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Determining covers, and covering properties

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1. Introduction

We assume that spaces are regular T_1, and maps are continuous and onto.

For a cover $\mathcal{P}$ of a space X, we recall that X *is determined by* $\mathcal{P}$ [6], if X has the *weak topology* with respect to $\mathcal{P}$ [4]; that is, $G \subset X$ is open in X if $G \cap P$ is open in P for each $P \in \mathcal{P}$. Here, we can replace "open" by "closed" twice. We call such a cover $\mathcal{P}$ a *determining cover* [37, 39] (or [11]). For some (basic) properties on weak topologies, see [4, 36], etc.

For a closed cover $\mathcal{F}$ of a space X, we recall that X *is dominated by* $\mathcal{F}$ [12] if $\mathcal{F}$ is a CP cover such that any $\mathcal{P} \subset \mathcal{F}$ is a determining cover of the union of $\mathcal{P}$. (Sometimes, we say that X has the *Whitehead weak topology*; *Morita weak topology* [15]); or *hereditarily weak topology*, with respect to $\mathcal{F}$). We call such a closed cover $\mathcal{F}$ a *dominating cover* [37, 39] (or [11]).

A collection $\mathcal{P}$ of sets in X is *closure-preserving* (abbreviated by CP), if for any subfamily $\mathcal{P}'$ of $\mathcal{P}$, $cl(\bigcup\{P : P \in \mathcal{P}'\}) = \bigcup\{clP : P \in \mathcal{P}'\}$. Also, $\mathcal{P}$ is *hereditarily closure-preserving* (abbreviated by HCP), if for any subcollection $\mathcal{P}' = \{P_\alpha : \alpha\}$ of $\mathcal{P}$, and any $\{A_\alpha : \alpha\}$ such that $A_\alpha \subset P_\alpha$, the collection $\{A_\alpha : \alpha\}$ is CP.

Locally finite closed cover $\Rightarrow$ *HCP closed cover* $\Rightarrow$ *Dominating cover* $\Rightarrow$ *Determining CP closed cover* $\Rightarrow$ *Determining cover* $\Leftarrow$ *Open cover.*

A space X is a *sequential space* (resp. *k-space*; *quasi-k-space* [18]) if X has a determining cover by all compact metric sets (resp. compact sets; countably compact sets). Here, we can replace "all" by "some". As is well-known, every sequential space (resp. k-space; quasi-k-space) is characterized as a quotient space of a locally compact metric space (resp. locally compact paracompact space; M-space). We recall that a space X has *countable tightness*, denoted $t(X) \leq \omega$, if whenever $x \in clA$, $x \in clC$ for some countable $C \subset A$; equivalently, X has a determining cover by countable sets (cf. [13]). Sequential spaces, or hereditarily separable spaces have countable tightness.

A space X having an increasing determining cover $\{X_n : n \in N\}$ is called the *inductive limit* of $\{X_n : n \in N\}$. When X_n are closed in X, $\{X_n : n \in N\}$ is a dominating cover of X. (But, every compact metric space having an increasing countable CP determimning closed cover (not a sequence) need no be a dominating cover). As is well-known, every CW-complex has a dominating cover by compact metric sets. For spaces dominated by metric sets, see [28], etc.

Remark 1.1. (1) Every space with a determining cover by sequential spaces (resp. k-spaces; quasi-k-spaces) is a sequential space (resp. k-space; quasi-k-space). While, every space with a dominating cover by paracompact spaces (resp. normal spaces) is paracompact (resp. normal); see [12] or [16].

(2) Every space with a CP cover by compact subsets is meta-compact, but not every normal space with a CP cover by finite sets is paracompact (see [45], etc.). While, every first contable, locally compact, separable, and σ-space having a determining CP closed cover by locally compact, metric sets need not be meta-compact nor normal.

In terms of weak topologies, the author has studied products of sequential spaces, k-spaces, and spaces having countable tightness, and studied topological properties of spaces having certain k-networks, topological groups, CW-complexes, GO-spaces, etc., in his (or joint) reports [9, 10, 26, 29, 31, 32, 34, 37, 38], etc., in RIMS *Kôkyûroku*, Research Institute for Mathematical Sciences Kyoto University. Concerning determining or dominating covers, see his (or joint) recent papers [11, 39, 40, 41], etc.

Concerning determining covers (containing dominating covers), the author had questions (Q1), (Q2), (Q3), (Q4) below. In RIMS Kyoto University, he gave a lecture related to (Q3) in 2004, and wrote [37] related to (Q3); and [38] related to (Q1), (Q2), and (Q3) for countable products of determining covers. For announcements or summaries related to answers to (Q1) $\sim$ (Q4), also see [39]. In particular, answers to (Q3) and results on countable products of determining covers will be appeared in [40].

In this paper, we shall consider question (Q4) below. The results except Theorems 2.7, 2.9, 2.10, 2.12, etc., would be (essentially) given in [39].

(Q1): Let $f : X \to Y$ be a map, and let $\mathcal{P}$ be a determining cover of X (resp. Y). Under what conditions, is $\{f(P) : P \in \mathcal{P}\}$ (resp. $\{f^{-1}(P) : P \in \mathcal{P}\}$) a determining cover of Y (resp. X) ?

(Q2): Let $\mathcal{P}$ be a determining cover of X. For a (or any) set $S \subset X$, under what conditions, is $\{P \cap S : P \in \mathcal{P}\}$ a determining cover of S ?

(Q3): Let $\mathcal{P}_i$ $(i = 1, 2)$ be a determining cover of X_i. Under what conditions, is $\{P_1 \times P_2 : P_i \in \mathcal{P}_i\}$ a determining cover of $X_1 \times X_2$?

(Q4): Let $\mathcal{P}$ be a determining cover of X. Under what conditions, does $\mathcal{P}$ have a nice determining refinement, or X have an open cover $[\mathcal{P}]^\circ$?

Here, a cover $\mathcal{A}$ of X is a *refinement* of a cover $\mathcal{P}$ if each element of $\mathcal{A}$ is contained in some element of $\mathcal{P}$. Also, for a cover $\mathcal{P}$ of X, let

$\mathcal{P}^\circ = \{intP : P \in \mathcal{P}\}$, and

$[\mathcal{P}] = \{S : S$ is a *finite* union of elements of $\mathcal{P}\}$.

Obviously, for a binary determining closed cover $\mathcal{P}$ of X by convergent sequences, $\mathcal{P}^\circ$ need not be an open cover of X.

For question (Q4), we have the following (negative) examples which are stated in [39]. Here, a collection $\mathcal{P}$ of sets in X is *point-countable* (resp. *point-finite*) if every $x \in X$ is in at most countably many (resp. finitely many) $P \in \mathcal{P}$.

Example 1.2. (1) A space X which has a countable and point-finite determining closed cover $\mathcal{F}$ by convergent sequences, but $\mathcal{F}$ has no CP refinements, hence no dominating refinements, and $[\mathcal{F}]^\circ$ is not a cover of X. Also, a space which has a countable dominating cover has no HCP refinements.

(2) A first countable σ-space X which has a point-finite closed and open determining cover $\mathcal{C}$ by metric sets, and a separable space Y which has a point-finite determining cover $\mathcal{F}$ by compact metric sets, but both of X and Y are not paracompact, so they have no dominating covers by paracompact sets. The covers $\mathcal{C}$ and $\mathcal{F}$ have a σ-discrete refinement, but these covers have no CP refinements, and no σ-CP determining refinements, and $[\mathcal{F}]^\circ$ is not a cover of Y.

(3) A Fréchet space X which has a HCP closed cover (hence, dominating cover) $\mathcal{F}$ by convergent sequences, but X has no point-countable determining covers by metric sets, thus $\mathcal{F}$ has no point-countable determining refinements, and $[\mathcal{F}]^\circ$ is not a cover of X.

(4) A CW-complex X which has a dominating cover (or, a point-finite determining cover) $\mathcal{F}$ by compact metric sets, but $\mathcal{F}$ has no σ-HCP or σ-locally countable determining refinements, and $[\mathcal{F}]^\circ$ is not a cover of X.

2. Results

A space X is *strongly Fréchet* [21] (= countably bi-sequential [13]), if for each decreasing sequence (A_n) in X with $x \in \bigcap\{clA_n : n \in N\}$, there exists a sequence $\{x_n : n \in N\}$ converging to the point x such that $x_n \in A_n$ $(n \in N)$. When the A_n are all the same set, then such a space is so-called *Fréchet* (= *Fréchet-Urysohn*).

A decreasing sequence (A_n) of non-empty sets of X is a *q-sequence* [13], if $C = \bigcap\{A_n : n \in N\}$ is countably compact in X such that each open set U

with $C \subset U$ contains some A_n (equivalently, for any $x_n \in A_n$, $\{x_n : n \in N\}$ has an accumulation point in C). A space X is *countably bi-quasi-k* [13] if, for each decreasing sequence (A_n) with $x \in clA_n$, there exists a q-sequence (B_n) such that $x \in cl(A_n \cap B_n)$ for each A_n. Also, X is *singly bi-quasi-k* if the A_n are all the same set.

Locally compact spaces, strongly Fréchet spaces, or M-spaces are countably bi-quasi-k. Countably bi-quasi-k-spaces, or Fréchet spaces are singly bi-quasi-k. Singly quasi-k-spaces are quasi-k. For these spaces and their peripheral spaces, see [13].

Theorem 2.1. (1) For an infinite cardinal α, let X be α-compact (i.e., every subset with size α has an accumulation point in X). Then every dominating or point-countable determining cover of X has a subcover with size $< \alpha$ ([44]).

(2) Let X be separable. Then every dominating cover of X has a countable determining subcover. When X is singly bi-quasi-k, every point-countable determining closed cover has a countable determining subcover ([43]).

Theorem 2.2. For a singly bi-quasi-k-space X, the following hold.

(1) Every dominating (or every countable determining closed) cover of X has a HCP closed refinement $\mathcal{C}$, and X is decomposed into spaces X_1 and X_2 (abbreviated by $X = X_1 + X_2$) such that X_1 is closed discrete in X, and $\mathcal{C}$ is locally finite on X_2.

(2) For a point-countable determining closed cover $\mathcal{F}$ of X, $\{\text{int}(\cup_{n=1}^{\infty} F_n) : F_n \in \mathcal{F}\}$ is an open cover of X. Also, $X = X_1 + X_2$ such that X_1 is closed discrete in X, and $[\mathcal{F}]^\circ$ covers X_2. When the cover $\mathcal{F}$ is point-finite in X, $[\mathcal{F}]^\circ$ is an open cover of X.

(3) When X is a countably bi-quasi-k-space, the cover $\mathcal{C}$ in (1) is locally finite in X, and the cover $[\mathcal{F}]^\circ$ in (2) is an open cover of X.

Corollary 2.3. Let X be a separable singly bi-quasi-k-space. Then evey point-countable determining closed cover of X has a countable HCP closed refinement $\mathcal{F}$, and $X = X_1 + X_2$ such that X_1 is closed discrete in X, and $\mathcal{F}$ is a locally finite cover on X_2.

Corollary 2.4. (1) Let X have a dominating cover $\mathcal{F}$ by metric sets. Then the following are equivalent.

(a) X is a singly bi-quasi-k-space.

(b) $\mathcal{F}$ has a HCP closed refinement.

(c) X is a closed image of a metric space.

(2) Let X have a point-countable determining closed cover $\mathcal{F}$ by locally separable metric sets, or let X be a locally separable space which has a point-

countable determining closed cover $\mathcal{F}$ by metric sets. Then the following are equivalent.

(a) X is a singly bi-quasi-k-space.

(b) $\mathcal{F}$ has a refinement which is a locally countable and σ-locally finite HCP closed cover (by separable metric sets).

(c) $\mathcal{F}$ has a HCP closed refiniment.

(d) X is a closed s-image of a locally separable, metric space.

Remark 2.5. (1) Similarly, for a space X having a point-countable determining closed cover $\mathcal{F}$ by metric sets, $\mathcal{F}$ has a refinement which is a locally countable and σ-locally finite HCP closed cover $\Leftrightarrow$ $\mathcal{F}$ has a HCP closed refiniment $\Leftrightarrow$ X is a closed (s-)image of a metric space. However, the first countable σ-space X in Example 1.2(2) has a point-finite closed and open determining cover by metric sets, but X is not normal. Hence, X is not a closed image of a metric space, and X doesn't have any dominating or σ-locally finite determining closed cover by metric sets.

(2) Every closed image of a countable metric space need not have a dominating or point-countable determining cover by metric sets ([44]).

Corollary 2.6. For a paracompact singly bi-quasi-k-space X, the following hold.

(1) Every point-finite determining closed cover of X has a locally finite closed refinement.

(2) Every point-countable determining closed cover of X has a σ-locally finite closed refinement $\mathcal{F}$.

A space X is an *A-space* [14] if, whenever (A_n) is a decreasing sequence in X with $x \in cl(A_n - \{x\})$, then there exist $B_n \subset A_n$ such that $\bigcup\{clB_n : n \in N\}$ is not closed in X. Also, X is an *inner-closed A-space* (resp. *inner-one A-space*) when the B_n are closed sets (resp. singletons). Also, X is respectively an *A'-space*; *inner-closed A'-space*; *inner-one A'-space* if we assume $\bigcap\{A_n : n \in N\} = \emptyset$ for the decreasing sequence (A_n) in the above. For a space X of non-measurable cardinality or $t(X) \leq \omega$, X is an A-space iff X is an A'-space, and we can add a prefix "inner-one" (or "inner-closed") twice ([14]).

Let us consider the following property (P) which is defined in [39].

(P): For each decreasing sequence (A_n) in X with $\bigcap\{clA_n : n \in N\} \neq \emptyset$, there exists a countably compact set K of X with $K \cap A_n \neq \emptyset$ for all $n \in N$.

In (P), when the countably compact set K is compact, (P) is just property

(3.1) in [6]. When the countably compact set K is a convergent sequence in X, (P) is precisely condition (C) in [24], and a space X satisfying (C) is called a *Tanaka space* in [17]. (For properties of Tanaka spaces and related spaces, see [41]). When there exist $x_n \in A_n$ such that the sequence $\{x_n : n \in N\}$ has an accumulation point in X, (P) is just condition (C^*) in [42].

Countably bi-quasi-k or Tanaka space $\Rightarrow$ Space having (P) $\Rightarrow$ Space satisfying (C^*) $\Leftrightarrow$ Inner-one A'-space $\Rightarrow$ Inner-closed A'-space $\Leftarrow$ Inner-closed A-space $\Leftarrow$ Inner-one A-space $\Leftarrow$ Countably bi-quasi-k-space.

Theorem 2.7. (1) For a quasi-k-space X, X is inner-closed $A \Leftrightarrow X$ is inner-one $A \Rightarrow X$ has property (P) $\Leftrightarrow X$ is inner-one $A' \Leftrightarrow X$ is inner-closed A'. When X has non-measurable cardinality or $t(X) \leq \omega$, these are all equivalent.

(2) For a space X, X is a Tanaka space iff X has property (P) and each countably compact set is sequentially compact. When X is sequential, X is a Tanaka space iff X is one of the spaces in (1).

Remark 2.8. Related to Theorem 2.7, there exists a countable inner-one A-space, but it is not a (quasi-)k-space, and it doesn't have property (P) (cf. [14]). While, under the existence of a measurable cardinal, there exists a Tanaka space (hence having (P)), but it is not an inner-one A-space, not a quasi-k-space (cf. [13]).

Theorem 2.9. (1) Let X be a Fréchet space, or a sequential hereditarly normal space. Then X is an inner-closed A'-spac iff X is strongly Fréchet.

(2) Let X be a space whose points are G_δ-sets. Then X has property (P) $\Leftrightarrow X$ is strongly Fréchet $\Leftrightarrow X$ is a Tanaka space. When X is a quasi-k-space (equivalently, sequential space), we can replace "X has property (P)" by "X is an inner-closed A'-space".

(3) Let X be a closed image of a countably bi-quasi-k-space, or let X be a quotient s-image of a meta-Lindelöf countably-bi-quasi-k-space, but $t(X) \leq \omega$. Then X is an inner-closed A'-space iff X is a countably bi-quasi-k-space.

We recall that a cover $\mathcal{P}$ of a space is a *k-network* if whenever, for any compact set K and any open set U with $K \subset U$, $K \subset A \subset U$ for some $A \in [\mathcal{P}]$. A space X is an *$\aleph$-space* if it has a σ-locally finite k-network. Open bases are k-networks. Among k-spaces, for any k-network $\mathcal{P}$, $[\mathcal{P}]$ is a determining cover, and so is $\mathcal{P}$ when $\mathcal{P}$ is closed. Quotient s-images or closed images of metric spaces; or spaces having a dominating or point-countable determining cover by metric sets have point-countable k-networks. For a survey on k-networks, see [33]. For spaces with certain point-countable covers or certain k-networks, also see [6, 27], etc.

Theorem 2.10. Let X be an inner-closed A'-space. Then (1), (2), and (3) below hold.

(1) For a cover $\mathcal{P}$ of X satisfying the following (a) or (b), $[\mathcal{P}]^\circ$ is an open cover of X.

(a) $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in N\}$, $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, is a σ-locally countable determining cover of X such that each $\mathcal{P}_n$ is a determing cover of the union of $\mathcal{P}_n$. In particular, $\mathcal{P}$ is a star-countable determining cover, generally a locally countable determining cover, or $\mathcal{P}$ is a σ-locally finite determining cover.

(b) $t(X) \leq \omega$, and $\mathcal{P}$ is a point-countable cover such that $[\mathcal{P}]$ is a determining cover (in particular, $\mathcal{P}$ is a determining cover).

(2) For a dominating cover $\mathcal{F}$ of X, $\mathcal{F}$ has a point-finite determining closed refinement, and $[\mathcal{F}]^\circ$ is an open cover of X when $t(X) \leq \omega$.

(3) Let X be a k-space, and $\mathcal{P}$ be a point-countable k-network for X. For each open set V of X, let $\mathcal{V} = \{P \in \mathcal{P} : P \subset V\}$. Then $[\mathcal{V}]^\circ$ is an open cover of V. Thus X has a point-countable base (in view of [2]).

Let us recall canonical sequential spaces, the *sequential fan* S_ω and the *Arens' space* S_2. Let L be the convergent sequence $\{1/n : n \in N\} \cup \{0\}$. Let S_ω be the space obtained from the disjoint union $\Sigma\{L_n : n \in N\}$ of copies of the sequence L by identitying all the limit points to a single point. Let S_2 be the space obtained from the disjoint union $\Sigma\{L_n : n = 0, 1, \cdots\}$ of copies of the sequence L, by identifying each $1/n \in L_0$ with $0 \in L_n$ $(n \geq 1)$.

Obviously, any inner-closed A'-space contains no closed copy of S_ω, and no S_2. For a space X with $t(X) \leq \omega$, X contains no closed copy of S_ω (resp. S_2) iff X contains no copy of S_ω (resp. S_2) ([19]). For a sequential space X, X contains no (closed) copy of S_ω iff X is an A-space ([27]), and for a Fréchet space X, X is strongly Fréchet iff X contains no (closed) copy of S_ω. For spaces which contain a copy of S_ω or S_2, see [28], etc.

Remark 2.11. In Theorems 2.9 and 2.10, when the X is sequential, some results there remain true if we replace "X is an inner-closed A'-space" by "X contains no (closed) copy of S_ω, and no S_2". Indeed, this holds for cases where (a) X is a sequential space such that X is hereditarly normal, or all points of X are G_δ-sets (in view of [27]), (b) X is a sequential space which is a closed image of a countably bi-quasi-k-space (under X containing no (closed) copy of S_ω), and (c) X is a k-space with a point-countable k-network (in view of [7]). We note that every compact sequential space (hence it contains no copy of S_ω, and no S_2) need not be Fréchet.

Theorem 2.12. Let Y be a closed image of a paracompact space X

such that X is a countably bi-quasi-k-space, or an inner-closed A-space with $t(X) \leq \omega$. Then, each point-countable determining closed cover $\mathcal{F}$ of Y has a locally countable HCP closed refinement which is a σ-locally finite cover.

Remark 2.13. In Theorem 2.12, if the cover $\mathcal{F}$ is not necessarily closed, $\mathcal{F}$ has at least a HCP refinement if X is an inner-closed A-space with $t(X) \leq \omega$. Here, when $Y = X$, the refinement can be chosen to be locally finite. Similarly, for point-countable determining covers, certain analogues would hold in some other results (as in Theorem 2.18 later).

Theorem 2.14. Let $t(X) \leq \omega$, and X^ω be a quasi-k-space. Then X is inner-one A ([25]), equivalently, X has property (P). Here, if X^ω is a k-space, we can replace "X" by "X^ω".

Corollary 2.15. Let $t(X) \leq \omega$, and let X have a dominating cover $\mathcal{P}$, or a point-countable cover $\mathcal{P}$ with $[\mathcal{P}]$ a determining cover. If X^ω is a quasi-k-space, then $[\mathcal{P}]^\circ$ is an open cover of X.

Corollary 2.16. Let X have a dominating or point-countable determining closed cover $\mathcal{F}$ by locally compact spaces (resp. first countable spaces). Then for (a) $\sim$ (e) below, the implications (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), and (d) $\Leftrightarrow$ (e) $\Rightarrow$ (b) hold. When $t(X) \leq \omega$, (a) $\sim$ (e) are equivalent. Here, we can omit "$t(X) \leq \omega$" for the parenthetic part.

(a) X^ω is a quasi-k-space.
(b) X^ω is a k-space.
(c) $[\mathcal{F}]^\omega$ is a determining cover of X^ω.
(d) $[\mathcal{F}]^\circ$ is an open cover of X.
(e) X is a locally compact space (resp. first countable space).

Remark 2.17. (CH). "$t(X) \leq \omega$" is essential in Corollary 2.16 (for the implication (b) $\Rightarrow$ (d) or (e)). Indeed, under (CH), there exists a space X having a countable HCP (hence, dominating) cover $\mathcal{F}$ by compact sets, and X^ω is a k-space, but X is not locally compact ([3]). Hence, $[\mathcal{F}]^\omega$ is a determing cover of X^ω, but $[\mathcal{F}]^\circ$ is not an open cover of X.

Theorem 2.18. (1) Let X be a σ-space. Then every dominating or point-countable determining closed cover of X has a refinement which is a σ-locally finite closed network. When X is a k-and-$\aleph$-space, the refinement can be chosen to be a determining cover which is a σ-locally finite closed k-network.

(2) (a) Let X be a k-space having a σ-HCP (closed) k-network. Then every dominating or point-countable determining closed cover of X has a determining refinement which is a σ-HCP closed k-network.

(b) Let X be a k-space having a point-countable closed k-network. Then every dominating cover of X has a determining refimement which is a point-countable closed k-network.

Remark 2.19. In Theorem 2.18(1), for X being a σ-space, we can not add a prefix "determining" before "closed refinement" in view of Example 1.1(2) or Remark 2.5(1). For a cosmic space (i.e., space with a countable network), every dominating cover has a countable determining subcover. But, under (CH), not every point-finite determining closed cover of a cosmic space X by separable metric sets has a σ-CP determining closed refinement (in view of [30], here the space X is regular under (CH) ([20]).

Corollary 2.20. Let X have a dominating cover $\mathcal{F}$ by metric sets. Then the following are equivalent.

(a) X has a point-countable closed k-network.

(b) $\mathcal{F}$ has a point-countable determining closed refinement (which is a k-network consisting of metric sets).

(c) Every dominating cover of X has a point-countable determining closed refinement (consisting of metric sets).

For a space X, a collection $T_C = \{T_x : x \in X\}$ is a *weak base* [1] if each $T \in T_x$ contains the point x, and each T_x is closed under finite intersections; and $G \subset X$ is open in X iff for each $x \in G$, there exists $Q(x) \in T_x$ such that $Q(x) \subset G$. A space X is *g-first countable* [22] (or X satisfies *the weak first axiom of countability* [1]) if X has a weak base $T_C = \{T_x : x \in X\}$ with each T_x countable. Every weak base of a sequential space is a determining cover. First countable spaces are g-first countable, and the converse holds among Fréchet spaces; see [1].

Theorem 2.21. Let X have a dominating cover $\mathcal{F}$ by first countable spaces. Then the following are equivalent.

(a) X is g-first countable.

(b) X has a point-finite determining closed cover by first countable spaces.

(c) $\mathcal{F}$ has a point-finite determining closed refinement.

A space X is *g-metrizable* [22] if X has a σ-locally finite weak base. Every g-metrizable space is metrizable iff it is Fréchet [22] (or singly bi-quasi-k). For a space X, X is g-metrizable $\Leftrightarrow$ X is a g-first countable $\aleph$-space ([5]) $\Leftrightarrow$ X has a σ-HCP weak base ([8]).

Corollary 2.22. Let X have a dominating or point-countable determining closed cover $\mathcal{F}$ by metric sets. Then the following are equivalent.

(a) X is g-metrizable.

(b) X has a point-finite and σ-locally finite determining closed cover by metric sets.

(c) $\mathcal{F}$ has a point-finite and σ-locally finite determining closed refinement.

(d) Every dominating or point-countable determining closed cover of X has a point-finite and σ-locally finite determining closed refinement (by metric sets).

3. Questions

First, let us give a question for σ-spaces in terms of determining covers, in view of Remark 1.1, etc. Every space with a dominating cover by σ-spaces is a σ-space ([23], etc.). Also, every separable space with a CP closed cover $\mathcal{F}$ by σ-spaces is a σ-space. When the elements of $\mathcal{F}$ are compact metric spaces, we can replace "separable" by "locally separable (or locally Lindelöf)". While, every Lindelöf space with a point-countable determining cover by cosmic spaces is cosmic. But, every space with a point-finite determining closed cover by metric sets need not even be a space whose closed sets are G_δ-sets. Here, under the space being *Hausdorff*, we can replace "metric sets" by "compact metric sets". These are stated in [11, 35], etc. In view of the above, we have the following question. (a) (resp. (c)) was posed in [37] (resp. [11, 35], etc.).

Question 3.1. (a) Is every space with a determining CP closed cover by σ-spaces (or compact metric spaces) a σ-space ?

(b) Is every space X with a point-countable determining cover $\mathcal{F}$ by σ-spaces is a σ-space ? In particular,

(c) When the cover $\mathcal{F}$ is point-finite and consists of compact metric sets (equivalently, X is a quotient compact image of a locally compact metric space), is a σ-space (or a space whose points are G_δ-sets) ?

Next, let us give some questions in view of Section 2. The author has the following question in view of Corollary 2.6, and Corollary 2.4(2) with Remark 2.5(1). (a) was asked in [39], and it is positive when X is a closed image of a paracompact countably bi-quasi-k-space by Theorem 2.12.

Question 3.2. Let X be a paracompact singly bi-quasi-k-space, and let $\mathcal{F}$ be a point-countable determining closed cover of X.

(a) Does $\mathcal{F}$ have a refinement which is a σ-locally finite determining closed cover ? In particular,

(b) When the cover $\mathcal{F}$ consists of metric sets, is (a) positive ?

The author has the following question in view of Remark 2.11.

Question 3.3. Let X be a sequential space (in particular, X be a quotient s-image of a paracompact first countable space). If X contains no (closed) copy of S_ω, and no S_2, is X inner-one A ?

In view of Corollary 2.22, the following question has been posed in [39].

Question 3.4. Let X be a g-metrizable space. For each increasing countable dominating cover $\mathcal{F}$ of X, does X have a point-finite determining (closed) refinement of $\mathcal{F}$?

References

[1] A. V. Arhangel'skiĭ, Mappings and spaces, Russian Math. Surveys, 21(1966), 115-162.

[2] D. Burke and E. Michael, On certain point-countable covers, Pacific J. Math., 64(1976), 79-92.

[3] H. Chen, An answer to a conjecture on the countable products of k-spaces, Proc. Amer. Math. Soc., 123(1995), 583-587.

[4] J. Dugundji, Topology, Allyn and Bacon, INC., Boston, 1966.

[5] L. Foged, On g-metrizability, Pacific J. Math., 98(1982), 327-332.

[6] G. Gruenhage, E. Michael and Y. Tanaka, Spaces determined by point-countable covers, Pacific J. Math., 113(1984), 303-332.

[7] S. Lin, A note on the Arens' space and sequential fan, Topology and Appl., 81(1997), 185-196.

[8] C. Liu, On weak bases, Topology and Appl., 150(2005), 91-99.

[9] C. Liu, M. Sakai and Y. Tanaka, Topological groups, k-networks, and weak topology, in: Research in General and Geometric Topology, RIMS Kyoto University, 1126(2000), 39-44.

[10] C. Liu and Y. Tanaka, Spaces having σ-compact-finite k-networs, in: General Topology and Related Problems, RIMS Kyoto University, 953(1996), 1-4.

[11] C. Liu and Y. Tanaka, Spaces and Mappings: Special networks, in: Open problems in Topology 2, E. Pearl ed., Elsevier Science B. V., to appear.

[12] E. Michael, Continuous selections. I, Annals of Math., 63(1956), 361-382.

[13] E. Michael, A quintuple quotient quest, General topology and Appl., 2(1972), 91-138.

[14] E. Michael, R. C. Olson and F. Siwiec, A-spaces and countably bi-quotient maps, Dissertationes Math., 133(1976), 4-43.

[15] K. Morita, On spaces having the weak topology with respect to closed coverings, Proc. Japan Acad., 29(1953), 537-543.

[16] K. Morita, On spaces having the weak topology with respect to closed coverings, II, Proc. Japan Acad., 30(1954), 711-717.

[17] F. Mynard, More on strongly sequential spaces, Comment. Math. Univ., Carolinae, 43(2002), 525-530.

[18] J. Nagata, Quotient and bi-quotient spaces of M-spaces, Proc. Japan Acad., 45(1969), 25-29.

[19] T. Nogura and Y. Tanaka, Spaces which contain a copy of S_ω or $S2$, and their applications, Topology and Appl., 30(1988), 51-62.

[20] M. Sakai, A special subset of the real line and regularity of weak topologies, Topology Proc., 23(1998), 281-287.

[21] F. Siwiec, Sequence-converging and countably bi-quotient mappings, Gen. Topology and Appl., 1 (1971), 143-154.

[22] F. Siwiec, On defining a space by a weak base, Pacific J. Math., 52(1974), 233-245.

[23] Y. Tanaka, On local properties of topological spaces, Science Reports of the Tokyo Kyoiku Daigaku, Sect. A., 11(1972), 106-116.

[24] Y. Tanaka, Products of sequential spaces, Proc. Amer. Math., Soc., 54(1976), 371-375.

[25] Y. Tanaka, Some necessary conditions for products of k-spaces, Bull. Tokyo Gakugei Univ., 30(1978), 1-16.

[26] Y. Tanaka, On products of countable tightness, in: Shape Theory and Topological spaces, RIMS Kyoto University, 445(1981), 22-28.

[27] Y. Tanaka, Metrizability of certain quotient spaces, Fund. Math., 119(1983), 157-168.

[28] Y. Tanaka, Metrization II, In: K. Morita and J. Nagata eds., Topics in General Topology, Elsevier Science Publishers B. V., (1989), 275-314.

[29] Y. Tanaka, k-networks, and covering properties of CW-complexes, in: General Topology and around it, RIMS Kyoto University, 758(1991), 44-48.

[30] Y. Tanaka, Products of k-spaces having point-countable k-networks, Topology Proc., 22(1997), 305-329.

[31] Y. Tanaka, Products of k-spaces, and k-networks, in: General and Geometric Topology, RIMS Kyoto University, 1074(1999), 32-37.

[32] Y. Tanaka, Metrizability of GO-spaces, and k-networks, in: Unsolved Problems and its Progress in General・Geometric Topology, RIMS Kyoto University, 1107(1999), 57-62.

[33] Y. Tanaka, Theory of k-networks II, Questions and Answers in General Topology, 19(2001), 27-46.

[34] Y. Tanaka, Products of k-spaces, and questions, in: Problems and applications in General and Geometric Topology, RIMS Kyoto University, 1303(2003), 12-19.

[35] Y. Tanaka, Questions on quotient compact images of metric spaces, and symmetric spaces, in: General and Geometric Topology and Related Topics, RIMS Kyoto University, 1370(2004), 117-125.

[36] Y. Tanaka, Quotient spaces and decompositions, In Encyclopedia of General Topology, K. P. Hart, J. Nagata and J. E. Vaughan eds, Elsevier Science B. V., 2004, 43-46.

[37] Y. Tanaka, Products of weak topologies, in: Set Theoretic and Geometric Topology and Its Applications, RIMS Kyoto University, 1419(2005), 1-9.

[38] Y. Tanaka, Weak Topologies, and determining covers, in: General and Geometric Topology and Geometric Group Theory, RIMS Kyoto University, 1492(2006), 84-93.

[39] Y. Tanaka, Determining covers and dominationg covers, Questions and Answers in General Topology, 24(2006), 123-141.

[40] Y. Tanaka, k-spaces, and products of weak topologies, Topology Proc., to appear.

[41] Y. Tanaka, Tanaka spaces and products of sequential spaces, Comment. Math. Univ. Carolinae, to appear.

[42] Y. Tanaka and Y. Shimizu, Products of k-spaces, and special countable spaces, Tsukuba J. Math., 27(2003), 405-415.

[43] Y. Tanaka and I. Yoshioka, Spaces determined by point-countable covers, and singly bi-quasi-k-spaces, Questions and Answers in General Topology, 23(2005), 35-54.

[44] Y. Tanaka and Zhou Hao-xuan, Spaces determined by metric subsets, and their character, Questions and Answers in General Topology, 3(1985/86), 145-160.

[45] Y. Yajima, Topological games and applications, In: K. Morita and J. Nagata eds., Topics in General Topology, Elsevier Science Publishers B. V., (1989), 523-562.

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