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ON FIBREWISE COVERING UNIFORMITIES

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1. Introduction

In this report, by using coverings we introduce the concepts of fibrewise covering uniform spaces and its generalizations, and study the fibrewise completion theory which is the extended one of the classical theory established in topological spaces, [1] Chapter 8 and [5] Sections 1 and 2.

Motivation and significance of this report are:

(1): Developing the fibrewise covering uniformity theory corresponding the fibrewise (entourage) uniformity theory by James [4].

(2): An application of conjugate pairs of coverings ([2]) to fibrewise covering uniformity theory: The fibrewise entourage uniformity theory ([4]) is symmetric in local, but asymmetric in global. So, the conjugate pair covering theory just fits to our theory.

(3): Developing the fibrewise generalized uniformity theory corresponding the generalized uniformity theory by Morita ([5]): **(1)** enables us to develope the fibrewise generalized uniformity theory.

Section 3 devotes to **(1)** and **(2)**. Sections 4 and 5 devote to **(3)**. We omit all proofs of Propositions, Theorems and Lemmas.

2. Preliminaries

In this section, we refer to the notations used in the latter sections, further the notions and notations in Fibrewise Topology. Throughout this report, we will use the abbreviation *nbd(s)* for *neighborhood(s)*.

Let (B,τ) be a fixed topological space with a fixed topology τ. For the base space (B,τ), TOP_B is the fibrewise category over B. (Cf. TOP is the topological category.) Note that regularity is assumed in Propositions 3.7 and 3.9.

A *fibrewise set* (resp. *space*) over B consists of a set (resp. topological space) X together with a (resp. continuous) function $p : X \to B$ (called the *projection*). Throughout this report, for fibrewise sets X and Y over B the projections are $p : X \to B$ and $q : Y \to B$, respectively. For each point $b \in B$, the *fibre* over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset B' of B we regard $X_{B'} = p^{-1}B'$ as a fibrewise set over B' with the projection determined by p. For a filter (base) $\mathcal{F}$ in X, we denote that $p_*(\mathcal{F})$ is the filter generated by the set $\{p(F)|F \in \mathcal{F}\}$.

Definition 2.1. Let $p : X \to B$ be the continuous projection.
([4] Definition 2.2.) The fibrewise space X over B is *fibrewise* R_0 if for each point $x \in X_b$, where $b \in B$, and each nbd V of x in X, there exists $W \in N(b)$ such that $X_W \cap \mathrm{Cl}\{x\} \subset V$, where Cl is the closure operator.

The concept of tied filters (or b-filters) plays a central role in this report, which is defined as follows.

Definition 2.2. ([4] Section 4.) For a fibrewise set X over B, by a *b-filter* (or *tied filter*) on X we mean a pair $(b, \mathcal{F})$, where $b \in B$ and $\mathcal{F}$ is a filter on X such that b is a limit point of the filter $p_*(\mathcal{F})$ on B.

3. Fibrewise uniform spaces — entourages vs. coverings

First, we recall the definition of fibrewise uniform structures in [4].

Definition 3.1 (James[4]). Let X be a fibrewise set over B. Δ is the diagonal of X. A *fibrewise uniform structure* on X is a filter Ω on $X \times X$ satisfying following three conditions:

(1) $\Delta \subset D$, for every $D \in \Omega$.
(2) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that $X_W^2 \cap E \subset D^{-1}$.
(3) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that

$$(X_W^2 \cap E) \circ (X_W^2 \cap E) \subset D$$

The fibrewise set X with fibrewise uniform structure Ω is called by a *fibrewise uniform space*, and we denote (X, Ω). The members of Ω are called *entourages*.

For defining the concept of fibrewise covering uniformities, we first define the concept of conjugate pair of coverings, which was introduced in [2] and plays the essential roles in this report. (T.E.Gantner and R.C. Steinlage [2] introduced the concept for studying quasi-uniform spaces.) This concept enables us **(1)** and **(2)** in section 1.

We first begins with the following definitions.

Definition 3.2. (Gantner and Steinlage[2])
(1) Let X be a set, $\mathcal{U} = \{U_\alpha|\alpha \in \Lambda\}$ and $\mathcal{U}' = \{U'_\alpha|\alpha \in \Lambda\}$ be two coverings of X with the same index set Λ. Then we say the pair $(\mathcal{U}, \mathcal{U}')$ a *conjugate pair of*

coverings of a set X if it satisfies the following condition: For each $x \in X$ there exists $\alpha \in \Lambda$ such that $x \in U_\alpha \cap U'_\alpha$.

For the conjugate pair of coverings $(\mathcal{U},\mathcal{U}')$, if we write $(U,U') \in (\mathcal{U},\mathcal{U}')$, we mean that there exists $\alpha \in \Lambda$ such that $(U,U') = (U_\alpha, U'_\alpha)$.

(2) For sets X and Y with $Y \subset X$, let $(\mathcal{U},\mathcal{U}')$ be a conjugate pair of coverings of X and $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}, \mathcal{U}' = \{U'_\alpha | \alpha \in \Lambda\}$, and $(\mathcal{V},\mathcal{V}')$ be a conjugate pair of coverings of Y and $\mathcal{V} = \{V_\beta | \beta \in \Gamma\}$, $\mathcal{V}' = \{V'_\beta | \beta \in \Gamma\}$. We say that $(\mathcal{V},\mathcal{V}')$ is a *refinement* of $(\mathcal{U},\mathcal{U}')$ if for each $\beta \in \Gamma$ there exists $\alpha \in \Lambda$ such that $V_\beta \subset U_\alpha, V'_\beta \subset U'_\alpha$. We denote $(\mathcal{V},\mathcal{V}') < (\mathcal{U},\mathcal{U}')$.

(3) For three sets X, Y, A with $A \subset Y \subset X$, let $(\mathcal{U},\mathcal{U}')$ and $(\mathcal{V},\mathcal{V}')$ be conjugate pairs of coverings of X and Y, respectively. Let

$$\mathrm{st}'(A,\mathcal{V}) := \cup\{V_\beta \in \mathcal{V} | V'_\beta \cap A \neq \emptyset\}$$
$$\mathrm{st}'(A,\mathcal{V}') := \cup\{V'_\beta \in \mathcal{V}' | V_\beta \cap A \neq \emptyset\}.$$

Then we say that $(\mathcal{V},\mathcal{V}')$ is a *star refinement* of $(\mathcal{U},\mathcal{U}')$ if for each $(V,V') \in (\mathcal{V},\mathcal{V}')$ there exists $(U,U') \in (\mathcal{U},\mathcal{U}')$ such that

$$\mathrm{st}'(V,\mathcal{V}) \cup \mathrm{st}'(V',\mathcal{V}') \subset U \cap U'.$$

(4) For a conjugate pair of coverings $(\mathcal{U},\mathcal{U}')$ of X and $A \subset X$, we define the *restriction* of $(\mathcal{U},\mathcal{U}')$ to A as follows:

$$(\mathcal{U},\mathcal{U}')|_A := (\{U_\alpha \cap A | \alpha \in \Lambda\}, \{U'_\alpha \cap A | \alpha \in \Lambda\}).$$

(5) For a conjugate pair of coverings $(\mathcal{U},\mathcal{U}')$ of a set, we define the *doublewedge* $\mathcal{U} \Lleftarrow\!\!\!\wedge \mathcal{U}'$ of $(\mathcal{U},\mathcal{U}')$ as follows:

$$\mathcal{U} \mathbin{\wedge\!\!\!\wedge} \mathcal{U}' := \{U \cap U' | (U,U') \in (\mathcal{U},\mathcal{U}')\}.$$

Let X be a fibrewise set over B and $W \in \tau$. Let μ_W be a non-empty family of conjugate pairs of coverings of X_W, and $\{\mu_W\}_{W\in\tau}$ the system of μ_W, $W \in \tau$. We say that $\{\mu_W\}_{W\in\tau}$ is a *system of conjugate pairs of coverings* of $\{X_W\}_{W\in\tau}$. (For this, we briefly use the notations $\{\mu_W\}$ and $\{X_W\}$.)

In this definition, for $A \subset X_W$ and $(\mathcal{U},\mathcal{U}') \in \mu_W$, from the definition of conjugate pair of coverings (Definition 3.2 (1)) it is easy to see that $A \subset \mathrm{st}'(A,\mathcal{U})$, $A \subset \mathrm{st}'(A,\mathcal{U}')$, and $\mathrm{st}(A,\mathcal{U} \mathbin{\wedge\!\!\!\wedge} \mathcal{U}') \subset \mathrm{st}'(A,\mathcal{U}) \cap \mathrm{st}'(A,\mathcal{U}')$, where $\mathrm{st}(A,\mathcal{U} \mathbin{\wedge\!\!\!\wedge} \mathcal{U}')$ is the usual star.

Definition 3.3. Let X be a fibrewise set over B, and $\mu = \{\mu_W\}$ be a system of conjugate pairs of coverings of $\{X_W\}$. We say that the system $\{\mu_W\}$ is a *fibrewise covering uniformity* (and a pair (X,μ) or $(X,\{\mu_W\})$ is a *fibrewise covering uniform space*) if the following conditions are satisfied:

(1) Let $(\mathcal{U},\mathcal{U}')$ be a conjugate pair of coverings of X_W, there exists $(\mathcal{V},\mathcal{V}') \in \mu_W$ such that $(\mathcal{V},\mathcal{V}') < (\mathcal{U},\mathcal{U}')$, then $(\mathcal{U},\mathcal{U}') \in \mu_W$.

(2) For each $(\mathcal{U}_i,\mathcal{U}_i') \in \mu_W, i = 1,2$, there exists $(\mathcal{U}_3,\mathcal{U}_3') \in \mu_W$ such that $(\mathcal{U}_3,\mathcal{U}_3') < (\mathcal{U}_i,\mathcal{U}_i'), i = 1,2$.
(3) For each $(\mathcal{U},\mathcal{U}') \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $(\mathcal{V},\mathcal{V}') \in \mu_{W'}$ such that $W' \subset W$ and $(\mathcal{V},\mathcal{V}')$ is a star refinement of $(\mathcal{U},\mathcal{U}')$.
(4) For each $(\mathcal{U},\mathcal{U}') \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $(\mathcal{V},\mathcal{V}') \in \mu_{W'}$ such that $W' \subset W$ and $(\mathcal{V}',\mathcal{V}) < (\mathcal{U},\mathcal{U}')$.
(5) For $W' \subset W$, $\mu_{W'} \supset \mu_W|X_{W'}$, where

$$\mu_W|X_{W'} = \{(\mathcal{U},\mathcal{U}')|_{X_{W'}} | (\mathcal{U},\mathcal{U}') \in \mu_W\}.$$

For the fibrewise uniform structure Ω of a fibrewise uniform space (X,Ω), we now construct a system $\mu(\Omega) = \{\mu(\Omega)_W\}$ of conjugate pairs of coverings of $\{X_W\}$.

Construction 3.4. Let (X,Ω) be a fibrewise uniform space . Then we shall construct a system $\mu(\Omega) = \{\mu(\Omega)_W\}$ of conjugate pairs of coverings of $\{X_W\}$ for every $W \in \tau$, as follows: For $D \in \Omega$ and $W \in \tau$, let
$\mathcal{U}(D,W) := \{D^{-1}(x) \cap X_W | x \in X_W\}$, $\mathcal{U}'(D,W) := \{D(x) \cap X_W | x \in X_W\}$.
Then it is easy to see that the pair $(\mathcal{U}(D,W),\mathcal{U}'(D,W))$ is a conjugate pair of coverings of X_W. Let $\mu(\Omega)_W$ be the family of all conjugate pairs $(\mathcal{U},\mathcal{U}')$ of coverings of X_W satisfying that there exists $D \in \Omega$ such that $(\mathcal{U}(D,W),\mathcal{U}'(D,W)) < (\mathcal{U},\mathcal{U}')$.

Then we can prove the following proposition.

Proposition 3.5. Let (X,Ω) be a fibrewise uniform space. Then the system $\mu(\Omega) = \{\mu(\Omega)_W\}$ which is constructed above is a fibrewise covering uniformity.

Conversely, for a fibrewise covering uniformity $\mu = \{\mu_W\}$ we shall construct a fibrewise uniform structure Ω_μ as follows.

Construction 3.6. Let $\mu = \{\mu_W\}$ be the fibrewise covering uniformity of a fibrewise covering uniform space (X,μ). For $(\mathcal{U},\mathcal{U}') \in \mu_W$, let

$$D(\mathcal{U},\mathcal{U}') := \cup\{U_\alpha \times U_\alpha' | (U_\alpha, U_\alpha') \in (\mathcal{U},\mathcal{U}')\}.$$

Let Ω_μ be the family of all subsets $D \subset X \times X$ satisfying following condition:
For a finite open cover $\{W_1, \cdots, W_n\}$ of B, for each $i = 1, \cdots, n$, there exists $(\mathcal{U}_i,\mathcal{U}_i') \in \mu_{W_i}$ such that $D(\mathcal{U}_1,\mathcal{U}_1') \cup \cdots \cup D(\mathcal{U}_n,\mathcal{U}_n') \subset D$.

Remark: If B is compact, the definition of Ω_μ from μ is just suitable to this construction. But, in general, there are many ways of constructions of Ω_μ. For example, if we construct the members of Ω_μ for any open cover of B as the same methods in the above, then we have from Definition 3.3 (4) that $D \in \Omega_\mu$ implies $D^{-1} \in \Omega_\mu$, so Ω_μ obviously satisfies the condition (2) in Definition 3.1.

Then we have the following.

Proposition 3.7. Assume that B is a regular space. Let (X, μ) be a fibrewise covering uniform space, where $\mu = \{\mu_W\}$. Then the family Ω_μ which is the one of Construction 3.6 is a fibrewise uniform structure.

We conclude this section by discussing the fibrewise topologies with respect to Ω and μ of (X, Ω) and (X, μ) respectively, and by considering the relationships of fibrewise topologies of (X, Ω) and $(X, \mu(\Omega))$ (or, (X, μ) and (X, Ω_μ)).

For a fibrewise uniform space (X, Ω), the *fibrewise uniform topology* $\tau(\Omega)$ was defined and discussed in [4; Section 13] as follows: For every $x \in X$, $p(x) = b$, let $\mathcal{N}_x(\Omega) = \{X_W \cap D(x) | W \in N(b), D \in \Omega\}$. Then $\{\mathcal{N}_x(\Omega) | x \in X\}$ satisfies the conditions of nbd system ([4] Section 13).

On the other hand, for fibrewise covering uniform space (X, μ) we shall define the *fibrewise covering uniform topology* $\tau(\mu)$ as follows:

For every $x \in X$, $p(x) = b$, let $\mathcal{N}_x(\mu)$ be the family of all subsets which contains

$\mathrm{st}(x, \mathcal{U} \curlywedge \mathcal{U}')$ for some $(\mathcal{U}, \mathcal{U}') \in \mu_W$, $W \in N(b)$.

Then we can prove that $\{\mathcal{N}_x(\mu) | x \in X\}$ satisfies the axiom of nbd system and it defines the fibrewise covering uniform topology $\tau(\mu)$, as follows.

Proposition 3.8. For a fibrewise covering uniform space (X, μ), $\{\mathcal{N}_x(\mu) | x \in X\}$ satisfies the axiom of nbd system.

Finally, we can prove the following proposition.

Proposition 3.9. (1) For a fibrewise uniform space (X, Ω), it holds that $\tau(\Omega) = \tau(\mu(\Omega))$.
(2) Assume that B is regular. For a fibrewise covering uniform space (X, μ), where $\mu = \{\mu_W\}$, it holds that $\tau(\mu) = \tau(\Omega_\mu)$.

4. Fibrewise generalized uniformities

In the previous section 3, we considered fibrewise covering uniformities and fibrewise covering uniform spaces and the fibrewise uniform topology. In this section, by weakening the condition (3) of fibrewise covering uniformity (Definition 3.3), we define fibrewise generalized uniformities (and fibrewise generalized uniform spaces). This concept is the fibrewise version of generalized uniform spaces in [5].

In this section, unless otherwise stated, we exclusively use that X is a fibrewise set over B and $\mu = \{\mu_W\}$ is a system of conjugate pairs of coverings of $\{X_W\}$.

To weaken the condition (3) of Definition 3.3, we define a following concept.

Let $\{\mu_W\}$ be a system of conjugate pairs of coverings of $\{X_W\}$. For an open set W of B and $Y \subset X$, let

$$\mathrm{Int}_{\mu_W} Y := \{x \in X_W | \exists W' \in N(p(x)), \exists (\mathcal{U}, \mathcal{U}') \in \mu_{W'} \text{ such that } W' \subset W, \mathrm{st}(x, \mathcal{U} \Lsh \mathcal{U}') \subset Y\}.$$

For a collection $\mathcal{U}$ of subsets of X, let

$$\mathrm{Int}_{\mu_W} \mathcal{U} := \{\mathrm{Int}_{\mu_W} U | U \in \mathcal{U}\}.$$

Note that, since X is not yet a topological space, $\mathrm{Int}_{\mu_W} Y$ is not the interior of $Y \cap X_W$. But by Proposition 4.7, those are combined.

Definition 4.1. Let $\mu = \{\mu_W\}$ be a system of conjugate pairs of coverings of $\{X_W\}$. Then $\mu = \{\mu_W\}$ is called a *fibrewise generalized uniformity* (we briefly say *fibrewise g-uniformity*) if it satisfies (1), (2), (4) and (5) of Definition 3.3 and

(FGU) For each $b \in B, W \in N(b)$ and $(\mathcal{U}, \mathcal{U}') \in \mu_W$, there exist $W' \in N(b)$ and $(\mathcal{V}, \mathcal{V}') \in \mu_{W'}$ such that $W' \subset W$ and $(\mathrm{Int}_{\mu_{W'}} \mathcal{U}, \mathrm{Int}_{\mu_{W'}} \mathcal{U}')$ is a conjugate pair of coverings of $X_{W'}$ and $(\mathcal{V}, \mathcal{V}') < (\mathrm{Int}_{\mu_{W'}} \mathcal{U}, \mathrm{Int}_{\mu_{W'}} \mathcal{U}')$.

The pair (X, μ) (or $(X, \{\mu_W\})$) is called a *fibrewise generalized uniform space* (we briefly say a *fibrewise g-uniform space*).

Remark 4.2. In the condition (FGU) above, it is evident that $(\mathrm{Int}_{\mu_{W'}} \mathcal{U}, \mathrm{Int}_{\mu_{W'}} \mathcal{U}') \in \mu_{W'}$ and $(\mathrm{Int}_{\mu_{W'}} \mathcal{U}, \mathrm{Int}_{\mu_{W'}} \mathcal{U}') < (\mathcal{U}, \mathcal{U}')$. Further, $(\mathrm{Int}_{\mu_W} \mathcal{U}, \mathrm{Int}_{\mu_W} \mathcal{U}')$ is a conjugate pair of coverings of X_W, but we don't know that $(\mathrm{Int}_{\mu_W} \mathcal{U}, \mathrm{Int}_{\mu_W} \mathcal{U}') \in \mu_W$ or not.

Proposition 4.3. A fibrewise covering uniformity is a fibrewise g-uniformity.

Definition 4.4. (1) Let $\{\mu_W\}$ be a fibrewise g-uniformity and $\{\mu_W^0\}$ be a system of conjugate pairs of coverings of $\{X_W\}$ satisfying that $\mu_W^0 \subset \mu_W$ for all $W \in \tau$, and $\mu_{W'}^0 \supset \mu_W^0 |_{X_{W'}}$ for every $W' \subset W$.

We say that $\{\mu_W^0\}$ is a *base* for $\{\mu_W\}$ if for each W and $(\mathcal{U}, \mathcal{U}') \in \mu_W$ there exists $(\mathcal{V}, \mathcal{V}') \in \mu_W^0$ such that $(\mathcal{V}, \mathcal{V}') < (\mathcal{U}, \mathcal{U}')$.

(2) Let $\{\mu_W^0\}$ be a system of conjugate pairs of coverings of $\{X_W\}$. We say that $\{\mu_W^0\}$ is a *fibrewise g-uniformity base* if $\{\mu_W^0\}$ satisfies (2), (4), (5) of Definition 3.3 and (FGU).

Unless otherwise stated, we use the notation $\{\mu_W^0\}$ for a fibrewise g-uniformity base.

Proposition 4.5. (1) A base for a fibrewise g-uniformity is a fibrewise g-uniformity base.

(2) Let $\{\mu_W^0\}$ be a fibrewise g-uniformity base. For each $W \in \tau$, let μ_W be the family of conjugate pairs of coverings of X_W such that each $(\mathcal{U}, \mathcal{U}') \in \mu_W$ there exists $(\mathcal{V}, \mathcal{V}') \in \mu_W^0$ satisfying $(\mathcal{V}, \mathcal{V}') < (\mathcal{U}, \mathcal{U}')$. Then $\{\mu_W\}$ is a fibrewise g-uniformity such that $\{\mu_W^0\}$ is a base for $\{\mu_W\}$.

Proposition 4.6. For each $x \in X$, let

$$\mathcal{N}(x) := \{O \subset X | \exists W \in N(p(x)), \exists(\mathcal{U},\mathcal{U}') \in \mu_W \text{ such that } \mathrm{st}(x,\mathcal{U} \Cap \mathcal{U}') \subset O\}$$

Then $\{\mathcal{N}(x)\}$ is a nbd system on X. (We denote the fibrewise topology defined by this system as $\tau(\{\mu_W\})$).

Moreover, Int_{μ_B} defined in the above is the interior operator of $\tau(\{\mu_W\})$.

About Int_{μ_B} and Int_{μ_W}, we have the following.

Lemma 4.7. For any $Y \subset X$ and each open set W of B, it holds that

$$\mathrm{Int}_{\mu_W} Y = X_W \cap \mathrm{Int}_{\mu_B} Y.$$

We conclude this section by considering the fibrewise g-uniformity base consisted of a system of conjugate pairs of *open* coverings, and by proving a characterization of fibrewise R_0-ness. For the definition of fibrewise R_0, see Definition 2.1.

Theorem 4.8. Let (X, τ_X) be a fibrewise space over B and $\{\mu_W^0\}$ be a system of conjugate pairs of open coverings of $\{X_W\}$ that satisfies (2), (4), (5) of Definition 3.3. Then $\{\mu_W^0\}$ is a fibrewise g-uniformity base compatible with the topology τ_X if and only if

$$\mathcal{B}(x) := \{\mathrm{st}(x,\mathcal{U} \Cap \mathcal{U}') | W \in N(p(x)), (\mathcal{U},\mathcal{U}') \in \mu_W^0\}$$

is a base for nbds of x for all $x \in X$.

Theorem 4.9. Let X be a fibrewise space over B.

(1) If X admits a fibrewise g-uniformity compatible with the topology, then X is fibrewise R_0.

(2) Suppose that X is fibrewise R_0. For each open set W of B, let

$$\mu_W^0 = \{(\mathcal{U},\mathcal{U}') | (\mathcal{U},\mathcal{U}') \text{ is a conjugate pair of open coverings of } X_W\}.$$

Then $\{\mu_W^0\}$ is a fibrewise g-uniformity base on X compatible with the topology.

5. Fibrewise completions of fibrewise g-uniform spaces

Unless otherwise stated, we consider the theory in a fibrewise g-uniform space $(X, \{\mu_W\})$ with a fibrewise g-uniformity $\{\mu_W\}$, and use the notation $\{\mu_W^0\}$ as the fibrewise g-uniformity base of $\{\mu_W\}$. In this section, we devote to study the fibrewise completion of fibrewise g-uniform spaces. For a filter base, see Definition 2.3.

We now begin with stating some definitions and lemmas.

Definition 5.1. Let $\mathcal{F}$ be a b-filter base.

We say $\mathcal{F}$ is *Cauchy* if for each $W \in N(b)$ and $(\mathcal{U},\mathcal{U}') \in \mu_W$ there exist $F \in \mathcal{F}$ and $(U, U') \in (\mathcal{U};\mathcal{U}')$ such that $F \subset U \cap U'$.

Lemma 5.2. A nbd filter is a Cauchy b-filter.

We shall define the notion of weak star b-filters and study some basic properties for this notion. This notion plays the important roles in the fibrewise completion theory.

Definition 5.3. Let $\mathcal{F}$ be a Cauchy b-filter.

$\mathcal{F}$ is a *weak star b-filter* with respect to $\{\mu_W^0\}$ if for each $F \in \mathcal{F}$ there exist $W \in N(b)$ and $(\mathcal{U}, \mathcal{U}') \in \mu_W^0$ such that $U \cap U' \subset F$ for each $U \cap U' \in (\mathcal{U} \Cap \mathcal{U}') \cap \mathcal{F}$, that is, $\cup((\mathcal{U} \Cap \mathcal{U}') \cap \mathcal{F}) \subset F$.

Lemma 5.4. For each weak star b-filter $\mathcal{F}, W \in N(b)$ and $(\mathcal{U}, \mathcal{U}') \in \mu_W$, we have

$$(\mathcal{U} \Cap \mathcal{U}') \cap \mathcal{F} \neq \emptyset.$$

Lemma 5.5. Let $\{\mu_W^0\}$ and $\{\mu_W^1\}$ be bases for fibrewise g-uniformity $\{\mu_W\}$. Then $\mathcal{F}$ is a weak star b-filter with respect to $\{\mu_W^0\}$ if and only if $\mathcal{F}$ is a weak star b-filter with respect to $\{\mu_W^1\}$.

By this lemma, we can remove "with respect to $\{\mu_W^0\}$" in Definition 5.10 (1), and we have the following definition.

Definition 5.6. A Cauchy b-filter $\mathcal{F}$ is a *weak star b-filter* if $\mathcal{F}$ is a weak star b-filter with respect to $\{\mu_W^0\}$ for some base $\{\mu_W^0\}$.

Definition 5.7. $(X, \{\mu_W\})$ is said to be *fibrewise complete* if every weak star b-filter $(b \in B)$ with respect to $\{\mu_W^0\}$ converges.

Lemma 5.8. Let $\{\mu_W^0\}$ and $\{\mu_W^1\}$ be bases for fibrewise g-uniformity $\{\mu_W\}$. Then a weak star b-filter with respect to $\{\mu_W^0\}$ converges if and only if a weak star b-filter with respect to $\{\mu_W^1\}$ converges.

Definition 5.9. Let $(X, \{\mu_W\})$ and $(Y, \{\nu_W\})$ be fibrewise g-uniform spaces and $X \subset Y$. $(Y, \{\nu_W\})$ is a *fibrewise completion* of $(X, \{\mu_W\})$ if

(1) $(Y, \{\nu_W\})$ is fibrewise complete,
(2) $\{\nu_W|_X\} = \{\mu_W\}$,
(3) $(X, \tau(\{\mu_W\}))$ is dense in $(Y, \tau(\{\nu_W\}))$.

Now we construct a fibrewise completion of a fibrewise g-uniform space $(X, \{\mu_W\})$.

Let Θ be the set of all weak star b-filters with respect to $\{\mu_W^0\}$ which do not converge and let $X^* := X \cup \Theta$. For $G \subset X$, we define

$$G^* := G \cup \{\mathcal{F} \in \Theta | G \in \mathcal{F}\}.$$

To obtain the theorem (Theorem 5.18) that X^* is a fibrewise completion of X, we use the following lemmas (Lemmas 5.10 – 5.17).

Lemma 5.10. Let G and H be subsets of X. Then

(1) $G \subset H$ if and only if $G^* \subset H^*$.
(2) $(G \cap H)^* = G^* \cap H^*$.

Let (X, μ) be a fibrewise g-uniform space with the fibrewise topology $\tau(\mu)$. We now define the projection $p^* : X^* \to B$ as follows:

$$p^*(y) = \begin{cases} p(y) & (y \in X) \\ b & (y = \mathcal{F} \in \Theta). \end{cases}$$

Then noting $(X^*)_W = (p^*)^{-1}(W)$, the family

$$\{G^* \cap (X^*)_W | G \in \tau(\mu), W \in \tau\}$$

is a base for a topology on X^* because of Lemma 5.10. We denote the topology generated by this base by $\tau(\mu)^*$. Then since p^* is continuous, hence $\tau(\mu)^*$ is the fibrewise topology of X^*.

Lemma 5.11. Let $\{\mu_W^0\}$ be a fibrewise g-uniformity base. For each $(\mathcal{U}, \mathcal{U}') \in \mu_W^0$, let

$$\mathcal{U}^* := \{U^* \cap (X^*)_W | U \in \mathcal{U}\}, \quad \mathcal{U}'^* := \{U'^* \cap (X^*)_W | U' \in \mathcal{U}'\}.$$

Then $(\mathcal{U}^*, \mathcal{U}'^*)$ is a conjugate pair of coverings of $(X^*)_W$.

We know from the following lemma that X is dense in X^*.

Lemma 5.12. For each open set O of X^*, each $y \in O$ and $b = p^*(y)$, there exist $W \in N(b)$ and $(\mathcal{U}, \mathcal{U}') \in \mu_W$ such that $\mathrm{st}(y, \mathcal{U}^* \curlywedge \mathcal{U}'^*) \subset O$.

Lemma 5.13. Let G be an open set of X. For each open set W of B and each $(\mathcal{U}, \mathcal{U}') \in \mu_W$,

$$\mathrm{st}(G^*, \mathcal{U}^* \curlywedge \mathcal{U}'^*) \subset [\mathrm{st}(G, \mathcal{U} \curlywedge \mathcal{U}')]^*.$$

For a fibrewise g-uniformity base $\{\mu_W^0\}$, let

$$(\mu_W^0)^* := \{(\mathcal{U}^*, \mathcal{U}'^*) | (\mathcal{U}, \mathcal{U}') \in \mu_W^0\}.$$

Then we have the following lemmas.

Lemma 5.14. For $U \subset X$,

$$(\mathrm{Int}_{\mu_W^0} U)^* \cap (X^*)_W \subset \mathrm{Int}_{(\mu_W^0)^*} U^*.$$

Lemma 5.15. Let $\{\mu_W^0\}$ be a fibrewise g-uniformity base compatible with the topology of X. Then

(1) $\{(\mu_W^0)^*\}$ is a fibrewise g-uniformity base compatible with the topology of X^*.
(2) Let $\mu^* = \{(\mu_W)^*\}$ be the fibrewise g-uniformity generated by $\{(\mu_W^0)^*\}$. Then $\{(\mu_W)^*\}$ does not depend on the choice of a base $\{\mu_W^0\}$.

From (1), we have $\tau(\mu)^* = \tau(\mu^*)$.

Lemma 5.16. (1) If $\mathcal{M}$ is a weak star b-filter with respect to $\{(\mu_W^0)^*\}$ then

$$R(\mathcal{M}) := \{F \subset X | \exists W \in N(b), \exists(\mathcal{U}, \mathcal{U}') \in \mu_W^0, \exists(U, U') \in (\mathcal{U}, \mathcal{U}') \text{ such that } U^* \cap U'^* \cap (X^*)_W \in \mathcal{M}, U \cap U' \subset F\}$$

is a weak star b-filter with respect to $\{\mu_W^0\}$.

(2) If $\mathcal{F}$ is a weak star b-filter with respect to $\{\mu_W^0\}$, then

$$E(\mathcal{F}) := \{F \subset X^* | \exists W \in N(b), \exists(\mathcal{U},\mathcal{U}') \in \mu_W^0, \exists(U,U') \in (\mathcal{U},\mathcal{U}') \text{ such that } U \cap U' \in \mathcal{F}, U^* \cap U'^* \cap (X^*)_W \subset F\}$$

is a weak star b-filter with respect to $\{(\mu_W^0)^*\}$.

(3) $E[R(\mathcal{M})] = \mathcal{M}$, $R[E(\mathcal{F})] = \mathcal{F}$.

Lemma 5.17. Let $\mathcal{M}$ and $R(\mathcal{M})$ be the same as in (1) of Lemma 5.16.

(1) If $R(\mathcal{M})$ converges to x in X, then $\mathcal{M}$ converges to x in X^*.

(2) Suppose that $R(\mathcal{M})$ does not converge to any point in X. Since $R(\mathcal{M}) \in \Theta$, $R(\mathcal{M})$ defines a point $y \in X^* - X$. Then $\mathcal{M}$ converges to y in X^*.

Now we can obtain the following.

Theorem 5.18. $(X^*, \{(\mu_W)^*\})$ is a fibrewise completion of $(X, \{\mu_W\})$.

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