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Kyoto University
HAUSDORFF HYPERSPACES OF EUCLIDEAN SPACES AND THEIR DENSE SUBSPACES

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Here, we introduce the results obtained in the paper [11] and related problems. We consider metric spaces and their hyperspaces endowed with the Hausdorff metric. Specifically, given a metric space $X = (X, d)$, we shall denote by $\text{Cld}(X)$ and $\text{Bd}(X)$ the hyperspaces consisting of all nonempty closed sets and of all nonempty bounded closed sets in $X$ respectively and we denote by $d_H$ the Hausdorff metric, which is infinite-valued on $\text{Cld}(X)$ if $X$ is unbounded. When $X$ is compact, the space $\text{Cld}(X) (= \text{Bd}(X))$ is equal to the hyperspace $\exp(X)$ of all nonempty compact sets with the Vietoris topology. Even if $X$ is noncompact, on the space $\exp(X)$, the Hausdorff metric topology coincides with the Vietoris topology. However, in case $X$ is noncompact, these topologies are very different on the spaces $\text{Cld}(X)$ and $\text{Bd}(X)$.

Vietoris hyperspaces $\exp(X)$ have been studied extensively for many years. Among the known results, let us mention the theorem of Curtis and Schori [8] (cf. [13, Chapter 8]), saying that $\exp(X)$ is homeomorphic to $(\cong)$ the Hilbert cube $Q = [-1, 1]^\omega$ if and only if $X$ is a Peano continuum, that is, it is compact, connected and locally connected. Later, Curtis [7] characterized non-compact metric spaces $X$ for which $\exp(X)$ is homeomorphic to the Hilbert cube minus a point $Q \setminus 0 (= Q \setminus \{0\})$ or the pseudo-interior $s = (-1, 1)^\omega$ of $Q$. In particular, $\text{Bd}(\mathbb{R}^m) = \exp(\mathbb{R}^m)$ is homeomorphic to $Q \setminus 0$. For more information concerning Vietoris hyperspaces, we refer to the book of Ilanes and Nadler [10].

It is well known that the hyperspace $\exp(X)$ is an ANR (AR) if and only if $X$ is locally connected (and connected). On the other hand, it is proved in [6] that the space $\text{Bd}(X)$ is an ANR (AR) whenever the metric on $X$ is almost convex, that is,

\footnote{It is well known that $s$ is homeomorphic to the separable Hilbert space $\ell_2$.}
for every $\alpha > 0, \beta > 0$ and for every $x, y \in X$ such that $d(x, y) < \alpha + \beta$, there exists $z \in X$ with $d(x, z) < \alpha$ and $d(z, y) < \beta$. This condition was further weakened in [12], which has turned out to be actually a necessary and sufficient one by Banakh and Voytsitskyy [3]. In the last paper, several equivalent conditions are given, which are too technical to mention them here. We refer to [3] for the details. On the other hand, Cld($X$) is not connected whenever $X$ is a metric space which is not totally bounded. For example, Cld($\mathbb{R}$) has $2^{|\mathbb{N}|}$ many components.

The completion of a metric space $X = \langle X, d \rangle$ is denoted by $\bar{X} = \langle \bar{X}, d \rangle$. Then Bd($X$) can be identified with the subspace of Bd($\bar{X}$), via the isometric embedding $A \mapsto \text{cl}_\bar{X} A$. Thus we shall often write Bd($X$) $\subseteq$ Bd($\bar{X}$), having in mind this identification. In this case, Bd($\bar{X}$) is the completion of Bd($X$). By such a reason, we also consider a dense subspace $D$ of a metric space $X = \langle X, d \rangle$. For each $0 \leq k < m$, let

$$\nu^m_k = \{ x = (x_i)_{i=1}^m \in \mathbb{R}^m : x_i \in \mathbb{R} \setminus \mathbb{Q} \text{ except for at most } k \text{ many } i \},$$

which is the universal space for completely metrizable subspaces in $\mathbb{R}^m$ of $\dim \leq k$.

In case $2k+1 < m$, $\nu^m_k$ is homeomorphic to the $k$-dimensional Nöbeling space $\nu^{2k+1}_k$, which is the universal space for all separable completely metrizable spaces. Note that $\nu^m_0 = (\mathbb{R} \setminus \mathbb{Q})^m \cong \mathbb{R} \setminus \mathbb{Q}$.

**Theorem 1.** Suppose $(m, k) = (1, 0)$ or $0 \leq k < m - 1$. Then,

$$\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(\nu^m_k) \rangle \cong \langle \mathbb{Q} \setminus 0, \mathbb{Q} \setminus 0 \rangle.$$

Consequently, Bd($\nu^m_k$) $\cong \ell_2$.

This can be derived from the following:

**Theorem 2.** Let $D$ be a dense $G_\delta$ set in $\mathbb{R}^m$ such that $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$ and in case $m > 1$ it is assumed that $D = p[D] \times \mathbb{R}$, where $p : \mathbb{R}^m \to \mathbb{R}^{m-1}$ is the projection onto the first $m - 1$ coordinates. Then, $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(D) \rangle \cong \langle \mathbb{Q} \setminus 0, \mathbb{Q} \setminus 0 \rangle$.

**Question 1.** In case $m > 1$, under the only assumption that $D \subseteq \mathbb{R}^m$ is a dense $G_\delta$ set and $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$, is the pair $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(D) \rangle$ homeomorphic to $\langle \mathbb{Q} \setminus 0, \mathbb{Q} \setminus 0 \rangle$? In particular, is the pair $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(\nu^m_{m-1}) \rangle$ homeomorphic to $\langle \mathbb{Q} \setminus 0, \mathbb{Q} \setminus 0 \rangle$?

We also consider the following dense subspaces of Bd($X$):

- Nwd($X$) — all nowhere dense closed sets;
- Perf($X$) — all perfect sets;\(^2\)

\(^2\)I.e., completely metrizable closed sets which are dense in itself.
- Cantor($X$) — all compact sets homeomorphic to the Cantor set.

In case $X = \mathbb{R}^m$, we can also consider the following subspace:

- $\mathfrak{N}(\mathbb{R}^m)$ — all closed sets of the Lebesgue measure zero.

For these spaces, we have the following:

**Theorem 3.** Let $\mathcal{F}$ be one of the following subspaces of $\text{Bd}(\mathbb{R}^m)$:

$$\text{Nwd}(\mathbb{R}^m), \text{Perf}(\mathbb{R}^m), \text{Cantor}(\mathbb{R}^m) \text{ and } \mathfrak{N}(\mathbb{R}^m).$$

Then, $(\text{Bd}(\mathbb{R}^m), \mathcal{F}) \cong (Q \setminus 0, s \setminus 0)$, hence $\mathcal{F} \cong \ell_2$.

To prove Theorems 2 and 3 above, we adopt the characterization of the pseudo-boundary $Q \setminus s$ of the Hilbert cube $Q$, see [5].

We also study the space $\text{Cld}(\mathbb{R})$. It is very different from the hyperspace $\exp(\mathbb{R})$. It is not hard to see that $\text{Cld}(\mathbb{R})$ has $2^{\aleph_0}$ many components, $\text{Bd}(\mathbb{R})$ is the only separable one and any other component has weight $2^{\aleph_0}$. Applying Toruńczyk's Characterization of Hilbert space [14] (cf. [15]), we can prove

**Theorem 4.** Let $\mathcal{H}$ be a nonseparable component of $\text{Cld}(\mathbb{R})$ which does not contain $\mathbb{R}$, $[0, +\infty)$, $(-\infty, 0)$. Then $\mathcal{H} \cong \ell_2(2^{\aleph_0})$.

**Question 2.** Does Theorem 4 hold even if $\mathcal{H}$ contains $\mathbb{R}$, $[0, \infty)$ or $(-\infty, 0]$?

**Question 3.** For $m > 1$, is $\text{Cld}(\mathbb{R}^m) \setminus \text{Bd}(\mathbb{R}^m)$ an $\ell_2(2^{\aleph_0})$-manifold?

Now, we consider the subspaces $\mathfrak{N}(\mathbb{R})$, $\text{Nwd}(\mathbb{R})$, $\text{Perf}(\mathbb{R})$ and $\text{Cld}(\mathbb{R} \setminus Q)$ of $\text{Cld}(\mathbb{R})$. Similarly to $\text{Bd}(\mathbb{R})$, it can be shown that those complements are $Z_\sigma$-sets in $\text{Cld}(\mathbb{R})$.

Due to Negligibility Theorem ([1], [9]), if $M$ is an $\ell_2(2^{\aleph_0})$-manifold and $A$ is a $Z_\sigma$-set in $M$ then $M \setminus A \cong M$. Thus, the following follows from Theorem 4:

**Corollary 5.** Let $\mathcal{H}$ be a nonseparable component of $\text{Cld}(\mathbb{R})$ which does not contain $\mathbb{R}$, $[0, +\infty)$, $(-\infty, 0)$. Then, the following spaces are homeomorphic to $\ell_2(2^{\aleph_0})$:

$$\mathcal{H} \cap \mathfrak{N}(\mathbb{R}), \mathcal{H} \cap \text{Nwd}(\mathbb{R}), \mathcal{H} \cap \text{Perf}(\mathbb{R}) \text{ and } \mathcal{H} \cap \text{Cld}(\mathbb{R} \setminus Q).$$

**Borel classes.** Given a metric space $(X, d)$, let $(\tilde{X}, d)$ be its completion. Then, the hyperspace $\text{Bd}(\tilde{X})$ is the completion of the hyperspace $\text{Bd}(X)$. Concerning Borel classes of hyperspaces, the following are also shown in the paper [11]:

1. $\text{Bd}(X)$ is $F_{\sigma\delta}$ in $\text{Bd}(\tilde{X})$ if $X$ is $\sigma$-compact.
2. $\text{Bd}(X)$ is $G_\delta$ in $\text{Bd}(\tilde{X})$ if $X$ is Polish.\(^3\)

\(^3\)I.e., separable and completely metrizable
(3) $\text{Bd}(X)$ is Polish for every Polish space $X$ in which bounded sets are totally bounded.

(4) $\text{Nwd}(X)$ is $G_\delta$ in $\text{Bd}(X)$ for every separable metric space $X$.

(5) $\text{Perf}(X)$ is $G_\delta$ in $\text{Bd}(X)$ if $X$ is separable and locally compact.

(6) $\text{Perf}(X)$ is $F_{\sigma\delta}$ in $\text{Bd}(X)$ for every Polish space $X$.

(7) $\text{Bd}(X)$ is analytic for every analytic metric space $X$ in which bounded sets are totally bounded.

Fix a dense set $X$ in a separable Banach space $E$ which admits the metric $d$ induced from the norm of $E$. Then $(X,d)$ is an almost convex metric space and therefore by a result of [6] the space $\text{Bd}(X)$ is an AR. In case $X$ is $G_\delta$, the space $\text{Bd}(X)$ is completely metrizable by (2). If additionally $E$ is finite-dimensional then $\text{Bd}(X)$ is Polish by (3). In case $X$ is $\sigma$-compact, by (1), $\text{Bd}(X)$ is absolutely $F_{\sigma\delta}$.

**Remarks.** Recently, Banakh and Voytsitskyy [4] proved that the space $\text{Cld}(X)$ (resp. $\text{Bd}(X)$) is homeomorphic to $\ell_2$ if and only if $X$ is a completely metrizable nowhere locally compact metric space such that each (resp. bounded) subset of $X$ is totally bounded and the completion $\tilde{X}$ of $X$ is connected and locally connected.

**References**


