\]
for all \((x, t) \in \Gamma_{\epsilon}(T_{\epsilon})\) with \(t > T_{\epsilon}\). Furthermore, by (2.15), (3.9), and 
\(\epsilon(1 + T_{\epsilon})^{1/2} = L + 1\), there exists a positive constant \(C_{4}\) such that
\[
(3.13) \quad W(x, T_{\epsilon}) \geq \frac{1}{2}(1 + T_{\epsilon})^{-\gamma - \underline{\alpha_{k}}}U_{k}(L) \geq \frac{1}{2}(1 + T_{\epsilon})^{-\gamma}\left(\frac{\epsilon}{L + 1}\right)^{\alpha_{k}}
\]
for all \((x, T_{\epsilon}) \in \Gamma_{\epsilon}(T_{\epsilon})\) and \(k \in \mathbb{N}_{0}\). By (3.7), (3.10), (3.12), and (3.13), we have (3.1)–(3.4), and the proof of Lemma 3.1 is complete. \(\square\)

Next we give the following lemmas on the estimates of derivatives of \(v_{\mu}^{k}\).

First, we estimate \(v\) and its time derivatives.

**Lemma 3.2** Assume that \(\psi\) is a radial function in \(\Omega_{L}\) such that \(\|\psi\|_{L^{p}(\Omega_{L})} = 1\) with \(p \geq 1\). Let \(N \geq 3\) and \(v\) be a solution of \((P_{\mu}^{k})\) with \(v(\cdot, 0) = \psi(\cdot)\) under the condition \((V_{\omega}^{\ell})\) with \(\omega \geq 0\). Put
\[
w(x, t) = F_{L}^{\psi_{k}}[(\partial_{t}v)(\cdot, t)](|x|).
\]
Then there exist positive constants \(T, \epsilon, \eta\), independent of \(k\), such that
\[
(3.14) \quad |\partial_{t}v(x, t)| \leq \eta^{\alpha_{k}}t^{-\frac{N - i}{2p}}g_{k}(t)U_{\omega, k}^{w, k}(|x|),
\]
\[
(3.15) \quad |\partial_{t}w(x, t)| \leq \eta^{\alpha_{k}}t^{-\frac{N - i}{2p} - 1}g_{k}(t)|x|^{2}U_{\omega, k}^{w, k}(|x|)
\]
for all \((x, t) \in D_{\epsilon}(T)\) and all \(i \in \mathbb{N}_{0}\) with \(2i \leq \ell + 1\).
Let $i \in \mathbb{N}_{0}$ and put $v_{i} = \partial_{t}^{i}v$. Let $T$ and $\epsilon$ be positive constants given in Lemma 3.1. Let $W$ be the function constructed in Lemma 3.1 with $\gamma = N/2p + i$. For any $\eta_{1} > 0$, we put
\[ \overline{v}_{i}(x, t) = \eta_{1}^{\alpha_{k}}W(x, t) \]
for all $(x, t) \in D_{\epsilon}(T)$. Then, taking a sufficiently large $T$ and $\eta_{1}$ if necessary, by Lemma 2.3, we have
\[ |v_{i}(x, t)| \leq \overline{v}_{i}(x, t) \quad \text{on} \quad \Gamma_{\epsilon}(T). \]
So, by the comparison principle, we have
\[ |v_{i}(x, t)| \leq \overline{v}_{i}(x, t) \quad \text{in} \quad D_{\epsilon}(T). \]
This inequality together with (2.8), (2.16), and (3.4) implies
\[ |v_{i}(x, t)| \leq \eta_{1}^{\alpha_{k}}t^{-\frac{N}{2p}-i}g_{k}(t)|x|^{-1}U_{\mu,L}^{\omega,k}(|x|) \]
for all $(x, t) \in D_{\epsilon}(T)$, and we obtain the inequality (3.14). On the other hand, since
\[ (\partial_{t}^{i}w)(x, t) = F_{L}^{V_{k}}[(\partial_{t}^{i+1}v)(\cdot, t)](|x|) \]
for all $(x, t) \in \Omega_{L} \times (0, \infty)$, by (2.17), (2.20) and (3.14), we have (3.15), and the proof of Lemma 3.2 is complete.

Furthermore we have the following lemma on the time derivatives of $\partial_{r}v$ and $\partial_{r}w$.

**Lemma 3.3** Assume the same assumptions as in Lemma 3.2. Then there exist positive constants $T$, $\eta$, and $\epsilon$, independent of $k$, such that
\[ |\partial_{t}^{i}v_{i}(x, t)| \leq \eta^{\alpha_{k}}t^{-\frac{N}{2p}-i}g_{k}(t)|x|^{-1}U_{\mu,L}^{\omega,k}(|x|), \]
\[ |\partial_{t}^{i}w_{i}(x, t)| \leq \eta^{\alpha_{k}}t^{-\frac{N}{2p}-1-i}g_{k}(t)|x|U_{\mu,L}^{\omega,k}(|x|) \]
for all $(x, t) \in D_{\epsilon}(T)$ and all $i \in \mathbb{N}_{0}$ with $2i \leq \ell + 1$.

**Proof.** By (2.17), (2.21), (3.14), and (3.16), we have (3.18). So we prove (3.17). Put $v_{i} = \partial_{t}^{i}v$ and $w_{i} = \partial_{t}^{i}w$. Then $v_{i}$ and $w_{i}$ satisfy
\[ \partial_{t}v_{i} = \Delta w_{i} - V_{k}(|x|)w_{i} \]
by the definition of $F_{L}^{V_{k}}$. By the uniqueness of the initial value problem for the ordinary differential equation, there exists a function $\zeta(t)$ in $(0, \infty)$ such that
\[ v_{i}(x, t) = \zeta(t)U_{\mu,L}^{V_{k}}(|x|) + w_{i}(x, t) \]
for all $(x, t) \in \Omega_L \times (0, \infty)$. Furthermore, by (2.17), (2.20), (3.14), (3.15), and (3.19), there exist constants $C_1, C_2, T, \eta_1,$ and $\epsilon$ such that

$$|\zeta(t)|U_k(\epsilon(1+t)^{1/2}) \leq |v_i(x, t)|_{|x|=\epsilon(1+t)^{1/2}} + |w_i(x, t)|_{|x|}\leq C_1 t^{-\frac{N}{2p}-i} + o_{2\eta_1^\alpha_{k}} t^{-\frac{N}{2p}-i}g_{k}(t)U_{+}^{\omega,k}(\epsilon(1+t)^{1/2})$$

for all $t \geq T$. This together with (3.11) implies that there exists a constant $\eta_2$ such that

$$|\zeta(t)| \leq \eta_2^\alpha_{k}t^{-\frac{N}{2p}-i}g_{k}(t), \quad t \geq T, \ k \in \mathbb{N}_0.$$

In addition, by (2.15), (3.18), and (3.19), there exists a constant $\eta_3$ such that such that

$$|\partial_r v_i(x, t)| \leq |\zeta(t)|(\partial_r U_{\mu,L}^{V_k})(|x|) + |\partial_r w_i(|x|, t)| \leq \eta_3^\alpha_{k}t^{-\frac{N}{2p}-1-i}g_{k}(t)|x|^{2-j}U_{+}^{\omega,k}(|x|)$$

for all $(x, t) \in D_\epsilon(T)$ and $k \in \mathbb{N}_0$. So we obtain (3.17), and the proof of Lemma 3.3 is complete. \(\square\)

We give upper estimates on the spatio-temporal derivatives of $v$ and $w$ and its proof is done in the similar way to the proofs of Lemmas 3.2 and 3.3.

**Lemma 3.4** Assume the same assumptions as in Lemma 3.2. Then there exist positive constants $T, \eta,$ and $\epsilon$, independent of $k$, such that

$$|\partial_t^i \partial_x^j v(x, t)| \leq \eta^\alpha_{k}t^{-\frac{N}{2p}-i}g_{k}(t)|x|^{-j}U_{+}^{\omega,k}(|x|),$$

$$|\partial_t^i \partial_x^j w(x, t)| \leq \eta^\alpha_{k}t^{-\frac{N}{2p}-1-i}g_{k}(t)|x|^{2-j}U_{+}^{\omega,k}(|x|)$$

for all $(x, t) \in D_\epsilon(T),$ $i \in \mathbb{N}_0$ with $2(i+1) \leq \ell+1$, and $j = 2, \ldots, \ell+2$.

Finally, we give estimates on the derivatives of $v$ for the case (1.12).

**Lemma 3.5** Assume that $\psi$ is a radial function such that $\|\psi\|_{L^p(\Omega_L)} = 1$ with $p \geq 1$. Let $v$ be the solution of $(P_{\mu}^k)$ with $v(\cdot, 0) = \psi(\cdot)$ and $k = 0$, under the condition (1.12). Then, for any $j \in \mathbb{N}_0^N$ with $|j| \geq n + 1$ and $i \in \mathbb{N}_0$, there exist positive constants $C$, $T$, and $\epsilon$ such that

$$|\partial_t^i \nabla_x^j v(x, t)| \leq Ct^{-\frac{N}{2p}-\frac{1}{2}-i-\frac{n}{2}}$$

for all $(x, t) \in D_\epsilon(T).$
PROOF. By (1.12), we have

\[ U_{\mu,L}^{V}(x) = c \left( \sum_{i=1}^{N} x_{i}^{2} \right)^{n'}, \quad U_{+}^{\omega}(r) = \left( \frac{r}{L} \right)^{n}, \quad g(t : \omega) = (1 + t)^{-\frac{n}{2}}, \]

where \( n = 2n' \) and \( c \) is a positive constant. (See also Proposition 2.3). Put \( v_{i}(x, t) = \partial_{t}^{i}v(x, t) \) and \( w_{i}(x, t) = F_{L}^{V}[v_{i+1}](|x|) \). Let \( j \in \mathbb{N}_{0}^{N} \) with \( |j| \geq n+1 \). Then \( \nabla_{x}^{j}U_{\mu,L}^{V}(|x|) \equiv 0 \) in \( \Omega_{L} \), and by (3.19), we have \( \nabla_{x}^{j}v_{i}(x, t) = \nabla_{x}^{j}w_{i}(x, t) \) for all \((x, t) \in \Omega_{L} \times (0, \infty) \). Therefore, by the radial symmetry of \( w_{i} \) and the inequality (3.22) with \( k = 0 \), there exist positive constants \( T \) and \( \epsilon \) such that

\[
|\nabla_{x}^{j}v_{i}(x, t)| \leq \sum_{m=1}^{|j|} \frac{|(\partial_{r}^{m}w_{i})(x,t)|}{|x|^{|j|-m}} \leq t^{-\frac{N}{2p}-\frac{\min\{\alpha(\omega),|j|\}}{2}} |x|^{n+2-|j|} \leq t^{-\frac{N}{2p}-\frac{1}{2}-i-\frac{n}{2}}
\]

for all \((x, t) \in D_{\epsilon}(T) \), and the proof of lemma 3.5 is complete. \( \square \)

REMARK 3.1 If the \( L^{p} \)-norm of the initial value is not 1, then all the right-hand terms in the estimates in Lemmas 3.2, 3.3 and 3.4 must be multiplied by \( \|\psi\|_{L^{p}(\Omega_{L})} \).

4 Upper bounds of derivatives of solutions

In this section, we prove the following two propositions, which are mentioned in Section 1 as upper estimates, by using lemmas given in the previous sections.

PROPOSITION 4.1 Assume the same assumptions as in Theorem 1.1. Then, for any \( p \geq 1 \) and \( j \in \mathbb{N}_{0}^{N} \) with \( |j| \leq \ell + 1 \),

\[
\|\nabla_{x}^{j}G_{\mu}^{V}(t)\|_{p \rightarrow \infty} \leq t^{-\frac{N}{2p} - \frac{\min\{\alpha(\omega),|j|\}}{2}}
\]

for all sufficiently large \( t \).

PROPOSITION 4.2 Assume the same assumptions as in Theorem 1.2. Then, for any \( p \geq 1 \) and \( j \in \mathbb{N}_{0}^{N} \) with \( |j| \geq n+1 \),

\[
\|\nabla_{x}^{j}G_{\mu}^{V}(t)\|_{p \rightarrow \infty} \leq t^{-\frac{N}{2p} - \frac{\alpha(\omega_{n} + \omega)}{2}}
\]

for all sufficiently large \( t \).
Proof of Proposition 4.1. Let $u^V_\mu$ be the solution of (1.1) with $\phi \in C_0(\Omega_L)$. By the same arguments as in [3] and [4], $\phi$ can be expanded in the Fourier series, that is, there exist radial functions $\{\phi_{k,i}\} \subset L^2(\Omega_L)$ such that

$$\phi(x) = \sum_{k=0}^{\infty} \sum_{i=1}^{l_k} \phi_{k,i}(|x|)Q_{k,i} \left( \frac{x}{|x|} \right) \quad \text{in} \quad L^2(\Omega_L).$$

Let $u^{k,i}_\mu$ be a solution of (1.1) with the initial data $\phi_{k,i}(|x|)Q_{k,i}(x/|x|)$ and $v^{k,i}_\mu$ a radial solution of $(P^k_\mu)$ with the initial data $\phi_{k,i}$. By the uniqueness of the solution of (1.1), we see that

$$u^{k,i}_\mu(x, t) = v^{k,i}_\mu(x, t)Q_{k,i} \left( \frac{x}{|x|} \right), \quad (x, t) \in \Omega_L \times (0, \infty),$$

where $k \in \mathbb{N}_0$ and $i = 1, \ldots, l_k$. On the other hand, by the standard elliptic regularity theorem and $\|Q_{k,i}\|_{L^2(\mathbb{S}^{N-1})} = 1$, for any $n \in \mathbb{N}$, we have

$$\|Q_{k,i}\|_{C^{2n}(\mathbb{S}^{N-1})} \leq (1 + \omega_k)^{n+1} \wedge (k+1)^{2n+2}$$

for all $k \in \mathbb{N}_0$ and $i = 1, \ldots, l_k$. Furthermore the eigenspace of $\Delta_{\mathbb{S}^{N-1}}$ corresponding to $\omega_\ell$ is spanned by the functions $\nabla_x^j |x|$ for $j \in \mathbb{N}_0^N$ with $|j| = \ell$, and we have

$$l_k \leq N^k.$$

By the orthogonality of $\{Q_{k,i}\}_{k,i}$, we have

$$\int_{\Omega_L} u^{k_1,i_1}_\mu(x, t)u^{k_2,i_2}_\mu(x, t)dx = 0$$

for all $t \geq 0$ if $(k_1, i_1) \neq (k_2, i_2)$. On the other hand, for any $t > 0$,

$$u^V_\mu(x, t) = \lim_{n \to \infty} \sum_{k=0}^{m} \sum_{i=1}^{l_k} v^{k,i}_\mu(x, t)Q_{k,i} \left( \frac{x}{|x|} \right)$$

holds uniformly for all $x \in \Omega_L$. Hence we have

$$\int_{\partial B(0,|x|)} u^V_\mu(x, t)Q_{k,i} \left( \frac{x}{|x|} \right) d\sigma = v^{k,i}_\mu(x, t) \int_{\partial B(0,|x|)} |Q_{k,i} \left( \frac{x}{|x|} \right)|^2 d\sigma$$

for all $(x, t) \in \Omega_L \times (0, \infty)$. Then, by (4.5) and the Jensen inequality, we have

$$|x|^{N-1} |v^{k,i}_\mu(x, t)|^p \leq (k+1)^{2p} \int_{\partial B(0,|x|)} |u^V_\mu(x, t)|^p d\sigma.$$
for all \((x, t) \in \Omega_L \times (0, \infty)\) and \(k \in \mathbb{N}_0\). So, by (2.9), we have

\[
(v^{k,i}(\cdot, t))_{L^p(\Omega_L)} \leq \left( \int_L^\infty r^{N-1} |v^{k,i}(r, t)|^p dr \right)^{1/p} \\
\leq (k+1)^2 \|u^{V}(\cdot, t)\|_{L^p(\Omega_L)} \leq (k+1)^2 \|\phi\|_{L^p(\Omega_L)}
\]

for all \(t > 0\) and \(k \in \mathbb{N}_0\).

Let \(j \in \mathbb{N}_0^N\) with \(|j| \leq \ell + 1\). Let \(k \in \mathbb{N}\) and \(i = 1, \ldots, l_k\). By (1.6), (4.4), and (4.5), we have

\[
|\nabla_x^j u^{k,i}(x, t)| \preceq (k+1)^{\ell+5} \sum_{m=0}^{|j|} \frac{|\partial_r^m v^{k,i}(x,t)|}{|x|^{|j|-m}}, \quad (x, t) \in \Omega_L \times (0, \infty).
\]

Since \(D_{\epsilon_1}(T) \subset D_{\epsilon_2}(T)\) if \(\epsilon_1 \leq \epsilon_2\), by Lemmas 3.2, 3.3, 3.4, Remark 3.1 and (4.9), there exist positive constants \(\eta_1, \eta_2, \eta_3\) such that

\[
\frac{|\partial_r^m v^{k,i}(x, t + t_0)|}{|x|^{|j|-m}} \preceq \eta_1^{\alpha_k} t^{-\frac{N}{2p}} g_k(t) U_{+}^{\omega, k}(|x|) |x|^{-|j|} \|v^{k,i}(\cdot, t_0)\|_{L^p(\Omega_L)} \leq (k+1)^2 \epsilon^{[\alpha_k - |j| + \eta_3^{\alpha_k} t^{-\frac{N}{2p}} - \frac{\min\{\alpha_k, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}
\]

for all \((x, t) \in D_\epsilon(T_*)\) with \(0 < \epsilon \leq \epsilon_*\), \(t_0 > 0\), and \(m = 0, 1, \ldots, |j|\), where \(\alpha_k = \alpha(\omega + \omega_k)\). Letting \(t_0 \to 0\), we obtain

\[
|\nabla_x^j u^{k,i}(x, t)| \preceq (k+1)^{\ell+5} \epsilon^{[\alpha_k - |j| + \eta_3^{\alpha_k} t^{-\frac{N}{2p}} - \frac{\min\{\alpha_k, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}
\]

for all \((x, t) \in D_\epsilon(T_*)\) with \(0 < \epsilon \leq \epsilon_*\) and \(m = 0, 1, \ldots, |j|\). This inequality together with (4.10) implies that

\[
|\nabla_x^j u^{0,i}(x, t)| \leq \frac{1}{2^{k}N^{k}} t^{-\frac{N}{2p}} \frac{\min\{\alpha_k, |j|\}}{2} \|\phi\|_{L^p(\Omega_L)}
\]

for all \((x, t) \in D_\epsilon(T_\epsilon)\), \(k \in \mathbb{N}\), and \(i = 1, \ldots, l_k\). Similarly, for the case \(k = 0\), we have

\[
|\nabla_x^j u^{0,1}(x, t)| = |\nabla_x^j v^{0,1}(x, t)| \leq \sum_{m=1}^{|j|} \frac{|\partial_r^m v^{0,1}(x,t)|}{|x|^{|j|-m}} \leq t^{-\frac{N}{2p}} \frac{\min\{\alpha_0, |j|\}}{2} \|\phi\|_{L^p(\Omega_L)}
\]
for all \((x, t) \in D_{\epsilon}(T_{\epsilon})\). By (4.6), (4.12), and (4.13), we obtain
\[
|\nabla_{x}^{j}u_{\mu}^{V}(x, t)| \leq \limsup_{m \to \infty} \sum_{k=0}^{m} \sum_{i=1}^{l_{k}} |\nabla_{x}^{j}u_{\mu}^{k,i}(x, t)| \leq t^{-\frac{N}{2p} - \frac{\min \{\alpha_{0}, |j|\}}{2}} \|\phi\|_{L^{p}(\Omega_{L})}
\]
for all \((x, t) \in D_{\epsilon}(T_{\epsilon})\). On the other hand, by Lemma 2.3, we have
\[
|\nabla_{x}^{j}u_{\mu}^{V}(x, t)| \leq t^{-\frac{N}{2p} - \frac{|j|}{2}} \|\phi\|_{L^{p}(\Omega_{L})}
\]
for all \((x, t) \not\in D_{\epsilon}(T_{\epsilon})\). Therefore, by (4.14) and (4.15), we obtain
\[
|\nabla_{x}^{j}u_{\mu}^{V}(x, t)| \leq t^{-\frac{N}{2p} - \frac{\min \{\alpha(\omega_{n} + \omega_{1}), |j|\}}{2}} \|\phi\|_{L^{p}(\Omega_{L})}
\]
for all \((x, t) \in \Omega_{L}\) with \(t \geq T_{\epsilon}\), where \(\phi \in C_{0}(\Omega_{L})\). Since \(C_{0}(\Omega_{L})\) is a dense subset of \(L^{p}(\Omega_{L})\), the inequality (4.16) holds for all \(\phi \in L^{p}(\Omega_{L})\), and the proof of Proposition 4.1 is complete. \(\square\)

**Proof of Proposition 4.2.** By (1.12), \(V\) satisfies the condition \((V_{\omega}^{\ell})\) with \(\omega = \omega_{n}\) and \(\ell = 0, 1, 2, \ldots\). Let \(j \in N_{0}^{N}\) with \(|j| \geq n + 1 = 2n' + 1\). Let \(u_{\mu}^{V}\) be the solution of (1.1) with \(\phi \in C_{0}(\Omega_{L})\) and \(u_{\mu}^{k,i}\) a function given in the proof of Proposition 4.1. By the same argument as in the proof of (4.13) and Lemma 3.5, for any sufficiently small \(\epsilon > 0\), there exists a positive constant \(T_{\epsilon}\) such that
\[
|\nabla_{x}^{j}u_{\mu}^{0,1}(x, t)| \leq t^{-\frac{N}{2p} - \frac{n+1}{2}} \|\phi\|_{L^{p}(\Omega_{L})}
\]
for all \((x, t) \in D_{\epsilon}(T_{\epsilon})\).

On the other hand, by the same argument as in the proof of (4.14), taking a sufficiently small \(\epsilon > 0\) if necessary, we have
\[
\limsup_{m \to \infty} \sum_{k=1}^{m} \sum_{i=1}^{l_{k}} |\nabla_{x}^{j}u_{\mu}^{k,i}(x, t)| \leq t^{-\frac{N}{2p} - \frac{\min \{\alpha(\omega_{n} + \omega_{1}), |j|\}}{2}} \|\phi\|_{L^{p}(\Omega_{L})}
\]
for all \((x, t) \in D_{\epsilon}(T_{\epsilon})\). We note that \(\alpha(\omega_{n} + \omega_{1}) \leq \alpha(\omega_{n}) + 1 = n + 1\). Therefore, by (4.17), (4.18), and \(|j| \geq n + 1\), we have
\[
|\nabla_{x}^{j}u_{\mu}^{V}(x, t)| \leq t^{-\frac{N}{2p} - \frac{\alpha(\omega_{n} + \omega_{1})}{2}} \|\phi\|_{L^{p}(\Omega_{L})}
\]
for all \((x, t) \in D_{\epsilon}(T_{\epsilon})\). Furthermore, by (4.15) and (4.19), taking a sufficiently small \(\epsilon\) if necessary, we have
\[
|\nabla_{x}^{j}u_{\mu}^{V}(x, t)| \leq t^{-\frac{N}{2p} - \frac{\alpha(\omega_{n} + \omega_{1})}{2}} \|\phi\|_{L^{p}(\Omega_{L})}
\]
for all \((x, t) \in \Omega_L \times (T, \infty)\), where \(\phi \in C_0(\Omega_L)\). Furthermore, since \(C_0(\Omega_L)\) is a dense subset of \(L^p(\Omega_L)\), we have the inequality (4.20) for all \(\phi \in L^p(\Omega_L)\), and the proof of Proposition 4.2 is complete. \(\square\)

5 Proofs of Theorems 1.1 and 1.2

In this section we consider the asymptotic behavior of the derivatives of the radial solution \(v\) of (1.1) for some initial data \(\psi \in C_0(\Omega_L)\) and complete proofs of Theorems 1.1 and 1.2.

**Proposition 5.1** Let \(R > 0\), \(\omega \geq 0\), and \(\psi(\not\equiv 0)\) be a nonnegative, radial function belonging to \(C_0(\Omega_R)\). Let \(v\) be a radial solution of

\[
\begin{cases}
\partial_t v = \Delta v - \frac{\omega}{|x|^2} v & \text{in } \Omega_R \times (0, \infty), \\
v(x, t) = 0 & \text{on } \partial \Omega_R \times (0, \infty), \\
v(x, 0) = \psi(x) & \text{in } \Omega_R.
\end{cases}
\]

Then, for any \(p \in [1, \infty]\),

\[
\|v(\cdot, t)\|_{L^p(\Omega_R)} \lesssim t^{-\frac{N}{2}(1 - \frac{1}{p}) - \frac{\alpha(\omega)}{2}}
\]

holds for all sufficiently large \(t\). Furthermore there exists a positive constant \(\epsilon_*\) such that, for any \(0 < \epsilon \leq \epsilon_*\),

\[
v(x, t) \big|_{|x| = \epsilon(1+t)^{1/2}} \lesssim \epsilon^\alpha(\omega) t^{-\frac{N + \alpha(\omega)}{2}}, \quad t > T
\]

holds with suitably chosen \(T = T(\epsilon)\).

**Proof.** Put

\[
z(y, s) = (1 + t)^{\frac{N + \alpha}{2}} v(x, t), \quad y = (1 + t)^{-\frac{1}{2}} x, \quad s = \log(1 + t),
\]

where \(\alpha = \alpha(\omega)\). Then the function \(z\) satisfies

\[
\begin{cases}
\partial_s z = \frac{1}{\rho} \text{div} (\rho \nabla y z) + \frac{N + \alpha}{2} z - \frac{\omega}{|y|^2} z & \text{in } W, \\
z = 0 & \text{on } \partial W, \\
z(y, 0) = \psi(y) & \text{in } \Omega_R,
\end{cases}
\]

where \(\rho(y) = \exp(|y|^2/4)\) and

\[
\Omega(s) = e^{-s/2} \Omega_R, \quad W = \bigcup_{0 < s < \infty} (\Omega(s) \times \{s\}), \quad \partial W = \bigcup_{0 < s < \infty} (\partial \Omega(s) \times \{s\}).
\]
Put
\[ \varphi(y) = c_0 |y|^{\alpha(\omega)} \exp(-|y|^2/4), \]
where \( c_0 \) is a positive constant such that \( \|\varphi\|_{L^2(\mathbb{R}^N, \rho dy)} = 1. \) Then, since
\[ \int_{\Omega_R} v(x, t) U_{1,R}^\omega(|x|) dx = \int_{\Omega_R} \phi(x) U_{1,R}^\omega(|x|) dx > 0, \quad t \geq 0, \]
by the same argument as in the proof of Lemma 6.1 in [4], we see that
\[ (5.6) \quad a \equiv \int_{\Omega_R} \phi(x) U_{1,R}^\omega(|x|) dx = \lim_{s \to \infty} \int_{\Omega(s)} z(y, s) \varphi(y) \rho(y) dy > 0. \]
Furthermore, by the same argument as in the proof of Lemmas 3.3 and 3.4 in [4], for any \( r_1 \) and \( r_2 \) with \( 0 < r_1 < r_2 \), we have
\[ (5.7) \quad \sup_{s > 0} \|z(\cdot, s)\|_{L^2(\Omega(s), \rho dy)} < \infty, \]
\[ (5.8) \quad \sup_{s > 0} \|z(\cdot, s)\|_{L^\infty(\{y:|y| \geq r_1\})} < \infty, \]
\[ (5.9) \quad \lim_{s \to \infty} \|z(\cdot, s) - a \varphi\|_{C(\{y: r_1 \leq |y| \leq r_2\})} = 0. \]
By (5.6), (5.7) and (5.9), we have \( \|z(\cdot, s)\|_{L^1(\Omega(s))} \approx 1 \) for all sufficiently large \( s \). So, by (5.8) and (5.9), for any \( p \in [1, \infty] \), we have \( \|z(\cdot, s)\|_{L^p(\Omega(s))} \approx 1 \) for all sufficiently large \( s \), and obtain (5.2).
On the other hand, by the same argument as in (3.19), there exists a function \( \zeta \) in \((0, \infty)\) such that
\[ (5.10) \quad v(x, t) = \zeta(t) U_{0,R}^V(|x|) + F_L^V[(\partial_t v)(\cdot, t)](|x|) \]
for all \((x, t) \in \Omega_R \times (0, \infty)\) with \( V = \omega/r^2 \). By (5.2) with \( p = \infty \), we may apply the same arguments as in the proof of Lemma 3.2 with \( \gamma = (N + \alpha(\omega))/2 \) to \( v \). Then we see that there exist positive constants \( \epsilon_* \) and \( T_* \) such that
\[ (5.11) \quad |F_L^V[(\partial_t v)(\cdot, t)](|x|)| \preceq t^{-N/2-\alpha(\omega)-1}|x|^\alpha(\omega)+2 \]
for all \((x, t) \in D_{\epsilon_*}(T_*)\). Therefore, by (2.18), (5.9), (5.10), (5.11), and the same arguments as in the deduction of (3.20), we may take a sufficiently small \( \tilde{\epsilon} \) so that
\[ (5.12) \quad \zeta(t) = [U_{0,R}^V(\tilde{\epsilon}(1 + t)^{1/2})]^{-1} [v(x, t) - F_L^V[\partial_t v](|x|)] \bigg|_{|x| = \tilde{\epsilon}(1 + t)^{1/2}} \]
\[ \approx \tilde{\epsilon}^{-\alpha} t^{-N/2 - \alpha(\omega) + 2} \left[ t^{-N/2 + \alpha(\omega) + 2} + O(\tilde{\epsilon}^{\alpha+2}) t^{-\frac{N+\alpha}{2}} \right] \approx t^{-N/2 - \alpha} \]
for all sufficiently large $t$. Then, by (5.10)–(5.12) and the similar argument as in (5.12), we have (5.3), and the proof of Proposition 5.1 is complete. $\square$

**Proof of Theorem 1.1.** Assume $(V_\omega^\ell)$. Let $\tilde{\omega}$ be a constant such that

$$\alpha(\tilde{\omega}) < \alpha(\omega) + 1.$$  

Then, by $(V_\omega^\ell)$-(i), we may take a sufficiently large $R$ so that

$$V(r) \leq \frac{\tilde{\omega}}{r^2}, \quad r \geq R.$$  

Let $p \geq 1$ and $\psi(\not\equiv 0)$ be a nonnegative, radial function belonging to $C_0(\Omega_R)$. Let $v$ be a solution of (5.1) with $\omega$ replaced by $\tilde{\omega}$. For any $T > 0$, let $u_T^V$ be a solution of (1.1) with the initial data $\phi(\cdot) = v(\cdot, T)/\|v(\cdot, T)\|_{L^p(\Omega_R)}$. Here we remark that

$$\|u_T^V(\cdot, 0)\|_{L^p(\Omega_L)} = 1.$$  

By the comparison principle, (5.2), and (5.3), for any sufficiently small $\epsilon > 0$, there exists a positive constant $T_\epsilon$ such that

$$u_T^V(x, T) \geq \frac{v(x, 2T)}{\|v(\cdot, T)\|_{L^p(\Omega_R)}} \geq \epsilon^\alpha(\tilde{\omega}) T^{-\frac{N}{2p}}$$

for all $(x, T) \in \Omega_L \times (T_\epsilon, \infty)$ with $|x| = \epsilon(1 + 2T)^{1/2} > \max\{R, 2L + 2\}$.

On the other hand, there exists a function $\zeta_V(t)$ such that

$$u_T^V(x, t) = \zeta_V(t)U_{\mu,L}^V(|x|) + F_L^V[\partial_t u_T^V](|x|)$$

for all $x \in \Omega_L$. By Lemmas 3.2–3.4 and (5.14), taking a sufficiently small $\epsilon$ and sufficiently large $T_\epsilon$ if necessary, we have

$$\zeta_V(T)U_{\mu,L}^V(|x|) \geq u_T^V(x, T) - |F_L^V[\partial_t u_T^V](|x|)| \geq C_1 \epsilon^\alpha(\tilde{\omega}) T^{-\frac{N}{2p}} - C_2 \epsilon^\alpha(\omega) + 2T^{-\frac{N}{2p}} \geq \epsilon^\alpha(\omega) + 1 T^{-\frac{N}{2p}}$$

for all $x \in \Omega_L$ with $L + 1 < |x| = \epsilon(1 + 2T)^{1/2} < \epsilon(1 + T)^{1/2}$ and $T \geq T_\epsilon$. Therefore, by (2.5) and (2.16), we have

$$\zeta_V(T) \geq T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}}$$
for all sufficiently large $T$. Therefore, by (5.16)-(5.18), there exist positive constants $C_3$ and $C_4$ such that

\begin{align}
|\nabla_x^j u_T^V(x, T)| \geq C_4 T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}} |\nabla_x^j U_{\mu,L}^V(x)| - C_4 T^{-\frac{N}{2p} - 1 - \frac{\alpha(\omega)}{2}} |x|^{2 + \alpha(\omega) - |j|}
\end{align}

for all $L < |x| \leq \epsilon(1 + T)^{1/2}$, $T \geq T_\epsilon$, and $j \in \mathbb{N}_0^N$ with $|j| \leq \ell$.

Let $j \in \mathbb{N}_0^N$ with $|j| \leq \ell$. By the assumption of Theorem 1.1 and Proposition 2.3, there exists a point $x_0 \in \Omega_L$ such that $(\nabla_x^j U_{V,L}^\mu)(x_0) \neq 0$. Then, by (5.19), there exist positive constants $C_5$ and $C_6$ such that

\begin{align}
|\nabla_x^j u_T^V(x_0, T)| \geq C_5 T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}} - C_6 T^{-\frac{N}{2p} - \underline{\alpha}_{2} \angle_{-1}} \omega \succeq T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}}
\end{align}

for all sufficiently large $T$. This inequality together with (5.14) implies

\begin{align}
\|\nabla_x^j G^V_\mu(T)\|_{\text{parrow}\infty} \succeq T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}}
\end{align}

for all sufficiently large $T$. This together with Proposition 4.1 implies (1.10) and (1.11), and the proof of Theorem 1.1 is complete. \(\square\)

**Proof of Theorem 1.2.** Let $u_T^{V_1}$ be a function given in the proof of Theorem 1.1 with $V(r) = (\omega_n + \omega_1)/r^2$.

Put

\[ u_T^V(x, t) = u_T^{V_1}(x, t) \frac{x_1}{|x|}. \]

Then $u_T^V$ is a solution of (1.1) with $V(r) = \omega_n/r^2$.

Let $j = (j_1, \ldots, j_N) \in \mathbb{N}_0^N$ with $|j| \geq n + 1$. Put $j' = (j_1 + 1, j_2, \ldots, j_N)$ and

\[ U_{\mu,L}^{\omega_n + \omega_1}(r) = \int_L^r U_{\mu,L}^{\omega_n + \omega_1}(s) ds \]

Then, by (2.5), we see that $U_{\mu,L}^{\omega_n + \omega_1}(r) \succeq r^{\alpha(\omega_n + \omega_1) + 1}$ for all sufficiently large $r$. If $\nabla_x^j u_T^{\mu,L}^{\omega_n + \omega_1}(|x|) \equiv 0$ in $\Omega_L$, then, we see that $U_{\mu,L}^{\omega_n + \omega_1}(r)$ is a polynomial. This contradicts $\alpha(\omega_n + \omega_1) \notin \mathbb{N}$ if $n \geq 1$. If $n = 0$, by (1.12),

\[ U_{\mu,L}^{\omega_n + \omega_1}(r) = U_{0,L}^{\omega_1}(r) = \frac{1}{N} \left( \frac{r}{L} \right)^{-(N-1)} + \frac{N - 1}{LN} r, \]

and $U_{\mu,L}^{\omega_n + \omega_1}(r)$ is not a polynomial. So we have

\[ \nabla_x^j U_{\mu,L}^{\omega_n + \omega_1}(|x|) = \nabla_x^j \left[ U_{\mu,L}^{\omega_n + \omega_1}(|x|) \frac{x_1}{|x|} \right] \neq 0 \text{ in } \Omega_L. \]
By the similar arguments in (5.16)–(5.20) and $\omega = \omega_n + \omega_1$, there exist positive constants $C_1$ and $C_2$ such that

$$|(\nabla^j_T \tilde{u}^V_T)(x_0, T)| \geq C_1T^{-\frac{N}{2}-\frac{\alpha(\omega_n+\omega_1)}{2}} - C_2T^{-\frac{N}{2}-\alpha(\omega_n+\omega_1)-1}$$

for all sufficiently large $T$. Furthermore, since $\|\tilde{u}^V_T(\cdot, 0)\|_{L^p(\Omega_L)} \approx 1$, we obtain

$$\|\nabla^j_T G^V_\mu(T)\|_{p \rightarrow \infty} \geq T^{-\frac{N}{2p} - \frac{\alpha(\omega_n+\omega_1)}{2}}$$

for all sufficiently large $T$. Therefore, this inequality together with Propositions 4.1 and 4.2 imply (1.13) and (1.14), and the proof of Theorem 1.2 is complete. □

6 Concluding remarks

As concluding remarks, we state some related topics. In the previous sections, we treat the exterior of a ball, however, we can treat the whole space and we can argue the movement of hot spots (the maximum points of a solution) with a potential $V$. According to the decay order of $V$, the behavior of hot spots varies. Such works are now in progress and we will discuss these topics later.

References


