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Nonexistence of self-similar singularities for the 3D incompressible Euler equations

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Abstract

We announce that there exists no self-similar finite time blowing up solution to the 3D incompressible Euler equations if the vorticity decays sufficiently fast near infinity in $\mathbb{R}^3$.

1 The self-similar singularities

We are concerned here on the following incompressible fluid equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3$.

$$\begin{cases} \dfrac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\nu\Delta v, \quad (x,t)\in\mathbb{R}^3\times(0,\infty) \\ \text{div } v=0, \quad (x,t)\in\mathbb{R}^3\times(0,\infty) \\ v(x,0)=v_0(x), \quad x\in\mathbb{R}^3 \end{cases}$$

where $v=(v_1,v_2,v_3)$, $v_j=v_j(x,t)$, $j=1,2,3$, is the velocity of the flow, $p=p(x,t)$ is the scalar pressure, and v_0 is the given initial velocity, satisfying div $v_0=0$. The constant $\nu\geq 0$ is called viscosity. If $\nu=0$ the system is called the Euler equations, while if $\nu>0$ the system is the Navier-stokes system.

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There are well-known results on the local existence of classical solutions(see e.g. [18, 13, 7] and references therein). The problem of finite time blow-up of the local classical solution is one of the most challenging open problem in mathematical fluid mechanics. On this direction there is a celebrated result on the blow-up criterion by Beale, Kato and Majda([1]). By geometric type of consideration some of the possible scenarios to the possible singularity has been excluded(see [8, 9, 10]. One of the main purposes of this paper is to exclude the possibility of self-similar type of singularities for the Euler system.

The system (E) has scaling property that if (v, p) is a solution of the system (E), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \tag{1.1}$$

are also solutions of (E) with the initial data $v_0^{\lambda,\alpha}(x) = \lambda^{\alpha} v_0(\lambda x)$. In view of the scaling properties in (1.1), the self-similar blowing up solution $v(x,t)$ of (E) should be of the form,

$$v(x,t) \;=\; \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \tag{1.2}$$

for $\alpha \neq -1$ and t sufficiently close to T_*. Substituting (1.2) into (E), we find that V should be a solution of the system

$$(SE)\begin{cases} \dfrac{\alpha}{\alpha+1}V + \dfrac{1}{\alpha+1}(x \cdot \nabla)V + (V \cdot \nabla)V = -\nabla P, \\ \mathrm{div}\ V = 0 \end{cases}$$

for some scalar function P, which could be regarded as the Euler version of the Leray equations introduced in [15]. The question of existence of nontrivial solution to (SE) is equivalent to that of existence of nontrivial self-similar finite time blowing up solution to the Euler system of the form (1.2). Similar question for the 3D Navier-Stokes equations was raised by J. Leray in [15], and answered negatively by the authors of [19], the result of which was refined later in [20]. Combining the energy conservation with a simple scaling argument, the author of this article showed that if there exists a nontrivial self-similar finite time blowing up solution, then its helicity should be zero. To the author's knowledge, however, the possibility of self-similar blow-up of the form (1.2) has never been excluded previously. In particular, due to lack of the laplacian term in the right hand side of the first equations of (SE), we cannot apply the argument of the maximum principle, which was crucial in the works in [19] and [20] for the 3D Navier-Stokes equations. Using a completely different argument from those used in [2], or [19], we prove here that there cannot be self-similar blowing up solution to (E) of the form (1.2), if the vorticity decays sufficiently fast near infinity. Before stating our main theorem we recall the notions of particle trajectory and the back-to-label map, which are used importantly in the recent work of [6]. Given a smooth velocity field $v(x,t)$, the particle trajectory mapping $a \mapsto X(a,t)$ is defined by the solution of the system of ordinary differential equations,

$$\frac{\partial X(a,t)}{\partial t} = v(X(a,t),t) \quad ; \quad X(a,0) = a.$$

The inverse $A(x,t) := X^{-1}(x,t)$ is called the back to label map, which satisfies $A(X(a,t),t) = a$, and $X(A(x,t),t) = x$.

Theorem 1.1 *There exists no finite time blowing up self-similar solution $v(x,t)$ to the 3D Euler equations of the form (1.2) for $t \in (0,T_*)$ with $\alpha \neq -1$, if v and V satisfy the following conditions:*

- *(i) For all $t \in (0,T_*)$ the particle trajectory mapping $X(\cdot,t)$ generated by the classical solution $v \in C([0,T_*);C^1(\mathbb{R}^3;\mathbb{R}^3))$ is a C^1 diffeomorphism from $\mathbb{R}^3$ onto itself.*
- *(ii) The vorticity satisfies $\Omega =$curl $V \neq 0$, and there exists $p_1 > 0$ such that $\Omega \in L^p(\mathbb{R}^3)$ for all $p \in (0,p_1)$.*

Remark 1.1 The condition (i), which is equivalent to the existence of the back-to-label map $A(\cdot,t)$ for our smooth velocity $v(x,t)$ for $t \in (0,T_*)$, is guaranteed if we assume a uniform decay of $V(x)$ near infinity, independent of the decay rate([5]).

Remark 1.2 Regarding the condition (ii), for example, if $\Omega \in L^1_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ and there exist constants R, K and $\varepsilon_1,\varepsilon_2 > 0$ such that $|\Omega(x)| \leq Ke^{-\varepsilon_1|x|^{\varepsilon_2}}$ for $|x| > R$, then we have $\Omega \in L^p(\mathbb{R}^3;\mathbb{R}^3)$ for all $p \in (0,1)$. Indeed, for all $p \in (0,1)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\Omega(x)|^p dx &= \int_{|x|\leq R} |\Omega(x)|^p dx + \int_{|x|>R} |\Omega(x)|^p\, dx \\ &\leq |B_R|^{1-p}\left(\int_{|x|\leq R} |\Omega(x)| dx\right)^p + K^p \int_{\mathbb{R}^3} e^{-p\varepsilon_1|x|^{\varepsilon_2}} dx < \infty,\end{aligned}$$

where $|B_R|$ is the volume of the ball B_R of radius R.

Remark 1.3 In the zero vorticity case $\Omega = 0$, from div $V = 0$ and curl $V = 0$, we have $V = \nabla h$, where $h(x)$ is a harmonic function in $\mathbb{R}^3$. Hence, we have an easy example of self-similar blow-up,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} \nabla h \left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right),$$

in $\mathbb{R}^3$, which is also the case for the 3D Navier-Stokes with $\alpha = 1$. We do not consider this case in the theorem.

Remark 1.4 If we assume that initial vorticity ω_0 has compact support, then the nonexistence of self-similar blow-up of the form given by (1.2) is immediate from the well-known formula, $\omega(X(a,t),t) = \nabla_a X(a,t)\omega_0(a)$(see e.g. [18]).

The proof of Theorem 1.1 will follow as a corollary of the following more general theorem, the proof of which is in [3].

Theorem 1.2 *Let $v \in C([0,T); C^1(\mathbb{R}^3;\mathbb{R}^3))$ be a classical solution to the 3D Euler equations generating the particle trajectory mapping $X(\cdot,t)$ which is a C^1 diffeomorphism from $\mathbb{R}^3$ onto itself for all $t \in (0,T)$. Suppose we have representation of the vorticity of the solution, by*

$$\omega(x,t) = \Psi(t)\Omega(\Phi(t)x) \qquad \forall t \in [0,T) \tag{1.3}$$

where $\Psi(\cdot) \in C([0,T);(0,\infty))$, $\Phi(\cdot) \in C([0,T);\mathbb{R}^{3\times 3})$ with $\det(\Phi(t)) \neq 0$ on $[0,T)$; $\Omega = \mathrm{curl}\, V$ for some V, and there exists $p_1 > 0$ such that $\Omega \in L^p(\mathbb{R}^3)$ for all $p \in (0,p_1)$. Then, necessarily either $\det(\Phi(t)) \equiv \det(\Phi(0))$ on $[0,T)$, or $\Omega = 0$.

The previous argument in the proof of Theorem 1.1 can also be applied to the following transport equations by a divergence-free vector field in $\mathbb{R}^n$, $n \geq 2$.

$$(TE) \begin{cases} \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = 0, \\ \mathrm{div}\, v = 0, \\ \theta(x,0) = \theta_0(x), \end{cases}$$

where $v = (v_1, \cdots, v_n) = v(x,t)$, and $\theta = \theta(x,t)$. In view of the invariance of the transport equation under the scaling transform,

$$v(x,t) \mapsto v^{\lambda,\alpha}(x,t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1}t),$$
$$\theta(x,t) \mapsto \theta^{\lambda,\alpha,\beta}(x,t) = \lambda^\beta \theta(\lambda x, \lambda^{\alpha+1}t)$$

for all $\alpha, \beta \in \mathbb{R}$ and $\lambda > 0$, the self-similar blowing up solution is of the form,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right), \tag{1.4}$$

$$\theta(x,t) = \frac{1}{(T_* - t)^{\beta}} \Theta\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \tag{1.5}$$

for $\alpha \neq -1$ and t sufficiently close to T_*. Substituting (1.4) and (1.5) into the above transport equation, we obtain

$$(ST)\begin{cases} \beta\Theta + \dfrac{1}{\alpha+1}(x\cdot\nabla)\Theta + (V\cdot\nabla)\Theta = 0, \\ \operatorname{div} V = 0. \end{cases}$$

The question of existence of suitable nontrivial solution to (ST) is equivalent to the that of existence of nontrivial self-similar finite time blowing up solution to the transport equation. We will establish the following theorem.

Theorem 1.3 *Let $v \in C([0,T_*); C^1(\mathbb{R}^n;\mathbb{R}^n))$ generate a C^1 diffeomorphism from $\mathbb{R}^n$ onto itself. Suppose there exist $\alpha \neq -1$, $\beta \in \mathbb{R}$ and solution (V,Θ) to the system (ST) with $\Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$ for some p_1, p_2 such that $0 < p_1 < p_2 \leq \infty$. Then, $\Theta = 0$.*

This theorem is a corollary of the following one.

Theorem 1.4 *Suppose there exists $T > 0$ such that there exists a representation of the solution $\theta(x,t)$ to the system (TE) by*

$$\theta(x,t) = \Psi(t)\Theta(\Phi(t)x) \qquad \forall t \in [0,T) \tag{1.6}$$

where $\Psi(\cdot) \in C([0,T);(0,\infty))$, $\Phi(\cdot) \in C([0,T);\mathbb{R}^{n\times n})$ with $\det(\Phi(t)) \neq 0$ on $[0,T)$; there exists $p_1 < p_2$ with $p_1, p_2 \in (0,\infty]$ such that $\Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Then, necessarily either $\det(\Phi(t)) \equiv \det(\Phi(0))$ and $\Psi(t) \equiv \Psi(0)$ on $[0,T)$, or $\Theta = 0$.

For the proof we refer [3].

2 The asymptotic self-similar singularities

In this section we generalize the previous notion, and consider the possibility of the asymptotic self-similar singularities. This notion was previously considered by Giga and Kohn in [11]. Here is the theorem for the Euler system.

Theorem 2.1 *Let $v \in C([0,T); B^{\frac{3}{p}+1}_{p,1}(\mathbb{R}^3))$ be a classical solution to the 3D Euler equations. Suppose there exist $p_1 > 0$, $\alpha > -1$, $\bar{V} \in C^1(\mathbb{R}^3)$ with $\lim_{R\to\infty} \sup_{|x|=R} |\bar{V}(x)| = 0$ such that $\bar{\Omega} =$curl $\bar{V} \in L^q(\mathbb{R}^3)$ for all $q \in (0, p_1)$, and the following convergence holds true:*

$$\lim_{t\nearrow T}(T-t)^{\frac{\alpha-3}{\alpha+1}} \left\| v(\cdot,t) - \frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}}\bar{V}\left(\frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}}\right)\right\|_{L^1} = 0, \qquad (2.1)$$

and

$$\lim_{t\nearrow T}(T-t) \left\| \omega(\cdot,t) - \frac{1}{T-t}\bar{\Omega}\left(\frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}}\right)\right\|_{\dot{B}^0_{\infty,1}} = 0. \qquad (2.2)$$

Then, $\bar{\Omega} = 0$, and $v(x,t)$ can be extended to a solution of the 3D Euler system in $[0, T+\delta] \times \mathbb{R}^3$, and belongs to $C([0,T+\delta]; B^{\frac{3}{p}+1}_{p,1}(\mathbb{R}^3))$ for some $\delta > 0$.

Remark 1.3 We note that Theorem 1.2 still does not exclude the possibility that the vorticity convergence to the asymptotically self-similar singularity is weaker than $L^\infty(\mathbb{R}^3)$ sense. Namely, a self-similar vorticity profile could be approached from a local classical solution in the pointwise sense in space, or in the $L^p(\mathbb{R}^3)$ sense for some p with $1 \le p < \infty$.

Next, we consider the asymptotic self-similar singularities for the Navier-Stokes equations. The following theorem for the case $p \in (3, \infty)$ was obtained by Hou and Li in [12]. In [4] we presented an alternative proof, which is very simple and elementary compared to the one given in [12].

Theorem 2.2 *Let $p \in [3, \infty)$, and $v \in C([0,T); L^p(\mathbb{R}^3))$ be a classical solution to (NS). Suppose there exists $\bar{V} \in L^p(\mathbb{R}^3)$ with $\nabla \bar{V} \in L^2_{loc}(\mathbb{R}^3)$ such that*

$$\lim_{t\nearrow T}(T-t)^{\frac{p-3}{2p}} \left\| v(\cdot,t) - \frac{1}{\sqrt{T-t}}\bar{V}\left(\frac{\cdot}{\sqrt{T-t}}\right)\right\|_{L^p} = 0. \qquad (2.3)$$

Then, $\bar{V} = 0$, and $v(x,t)$ can be extended to a solution of the Navier-Stokes equations in $[0, T+\delta] \times \mathbb{R}^3$ and belongs to $C([0,T+\delta]; L^p(\mathbb{R}^3))$ for some $\delta > 0$.

The following is a localized and improved version of the above theorem, the proof of which is in [4]

Theorem 2.3 *Let $p \in [3, \infty)$, and $v \in C([0,T); L^p(\mathbb{R}^3))$ be a classical solution to (NS). Suppose either one of the followings hold.*

(i) Let $q \in [3, \infty)$. Suppose there exists $\bar{V} \in L^p(\mathbb{R}^3)$ with $\nabla \bar{V} \in L^2_{loc}(\mathbb{R}^3)$ and $R \in (0, \infty)$ such that we have

$$\lim_{t \nearrow T} (T-t)^{\frac{q-3}{2q}} \sup_{t<\tau<T} \left\| v(\cdot, \tau) - \frac{1}{\sqrt{T-\tau}} \bar{V}\left(\frac{\cdot - z}{\sqrt{T-\tau}}\right) \right\|_{L^q(B(z, R\sqrt{T-t}))} = 0. \tag{2.4}$$

(ii) Let $q \in [2, 3)$. Suppose there exists $\bar{V} \in L^p(\mathbb{R}^3)$ with $\nabla \bar{V} \in L^2_{loc}(\mathbb{R}^3)$ such that (2.4) holds for all $R \in (0, \infty)$.

Then, $\bar{V} = 0$, and $v(x,t)$ is Hölder continuous near (z, T) in the space and the time variables.

Remark 1.5 We note that, in contrast to Theorem 1.4, the range of $q \in [2, 3)$ is also allowed for the possible convergence of the local classical solution to the self-similar profile.

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