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<thead>
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<th>Title</th>
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</tr>
</thead>
<tbody>
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Submetrizability and Interpolations

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All topological spaces considered here are Tychonoff. For a topological space $X$, $C(X)$ is the Banach space of all bounded real-valued continuous functions with the sup norm: $\|f\| = \sup\{|f(x)| : x \in X\}$ for $f \in C(X)$. The space $F(X \times \mathbb{R})$ is the hyperspace consisting of all finite subsets of the product space $X \times \mathbb{R}$. Of course, its topology is the Vietoris topology. Let $S(X)$ be the subspace of $F(X \times \mathbb{R})$ defined by

$$S(X) = \{(x_1, r_1), \cdots, (x_n, r_n) \} : x_i \neq x_j \quad \text{for} \quad i \neq j\}.$$

For each $n = 1, 2, \cdots$, define $F_n(X \times \mathbb{R})$ and $S_n(X)$ by:

$$F_n(X \times \mathbb{R}) = \{ D \in F(X \times \mathbb{R}) : \text{D has at most n points} \} ,$$

$$S_n(X) = S(X) \cap F_n(X \times \mathbb{R}).$$

For a point $D = \{(x_1, r_1), (x_2, r_2), \cdots, (x_n, r_n)\} \in S(X)$, a function $f_D$ in $C(X)$ is called an interpolation function for $D$ if

$$f_D(x_1) = r_1, f_D(x_2) = r_2, \cdots, f_D(x_n) = r_n$$

are satisfied. A map $\Theta : S(X) \rightarrow C(X)$ is called interpolation (algorithm) if $\Theta(D) = f_D$ is an interpolation function for each $D \in S(X)$. Further, if $\Theta$ satisfies the condition that the restriction $\Theta|_{S_n(X) - S_{n-1}(X)}$ is continuous for each $n = 1, 2, \cdots$, we call $\Theta$ to be a weakly continuous interpolation. Let a topology $\tau_1$ on a set $X$ is stronger than a topology $\tau_2$ on the same set. If $\Theta$ is a weakly continuous interpolation on $\tau_2$, then the same $\Theta$ is a weakly continuous interpolation on $\tau_1$. Every metrizable space has a weakly continuous interpolation, and hence every submetrizable space has a weakly continuous interpolation. In [1], a weakly continuous interpolation on a metric space $(X, d)$ is constructed as follows: For any $D = \{(x_1, r_1), \cdots, (x_n, r_n)\} \in S(X)$, let

$$m = \min\{d(x_i, x_j) : i \neq j\}$$
and

$$f_D(x) = \begin{cases} 0 & \text{if } d(x, x_i) \geq m/4 \text{ for each } i = 1, \ldots, n \\ r_i - 4r_i d(x, x_i) & \text{if } d(x, x_i) < m/4 \text{ for some } i = 1, \ldots, n \end{cases}$$

Then the map $\Theta : S(X) \rightarrow C(X)$ defined by $\Theta(D) = f_D$ is weakly continuous. For this interpolation, we can show the following. Here, for any distinct $p, q \in X$ $D_{pq}$ denotes the sample $\{(p, -1), (q, 1)\}$ in $S_2(X) - S_1(X)$.

**Proposition 1** The weakly continuous interpolation $\Theta : S(X) \rightarrow C(X)$ on a metric space $(X, d)$ constructed above satisfies the following condition: There exists a constant $M$ such that

$$||f_{D_{wx}} - f_{D_{zy}}|| \leq M \max\{||f_{D_{wy}} - f_{D_{xy}}||, ||f_{D_{xz}} - f_{D_{zy}}||\}$$

for any $x, y, z, w \in X$.

**Proof.** We can assume that metric function is bounded. Assume that $d(x, y) < 1$ for any $x, y \in X$. It suffices to show that $||f_{D_{wy}} - f_{D_{xy}}|| < \epsilon$, $||f_{D_{xz}} - f_{D_{zy}}|| < \epsilon$ imply $||f_{D_{wx}} - f_{D_{zy}}|| < 6\epsilon$.

Claim 1. Let $0 < \alpha < \frac{1}{2}$. If $d(z, y) < \alpha d(w, y)$, then $||f_{D_{wy}} - f_{D_{wp}}|| \leq 9\alpha$.

Since interpolation functions defined above are piecewise linear on the distance from data points, $||f_{D_{wy}} - f_{D_{wp}}||$ does not exceed the following 5 values:

1. At $z$, $||f_{D_{wy}}(z) - f_{D_{wy}}(z)|| = \frac{4}{d(w, y)}d(y, z) < \frac{4d(w, y)}{d(w, y)} = 4\alpha$.
2. At $y$, $||f_{D_{wy}}(y) - f_{D_{wy}}(y)|| = \frac{4}{d(w, y)}d(y, z) < \frac{4d(w, y)}{(1 - \alpha)d(w, y)} = \frac{4\alpha}{1 - \alpha} < 8\alpha$.
3. When the distance from $y$ is $\frac{d(w, y)}{4}$, $\frac{4}{d(w, y)}(\frac{d(w, y)}{4} - \frac{d(w, y)}{4}) = \frac{4\alpha}{1 + \alpha} < 9\alpha$.

4. When the distance from $z$ is $\frac{d(w, y)}{4}$, $\frac{4}{d(w, y)}(d(y, z) + \frac{d(w, y)}{4} - \frac{d(w, y)}{4}) = (\frac{4d(y, z)}{d(w, y)} + 1 - \frac{d(w, y)}{d(w, y)}) < \alpha + 1 - \frac{d(w, y)}{d(w, y)} < 2\alpha$.

5. When the distance from $w$ is $\frac{d(w, y)}{4}$ in case $d(w, z) \leq d(w, y)$, $\frac{4}{d(w, y)}(\frac{d(w, y)}{4} - \frac{d(w, y)}{4}) = 1 - \frac{d(w, y)}{d(w, y)} \leq 1 - \frac{d(w, y)}{d(w, y)} = 1 - \frac{d(w, y)}{d(w, y)} = \frac{d(w, y)}{1 + \alpha} < \alpha$.

Claim 2. Let $0 < \epsilon < 1$. If $||f_{D_{wy}} - f_{D_{zy}}|| < \epsilon$, $||f_{D_{xz}} - f_{D_{zy}}|| < \epsilon$, then $d(y, z) < \frac{\epsilon}{2}d(w, y)$.

Since $||f_{D_{wy}} - f_{D_{zy}}|| < \epsilon$, estimating the value at $z$, we obtain $d(y, z) \frac{4}{d(w, y)}d(w, z) < \epsilon$. Hence $d(x, w) < \frac{\epsilon}{4}d(w, y)$. Similarly, the value of $||f_{D_{zy}} - f_{D_{zy}}||$ at $z$ is $d(y, z) \frac{4}{d(w, y)}d(w, y) < \epsilon$. Hence it follows that $d(y, z) < \frac{\epsilon}{4}d(w, y)$. Then $d(y, z) < \frac{\epsilon}{4}(d(x, w) + d(w, y)) < \frac{\epsilon}{4}(d(w, y) + d(w, y)) = ((\frac{\epsilon}{4})^2 + \frac{\epsilon}{4})d(w, y) < \frac{\epsilon}{2}d(w, y)$. 


By claim 2, \(d(y, z) < \frac{\varepsilon}{2}d(w, y)\). Then \(\|f_{D_{wy}} - f_{D_{wz}}\| < \frac{9}{2}\varepsilon < 5\varepsilon\) by claim 1. Hence \(\|f_{D_{wz}} - f_{D_{wy}}\| \leq \|f_{D_{wz}} - f_{D_{wz}}\| + \|f_{D_{wy}} - f_{D_{wy}}\| < 5\varepsilon + \varepsilon = 6\varepsilon\).

Let us call a weakly continuous interpolation to be regular when the condition in the proposition is satisfied. A sequence \(\{G_n\}\) of open covers of \(X\) is called a \(G_\delta\)-diagonal if for any \(x \neq y \in X\), there exists \(n\) such that \(y \notin s(x, G_n)\). Further \(G_\delta\)-diagonal sequence \(\{G_n\}\) is called regular if for \(G, G' \in G_{n+1}\), \(G \cap G' \neq \emptyset\) implies that there exists \(G \in G_n\) such that \(G \cup G' \subset G\). It is well known that a space \(X\) is submetrizable if and only if \(X\) has a regular \(G_\delta\)-diagonal sequence (see [2]).

**Theorem 1** The following are equivalent.

1. \(X\) is submetrizable.
2. \(X \times (\omega + 1)\) has a weakly continuous regular interpolation.

**Proof.** If \(X\) is submetrizable, then \(X \times (\omega + 1)\) is also submetrizable. Hence it follows that (1) implies (2) from the proposition above.

Assume that (2) is satisfied. We will show that \(X\) has a regular \(G_\delta\)-diagonal sequence. Let \(\Theta : S(X \times (\omega + 1)) \to C(X \times (\omega + 1))\) be a weakly continuous interpolation which satisfies the regular condition. Let us recall the notation that \(D_{(p,m)(q,n)} = \{((p,m), -1), ((q,n), 1)\} \in S_2(X \times (\omega + 1)) - S_1(X \times (\omega + 1))\) for any \((p,m), (q,n) \in X \times (\omega + 1)\) and \(f_{D_{(p,m)(q,n)}} = \Theta(D_{(p,m)(q,n)})\). Then there exists a constant \(M\) such that

\[
\|f_{D_{(w,l)(z,k)}} - f_{D_{(x,i)(y,j)}}\| \leq M \max\{\|f_{D_{(w,1)(y,j)}} - f_{D_{(x,i)(y,j)}}\|, \|f_{D_{(*:*)(z,k)}} - f_{D_{(*:y)(z,k)}}\|\}
\]

for any \(D_{(x,i)(y,j)}, D_{(w,l)(y,j)}, D_{(x,i)(z,k)}, D_{(w,l)(z,k)} \in S_2(X \times (\omega + 1)) - S_1(X \times (\omega + 1))\). It can be also assumed that \(M > 1\).

Let \(G_n\) be the family of all open subsets \(U\) which satisfies

\[
\|f_{D_{(x,i)(y,w)}} - f_{D_{(x',i)(y',w)}}\| < \frac{1}{(2M)^n}
\]

for any \(x, y, x', y' \in U\) and any \(i = 0, 1, \cdots, n\).

**Claim 1.** \(G_n\) is an open cover of \(X\).

For any \(x \in X\), consider \(D_{(x,i)(x,\omega)}\) for \(i = 0, 1, \cdots, n\). Since \(\Theta\) is weakly continuous, for each \(i = 0, 1, \cdots, n\) there exists an open neighborhood \(U_i\) of \(x\) such that

\[
\|f_{D_{(u,i)(v,\omega)}} - f_{D_{(x,i)(z,\omega)}}\| < \frac{1}{2(2M)^n}
\]

for any \(u, v \in U_i\). Then \(U = \cap_{i=0}^n U_i \in G_n\) is an open neighborhood of \(x\) in \(X\).
Claim 2. $\{G_n\}$ is a $G_\delta$-diagonal sequence of $X$.

Assume that $\{G_n\}$ is not a $G_\delta$-diagonal sequence. Then there exist two distinct point $x_0, y_0$ such that for each $n$, $x_0, y_0 \in U_n$ for some $U_n \in \mathcal{G}_n$. Let us take $D_{(x_0, \omega)(y_0, \omega)} = \{((x_0, \omega), -1), ((y_0, \omega), 1)\}$. Then $f_{D_{(x_0, \omega)(y_0, \omega)}}((x_0, \omega)) = -1$. Let $W$ be a neighborhood of $D_{(x_0, \omega)(y_0, \omega)}$ in $S_2(X \times (\omega + 1)) - S_1(X \times (\omega + 1))$ such that $\|f_D - f_{D_{(x_0, \omega)(y_0, \omega)}}\| < 1$ for any $D \in W$. Then there exist $n$ such that $D_{(x_0, \iota)(y_0, \omega)} \in W$ for any $i \geq n$. Especially, for such $D_{(x_0, \iota)(y_0, \omega)}$, the value $|f_{D_{(x_0, \omega)}((x_0, \omega))} - f_{D_{(y_0, \omega)}((y_0, \omega))}| < 1$ at $(x_0, \omega)$, and hence

$$f_{D_{(x_0, \iota)(y_0, \omega)}}((x_0, \omega)) < 0.$$ 

On the other hand, since $x_0, y_0 \in U_i$ for $i \geq n$, $\|f_{D_{(x_0, \iota)(y_0, \omega)}} - f_{D_{(x_0, \omega)}((x_0, \omega))}\| < \frac{1}{(2M)^{n+1}}$ and $f_{D_{(x_0, \iota)(x_0, \omega)}}((x_0, \omega)) = 1$, it must be

$$f_{D_{(x_0, \iota)(y_0, \omega)}}((x_0, \omega)) > 0.$$ 

This is a contradiction.

Claim 3. $\{G_n\}$ is regular.

Assume that $U_1, U_2 \in \mathcal{G}_{n+1}$ satisfy $U_1 \cap U_2 \neq \emptyset$. Then there exist $p \in U_1 \cap U_2$. For any $x, y \in U_1 \cup U_2$, it is shown that $\|f_{D_{(x, \iota)(y, \omega)}} - f_{D_{(p, \iota)(p, \omega)}}\| < \frac{1}{2n+1M^n}$ for any $i = 0, 1, \cdots, n$. In fact, in case $x, y \in U_1$ or $x, y \in U_2$, it is obvious. In other case, since $\|f_{D_{(x, \iota)(p, \omega)}} - f_{D_{(p, \iota)(p, \omega)}}\| < \frac{1}{(2M)^{n+1}}$, $\|f_{D_{(p, \iota)(p, \omega)}} - f_{D_{(p, \iota)(p, \omega)}}\| < \frac{1}{(2M)^{n+1}}$, it follows that $\|f_{D_{(x, \iota)(y, \omega)}} - f_{D_{(p, \iota)(p, \omega)}}\| < \frac{1}{2n+1M^n}$ by the regularity condition of $\Theta$. Then for $i = 0, 1, \cdots, n$ and for any $x, y, x', y' \in U_1 \cup U_2$,

$$\|f_{D_{(x, \iota)(y, \omega)}} - f_{D_{(x', \iota)(y', \omega)}}\| \leq \|f_{D_{(x, \iota)(y, \omega)}} - f_{D_{(p, \iota)(p, \omega)}}\| + \|f_{D_{(p, \iota)(p, \omega)}} - f_{D_{(x', \iota)(y', \omega)}}\| \leq \frac{1}{2n+1M^n} + \frac{1}{2n+1M^n} < \frac{1}{(2M)^n}.$$ 

This shows that $U_1 \cup U_2 \in \mathcal{G}_n$.

In the proof of the above theorem, we used the regularity condition on the interpolation only to show the regularity of the $G_\delta$-diagonal sequence. Hence the following theorem is also obtained.

**Theorem 2** If $X \times (\omega + 1)$ has a weakly continuous interpolation, then $X$ has a $G_\delta$-diagonal. In particular, for a paracompact space $X$, $X \times (\omega + 1)$ has a weakly continuous interpolation if and only if $X$ has a $G_\delta$-diagonal.
Further, we used interpolation functions essentially for only $D \in S_2(X \times (\omega + 1)) - S_1(X \times (\omega + 1))$ to show the submetrizability of $X$ in Theorem 1. Let us call a map $\Theta_2 : S_2(X) - S_1(X) \to C(X)$ to be a continuous $S_2$-interpolation if it is continuous and $\Theta_2(D)$ is an interpolation function for every $D \in S_2(X) - S_1(X)$. Theorem 1 can be rewritten as the following.

**Theorem 3** The following are equivalent.

1. $X$ is submetrizable.
2. $X \times (\omega + 1)$ has a weakly continuous regular interpolation.
3. $X \times (\omega + 1)$ has a continuous regular $S_2$-interpolation.

**Remark.** It may be generally shown that a space $X$ has a weakly continuous interpolation if and only if $X$ has a continuous $S_2$-interpolation.

参考文献
