

Directed sets and inverse limits

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Abstract

We show that the tree property for directed sets is equivalent to the nontriviality of certain inverse limits.

1 Directed sets and cofinal types

First we review the basic facts about cofinal types.

Definition 1.1 Let $\langle D, \leq_D \rangle, \langle E, \leq_E \rangle$ be directed sets. A function $f: E \rightarrow D$ which satisfies

$$\forall d \in D \exists e \in E \forall e' \geq_E e [f(e') \geq_D d]$$

is called a *convergent function*. If such a function exists we write $D \leq E$ and say E is *cofinally finer than* D . \leq is transitive and is called the *Tukey ordering* on the class of directed sets. A function $g: D \rightarrow E$ which satisfies

$$\forall e \in E \exists d \in D \forall d' \in D [g(d') \leq_E e \rightarrow d' \leq_D d]$$

is called a *Tukey function*.

If there exists a directed set C into which D and E can be embedded cofinally, we say D is *cofinally similar with* E . In this case we write $D \equiv E$. \equiv is an equivalence relation, and the equivalence classes with respect to \equiv are the *cofinal types*.

Proposition 1.2 For directed sets D and E , the following are equivalent.

- (a) $D \equiv E$.
- (b) $D \leq E$ and $E \leq D$.

So we can regard \leq as an ordering on the class of all cofinal types.

Definition 1.3 For a directed set D ,

$$\begin{aligned} \text{add}(D) &\stackrel{\text{def}}{=} \min\{|X| \mid X \subseteq D \text{ unbounded}\}, \\ \text{cof}(D) &\stackrel{\text{def}}{=} \min\{|C| \mid C \subseteq D \text{ cofinal}\}. \end{aligned}$$

These are the *additivity* and the *cofinality* of a directed set. We restrict ourselves to directed sets D without maximum, so $\text{add}(D)$ is well-defined.

Proposition 1.4 For a directed set D (without maximum),

$$\aleph_0 \leq \text{add}(D) \leq \text{cof}(D) \leq |D|.$$

Furthermore, $\text{add}(D)$ is regular and $\text{add}(D) \leq \text{cf}(\text{cof}(D))$. Here cf is the cofinality of a cardinal, which is the same as the additivity of it.

Proposition 1.5 For directed sets D and E , $D \leq E$ implies

$$\text{add}(D) \geq \text{add}(E) \quad \text{and} \quad \text{cof}(D) \leq \text{cof}(E).$$

From the above proposition we see that these cardinal functions are invariant under cofinal similarity.

2 The width of a directed set

In the following, κ is always an infinite regular cardinal. If P is partially ordered set, we use the notation $X_{\leq a} = \{x \in X \mid x \leq a\}$ for X a subset of P and $a \in P$. As usual, for cardinals $\kappa \leq \lambda$, $\mathcal{P}_\kappa \lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$ is ordered by inclusion.

Definition 2.1 The *width* of a directed set D is defined by

$$\text{wid}(D) \stackrel{\text{def}}{=} \sup\{|X|^+ \mid X \text{ is a thin subset of } D\},$$

where 'a thin subset of D ' means

$$\forall d \in D [|X_{\leq d}| < \text{add}(D)].$$

The reason to consider this cardinal function is to give a characterization of the tree property. See [2, Theorem 7.1].

Example 2.2 The set of singletons $\{\{\alpha\} \mid \alpha < \lambda\}$ is thin in $\mathcal{P}_\kappa \lambda$, so we have $\text{wid}(\mathcal{P}_\kappa \lambda) \geq \lambda^+$. If κ is strongly inaccessible, then $\mathcal{P}_\kappa \lambda$ is thin in itself, which shows $\text{wid}(\mathcal{P}_\kappa \lambda) = (\lambda^{<\kappa})^+$.

Lemma 2.3 For a directed set D and a cardinal $\lambda \geq \kappa := \text{add}(D)$, the following are equivalent.

- (a) D has a thin subset of size λ .
- (b) $D \geq \mathcal{P}_\kappa \lambda$.
- (c) There exists an order-preserving function $f: D \rightarrow \mathcal{P}_\kappa \lambda$ with $f[D]$ cofinal in $\mathcal{P}_\kappa \lambda$.

Corollary 2.4 The width of a directed set depends only on its cofinal type.

Lemma 2.5 $\text{add}(D)^+ \leq \text{wid}(D) \leq \text{cof}(D)^+$.

3 The tree property for directed sets

In the following definition, if D is an infinite regular cardinal κ , a ' κ -tree on κ ' coincides with the classical ' κ -tree'. Moreover, an 'arbor' is a generalization of a 'well pruned tree'.

Definition 3.1 (κ -tree) ([1]) Let D denote a directed set. A triple $\langle T, \leq_T, s \rangle$ is said to be a κ -tree on D if the following holds.

- 1) $\langle T, \leq_T \rangle$ is a partially ordered set.
- 2) $s: T \rightarrow D$ is an order preserving surjection.
- 3) For all $t \in T$, $s \upharpoonright T_{\leq t}: T_{\leq t} \xrightarrow{\sim} D_{\leq s(t)}$ (order isomorphism).
- 4) For all $d \in D$, $|s^{-1}\{d\}| < \kappa$. We call $s^{-1}\{d\}$ the *level* d of T .

Note that under conditions 1)2)4), condition 3) is equivalent to 3')

$$3') \text{ (downwards uniqueness principle) } \forall t \in T \forall d' \leq_D s(t) \exists! t' \leq_T t [s(t') = d'].$$

We write $t \downarrow d$ for this unique t' .

If a κ -tree $\langle T, \leq_T, s \rangle$ satisfies in addition

$$5) \text{ (upwards access principle) } \forall t \in T \forall d' \geq_D s(t) \exists t' \geq_T t [s(t') = d'],$$

then it is called a κ -arbor on D .

Definition 3.2 (tree property) ([1]) Let $\langle D, \leq_D \rangle$ be a directed set and $\langle T, \leq_T, s \rangle$ a κ -tree on D . $f: D \rightarrow T$ is said to be a faithful embedding if f is an order embedding and satisfies $s \circ f = \text{id}_D$. If for each κ -tree T on D there is a faithful embedding from D to T , we say that D has the κ -tree property. If D has the $\text{add}(D)$ -tree property, we say simply D has the tree property.

Proposition 3.3 ([1]) Let D be directed set and let $\kappa = \text{add}(D)$. D has the tree property iff for any κ -arbor on D there is a faithful embedding into it.

Proposition 3.4 ([1]) Let D be directed set and let $\theta < \text{add}(D)$. For any θ -tree T on D , the number of faithful embeddings from D into T is less than θ .

Proposition 3.5 ([1]) Let D be directed set and let θ be a cardinal.

- (1) If $\theta < \text{add}(D)$ then D has the θ -tree property.
- (2) If $\theta > \text{add}(D)$ then D does not have the θ -tree property.

Thus we are interested in the case $\theta = \text{add}(D)$.

Proposition 3.6 ([2]) If E has the tree property, $D \leq E$ in the Tukey ordering and $\text{add}(D) = \text{add}(E)$, then D also has the tree property. Thus the tree property is a property about the cofinal type of a directed set.

Corollary 3.7 ([1]) If D has the tree property, then $\text{add}(D)$ has the tree property in the classical sense.

Theorem 3.8 ([1]) For a strongly inaccessible cardinal κ , the following are equivalent:

- (a) κ is strongly compact.
- (b) All directed sets D with $\text{add}(D) = \kappa$ have the tree property.

Condition (b) also holds for $\kappa = \aleph_0$.

4 Inverse limits

Now we give a characterization of the tree property in terms of various inverse systems.

Theorem 4.1 Let D be a directed set, and let θ be a cardinal. The following are equivalent:

- (a) D has the θ -tree property.
- (b) For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D, d \leq d' \rangle$ of sets satisfying $|A_d| < \theta$ for all $d \in D$, the inverse limit $\varprojlim_{d \in D} A_d$ is nonempty.
- (c) For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D, d \leq d' \rangle$ of groups (respectively of abelian groups or free abelian groups), satisfying $|A_d| < \theta$ for all $d \in D$ and $\exists d_0 \in D \forall d \geq d_0 [f_{d_0 d} \neq 0]$, the inverse limit $\varprojlim_{d \in D} A_d$ has a nonzero element.
- (d) For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D, d \leq d' \rangle$ of vector spaces, satisfying $\dim(A_d) < \theta$ for all $d \in D$ and $\exists d_0 \in D \forall d \geq d_0 [f_{d_0 d} \neq 0]$, the inverse limit $\varprojlim_{d \in D} A_d$ has a nonzero element.

Proof (a) \Rightarrow (b) Let $\langle A_d, f_{dd'} \mid d, d' \in D, d \leq d' \rangle$ be an inverse system of nonempty sets, such that $|A_d| < \theta$ for all $d \in D$. Without loss of generality, we may assume that $\langle A_d \mid d \in D \rangle$ is a disjoint family. Put $T := \bigcup_{d \in D} A_d$ and define $s: T \rightarrow D$ so that $s^{-1}\{d\} = A_d$ for any $d \in D$. For $t, u \in T$ define the ordering \leq_T on T so that

$$t \leq_T u \iff \text{if } t \in A_d, u \in A_{d'} \text{ then } d \leq_D d' \text{ and } f_{dd'}(u) = t.$$

Then $\langle T, \leq_T, s \rangle$ is a θ -tree on D , and $\varprojlim_{d \in D} A_d$ is the set of all faithful embeddings from D into T . Hence

- (a) implies (b).
- (b) \Rightarrow (a) Let $\langle T, \leq_T, s \rangle$ be a given θ -tree on D . Define $f_{dd'}: s^{-1}\{d'\} \rightarrow s^{-1}\{d\}$ so that $f_{dd'}(t) = t \downarrow d$.

Then $\langle s^{-1}\{d\}, f_{dd'} \mid d, d' \in D, d \leq d' \rangle$ is an inverse system of nonempty sets, and $\varprojlim_{d \in D} s^{-1}\{d\}$ is the set of all faithful embeddings from D into T .

(b) \Rightarrow (c) Let $\langle A_d, f_{dd'} \mid d, d' \in D, d \leq d' \rangle$ be a given inverse system of groups, and assume that $|A_d| < \theta$ for all $d \in D$ and that there is some $d_0 \in D$ such that $f_{d_0d} \neq 0$ for all $d \geq d_0$. Put

$$\begin{aligned} B_d &:= f_{dd'}[A_{d_0} \setminus \{0\}] && \text{for } d \geq d_0, \\ g_{dd'} &:= f_{dd'} \upharpoonright B_{d'} && \text{for } d' \geq d \geq d_0. \end{aligned}$$

Then $\langle B_d, g_{dd'} \mid d, d' \in D_{\geq d_0}, d \leq d' \rangle$ is an inverse system of nonempty sets. By (b), we can pick some $b \in \varprojlim_{d \geq d_0} B_d$. Since $D_{\geq d_0}$ is cofinal in D and D is directed, we can extend this b to a unique

$$a \in \left(\varprojlim_{d \in D} A_d \right) \setminus \{0\}.$$

(c) \Rightarrow (b) Let $\langle A_d, f_{dd'} \mid d, d' \in D, d \leq d' \rangle$ be an inverse system of nonempty sets such that $|A_d| < \theta$ for all $d \in D$. Since (a), and hence (b) is always true for $\theta = \aleph_0$, we may assume $\theta > \aleph_0$. For $d \in D$, let B_d be the free abelian group with generators in A_d , i.e.

$$B_d := \{b \in {}^{A_d}\mathbb{Z} \mid b(x) = 0 \text{ for all but finitely many } x \in A_d\}.$$

Let $\text{supt}(b) := \{x \in \text{dom}(b) \mid b(x) \neq 0\}$. We identify $b \in B_d$ with the expression $n_0x_0 + \cdots + n_kx_k$, where $\{x_0, \dots, x_k\} \supseteq \text{supt}(b)$ and $b(x) = \sum_{\substack{i \leq k \\ x_i = x}} n_i$ for $x \in A_d$. Clearly $|B_d| < \theta$. For $d \leq d'$ in D , put

$$\begin{array}{ccc} g_{dd'} : & B_{d'} & \rightarrow & B_d \\ & \cup & & \cup \\ & n_0x_0 + \cdots + n_kx_k & \mapsto & n_0f_{dd'}(x_0) + \cdots + n_kf_{dd'}(x_k). \end{array}$$

Then $\langle B_d, g_{dd'} \mid d, d' \in D, d \leq d' \rangle$ is an inverse system of free abelian groups, and $g_{dd'} \neq 0$ for any $d \leq d'$ in D . Thus by (c), there is some $b^* \in \left(\varprojlim_{d \in D} B_d \right) \setminus \{0\}$. Since $b^* \neq 0$, there is some $d_0 \in D$ such that $b^*(d_0) \neq 0$ for all $d \geq d_0$. Put

$$\begin{aligned} F_d &:= \text{supt } b^*(d) \cap f_{d_0d}^{-1}[\text{supt } b^*(d_0)] && \text{for } d \geq d_0, \\ h_{dd'} &:= f_{dd'} \upharpoonright F_{d'} && \text{for } d' \geq d \geq d_0. \end{aligned}$$

Note that $h_{dd'}[F_{d'}] = F_d$. Now $\langle F_d, h_{dd'} \mid d' \geq d \geq d_0 \rangle$ is an inverse system of nonempty finite sets. Since any directed set has the \aleph_0 -tree property, $\varprojlim_{d \geq d_0} F_d \neq \emptyset$. Take any $a \in \varprojlim_{d \geq d_0} F_d$. There is a unique

$a' \in \varprojlim_{d \in D} A_d$ which extends a .

(b) \Rightarrow (d) This is similar to the proof of (b) \Rightarrow (c).

(d) \Rightarrow (b) This is similar to the proof of (c) \Rightarrow (b). □

Corollary 4.2 *If G is the inverse limit of $\langle G_d, f_{dd'} \mid d, d' \in D, d \leq d' \rangle$ where each G_d is finite (i.e. G is a profinite group), then $G \neq 0$ iff $\exists d_0 \in D \forall d \geq d_0 [f_{d_0d} \neq 0]$.*

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