Borel classes dimensions

1 Introduction and results.

The classes of topological spaces are assumed to be

1. non-empty (we suppose that at least the empty space $\emptyset$ is a member),
and

2. monotone with respect to closed subsets.

The letter $\mathcal{P}$ is used to denote a such class and the following classes of spaces satisfy the conditions 1 and 2 above.

- The class of compact metrizable spaces $\mathcal{K}$.
- The class of $\sigma$-compact metrizable spaces $\mathcal{S}$.
- The class of completely metrizable spaces $\mathcal{C}$.
- The class of separable completely metrizable spaces $\mathcal{C}_0$. 
Let $X$ be a space and $A$, $B$ disjoint subsets of $X$. We recall that a closed set $C \subset X$ is said to be a partition between $A$ and $B$ in $X$ if there are disjoint open subsets $U$ and $V$ of $X$ such that $A \subset U$, $B \subset V$ and $C = X \setminus (U \cup V)$.

In [4] Lelek introduced the small inductive dimension modulo a class $\mathcal{P}$, $\mathcal{P}$-ind, which is a natural generalization of well known dimension functions such as the small inductive dimension $\text{ind}$ and the small inductive compactness degree $\text{cmp}$.

**Definition 1.1** Let $X$ be a regular $T_1$-space and $\mathcal{P}$ a class of spaces. Then we define the small inductive dimension modulo a class $\mathcal{P}$, $\mathcal{P}$-ind $X$, of $X$ as follows.

(i) $\mathcal{P}$-ind $X = -1$ iff $X \in \mathcal{P}$.

(ii) For a natural number $n$, $\mathcal{P}$-ind $X \leq n$ if for any point $x \in X$ and any closed subset $A$ of $X$ with $x \notin A$ there exists a partition $C$ between $x$ and $A$ in $X$ such that $\mathcal{P}$-ind $C < n$.

The small inductive dimension modulo a class $\mathcal{P}$ has a natural transfinite extension.

**Definition 1.2** Let $X$ be a regular $T_1$-space and $\alpha$ either an ordinal number or the integer $-1$. Then the small transfinite inductive dimension modulo $\mathcal{P}$, $\mathcal{P}$-trind $X$, of $X$ is defined as follows.

(i) $\mathcal{P}$-trind $X = -1$ iff $X \in \mathcal{P}$;

(ii) $\mathcal{P}$-trind $X \leq \alpha$ if for any point $x \in X$ and any closed subset $A$ of $X$ with $x \notin A$ there exists a partition $C$ between $x$ and $A$ in $X$ such that $\mathcal{P}$-trind $C < \alpha$.

(iii) $\mathcal{P}$-trind $X = \alpha$ if $\mathcal{P}$-trind $X \leq \alpha$ and $\mathcal{P}$-trind $X > \beta$ for any ordinal $\beta < \alpha$;

(iv) $\mathcal{P}$-trind $X = \infty$ if $\mathcal{P}$-trind $X > \alpha$ for any ordinal $\alpha$.

We notice the following.

- $\{\emptyset\}$-trind $X = \text{trind} X$, i.e., the small transfinite dimension.
\begin{itemize}
    \item $\mathcal{K}$-ind $X = \text{comp} X$ (and $\mathcal{K}$-trind $X = \text{trcomp} X$), i.e., the small (transfinite) compactness degree.
    \item $\mathcal{C}$-ind $X = \text{id} X$ (and $\mathcal{C}$-trind $X = \text{trid} X$), i.e., the small (transfinite) completeness degree.
    \item If $\mathcal{P}_2 \subset \mathcal{P}_1$, then $\mathcal{P}_1$-trind $X \leq \mathcal{P}_2$-trind $X$; in particular, $\text{trid} X \leq \text{trcomp} X \leq \text{trid} X$ holds.
\end{itemize}

Here, we shall consider on the absolute Borel classes. For each ordinal number $\alpha$, let $A(\alpha)$ and $\mathcal{M}(\alpha)$ be the absolute additive class $\alpha$ and the absolute multiplicative class $\alpha$, respectively. Further, $A(\alpha) \cap \mathcal{M}(\alpha)$ is said to be the absolute ambiguous class $\alpha$ and we write $AB = \bigcup\{A_\alpha : \alpha < \omega_1\}$. We notice that the absolute Borel classes in the universe of metrizable spaces satisfy the conditions 1 and 2.

Recall that in the universe of separable metrizable spaces, we have the following.

\begin{itemize}
    \item $A(0) = \{\emptyset\}$.
    \item $\mathcal{M}(0) = \mathcal{K}$.
    \item $A(1) = S$.
    \item $\mathcal{M}(1) = C_0$.
    \item A diagram of the hierarchy of absolute Borel classes:
\end{itemize}

\[
\begin{array}{ccccccc}
\mathcal{A}(1) = S & \mathcal{A}(2) & \ldots \\
\{\emptyset\} \subseteq \mathcal{K} \subseteq \mathcal{A}(1) \cap \mathcal{M}(1) & \subseteq & \mathcal{A}(2) \cap \mathcal{M}(2) & \subseteq & \ldots \\
\subseteq & \mathcal{M}(1) = C_0 & \subseteq & \mathcal{M}(2) & \ldots \\
\end{array}
\]

We have a trivial example which shows the difference between trind and trcmp: The Hilbert cube $I^\infty$ has $\text{trind} I^\infty = \infty$ and $\text{comp} I^\infty (= \text{id} I^\infty = S\text{-ind} I^\infty) = -1$. Furthermore, E. Pol constructed the following example.
Example 1.1 (E. Pol, [5]) There exists a \( \sigma \)-compact, completely metrizable space \( P \) such that \( \text{trcmp} P = \infty \) (i.e., \( \text{trind} P = \text{trcmp} P = \infty \) and \( \text{tricd} P = \mathcal{S} \text{-trind} P = \mathcal{A}(1) \cap \mathcal{M}(1) \text{-trind} P = -1 \)).

Thus, we may ask whether we can generalize Pol's example to every ordinal number \( \alpha < \omega_1 \).

It is well known that the small compactness degree \( \text{cmp} \) is related to an extension property, i.e., de Groot proved that a separable metrizable space \( X \) is rim-compact (i.e., \( \text{cmp} X \leq 0 \)) iff \( X \) has a metric compactification \( Y \) such that \( \dim(Y - X) \leq 0 \). Connect with this theorem, we introduce other two dimension-like functions.

Definition 1.3 Let \( \mathcal{P} \) be a class of spaces. We recall that a separable metrizable space \( Y \) is a \( \mathcal{P} \)-hull (resp. \( \mathcal{P} \)-kernel) of a separable metrizable space \( X \) if \( Y \in \mathcal{P} \) and \( X \subset Y \) (resp. \( Y \subset X \)). Then the small transfinite \( \mathcal{P} \)-deficiency, \( \mathcal{P} \text{-trdef} X \), and the small transfinite \( \mathcal{P} \)-surplus, \( \mathcal{P} \text{-trsur} X \), of a separable metrizable space \( X \) are defined by

\[
\mathcal{P} \text{-trdef} X = \min \{ \text{trind} (Y \setminus X) : Y \text{ is an } \mathcal{P} \text{-hull of } X \},
\]

\[
(\mathcal{P} \text{-def} X = \min \{ \text{ind} (Y \setminus X) : Y \text{ is an } \mathcal{P} \text{-hull of } X \}),
\]

\[
\mathcal{P} \text{-trsur} X = \min \{ \text{trind} (X \setminus Y) : Y \text{ is an } \mathcal{P} \text{-kernel of } X \},
\]

\[
(\mathcal{P} \text{-sur} X = \min \{ \text{ind} (X \setminus Y) : Y \text{ is an } \mathcal{P} \text{-kernel of } X \}).
\]

It is clear that the functions \( \mathcal{P} \text{-trdef} \) and \( \mathcal{P} \text{-trsur} \) are transfinite extensions of the functions \( \mathcal{P} \text{-def} \) and \( \mathcal{P} \text{-sur} \), respectively, which are discussed in [1]. It is also clear that if \( \mathcal{P} \subset \mathcal{P}_1 \), then \( \mathcal{P}_1 \text{-trdef} X \leq \mathcal{P}_2 \text{-trdef} X \) and \( \mathcal{P}_1 \text{-trsur} X \leq \mathcal{P}_2 \text{-trsur} X \).

Recall also that for the function \( \mathcal{K} \text{-def} \) is the well known compact deficiency \( \text{def} \). We will denote the transfinite extension \( \mathcal{K} \text{-trdef} \) of the compact deficiency \( \text{def} \) by \( \text{trdef} \).

Facts (cf. [1]). Let \( X \) be a separable metrizable space and \( \alpha \) an ordinal number. Then we have the following.
1. If \( \alpha = 0 \), then \( \mathcal{M}(0)-\text{ind} \ X \leq \mathcal{M}(0)-\text{def} \ X \leq \mathcal{M}(0)-\text{sur} \ X \) holds and the converse of the inequalities do not hold. (We notice that \( \mathcal{M}(0) = \mathcal{K} \) and so \( \mathcal{M}(0)-\text{ind} \ X = \text{cmp} \ X \) and \( \mathcal{M}(0)-\text{def} \ X = \text{def} \ X \).) We also notice that \( \mathcal{A}(0) = \{ \emptyset \} \) and hence \( \mathcal{A}(0)-\text{ind} \ X = \mathcal{A}(0)-\text{sur} \ X \) trivially holds and \( \mathcal{A}(0)-\text{def} \ X \) cannot be defined if \( X \neq \emptyset \).

2. If \( \alpha = 1 \), then \( \mathcal{A}(1)-\text{ind} \ X \leq \mathcal{A}(1)-\text{def} \ X = \mathcal{A}(1)-\text{sur} \ X \) and \( \mathcal{M}(1)-\text{ind} \ X = \mathcal{M}(1)-\text{def} \ X \leq \mathcal{M}(1)-\text{sur} \ X \) hold. The converses of the inequalities above do not hold. (We notice that \( \mathcal{A}(1) = \mathcal{S} \) and \( \mathcal{M}(1) = \mathcal{C}_0 \) and so \( \mathcal{M}(1)-\text{ind} \ X = \text{icd} \ X \).)

3. If \( \alpha \geq 2 \), then \( \mathcal{A}(\alpha)-\text{ind} \ X = \mathcal{A}(\alpha)-\text{def} \ X = \mathcal{A}(\alpha)-\text{sur} \ X \) and \( \mathcal{M}(\alpha)-\text{ind} \ X = \mathcal{M}(\alpha)-\text{def} \ X = \mathcal{M}(\alpha)-\text{sur} \ X \) hold.

M. Charalambous [2] showed that the equality \( \mathcal{M}(\alpha)-\text{def} \ X = \mathcal{M}(\alpha)-\text{ind} \ X \) can not be extended to the transfinite dimension for the case of \( \alpha = 1 \).

**Example 1.2 (M. Charalambous, [2])** There exists a separable metrizable space \( C \) such that \( C-\text{trdef} \ C = \omega_0 \) and \( \text{trind} \ C = \omega_0 \). (We notice that \( \mathcal{C}_0-\text{trdef} \leq \text{trind} \ X \) holds for every separable metrizable space.)

Thus, it seems to be natural that we ask whether for each ordinal number \( \alpha < \omega_1 \) there exists a separable metrizable space \( X \) such that \( \mathcal{M}(\alpha)-\text{trdef} \ X = \omega_0 \) and \( \mathcal{M}(\alpha)-\text{trind} \ X = \infty \) or \( \mathcal{A}(\alpha)-\text{trdef} \ X = \omega_0 \) and \( \mathcal{A}(\alpha)-\text{trind} \ X = \infty \).

Connect with the questions above, we have the following.

**Theorem 1.1** Let \( \alpha \) be any ordinal with \( 1 \leq \alpha < \omega_1 \).

1. There exist separable metrizable spaces \( X_\alpha, Y_\alpha \) and \( Z_\alpha \) such that
   
   (a) \( f \ X_\alpha, f \ Y_\alpha, f \ Z_\alpha \leq \omega_0 \), where \( f \) is either \( \text{trdef} \) or \( \mathcal{K}-\text{trsurr} \);

   (b) \( \mathcal{M}(\alpha)-\text{trind} \ X_\alpha = -1 \) and \( \mathcal{A}(\alpha)-\text{trind} \ X_\alpha = \infty \) (and hence \( \mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)-\text{trind} \ X_\alpha = \infty \));

   (c) \( \mathcal{A}(\alpha)-\text{trind} \ Y_\alpha = -1 \) and \( \mathcal{M}(\alpha)-\text{trind} \ Y_\alpha = \infty \) (and hence \( \mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)-\text{trind} \ Y_\alpha = \infty \));

   (d) \( \mathcal{M}(\alpha)-\text{trind} \ Z_\alpha = \mathcal{A}(\alpha)-\text{trind} \ Z_\alpha = \infty \) and \( \mathcal{A}(\alpha + 1) \cap \mathcal{M}(\alpha + 1)-\text{trind} \ Z_\alpha = -1 \).
(2) There does not exist a separable metrizable space $W_{\alpha}$ such that $A(\alpha)$-trind $W_{\alpha} \neq \infty$, $\mathcal{M}(\alpha)$-trind $W_{\alpha} \neq \infty$ and $A(\alpha) \cap \mathcal{M}(\alpha)$-trind $W_{\alpha} = \infty$.

**Theorem 1.2** There exists a separable metrizable space $X$ with $\mathcal{K}$-trsurf $X = \omega_0$ such that for each $1 \leq \alpha < \omega_1$ we have $B$-trind $X = \infty$ and $B$-trdef $X = B$-trsurf $X = \omega_0$, where $B = A(\alpha), \mathcal{M}(\alpha)$ or $A(\alpha) \cap \mathcal{M}(\alpha)$.

**Remark 1.1** By Theorems 1.1 and 1.2, it follows that the equalities $\mathcal{M}(\alpha)$-def $X = \mathcal{M}(\alpha)$-ind $X$ and $A(\alpha)$-surf $X = A(\alpha)$-ind $X$ can not be extended to transfinite-dimensional cases. For the spaces $X_{\alpha}, Y_{\alpha}$ and $Z_{\alpha}$ in Theorem 1.1, we additionally have that

- $\mathcal{M}(\alpha)$-trdef $X_{\alpha} = A(\alpha)$-trsurf $Y_{\alpha} = -1$;
- $\mathcal{M}(\alpha)$-trdef $Y_{\alpha} = \mathcal{M}(\alpha)$-trdef $Z_{\alpha} = A(\alpha)$-trsurf $X_{\alpha} = A(\alpha)$-trsurf $Z_{\alpha} = \omega_0$.

We refer the readers to the books [1], [3] and [7] for the dimensions modulo classes, dimension theory and the theory of Borel sets, respectively.

## 2 Outline of proofs.

All classes of topological spaces considered here are additionally assumed to be finitely additive. We will follow some idea of E. Pol [5]. Let $\mathcal{P}$ be a class of topological spaces. A space $X$ is said to have the property $(*)_{\mathcal{P}}$ if for every sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint compact subsets of $X$ there exist partitions $L_i$ between $A_i$ and $B_i$ in $X$ and an integer $N$ such that $\cap_{i=1}^{N} L_i \in \mathcal{P}$.

It is evident that the property $(*)_{\mathcal{P}}$ is closed hereditary.

We have two propositions on the property $(*)_{\mathcal{P}}$.

**Proposition 2.1** If a space $X$ is covered by a finite family of closed sets such that each element of this cover possesses property $(*)_{\mathcal{P}}$ then $X$ also possesses this property.

**Proposition 2.2** Let $X$ be a space. If $\mathcal{P}$-trind $X \neq \infty$ then $X$ possesses property $(*)_{\mathcal{P}}$. 
Let $\mathbb{I}^\infty = \{(x_i) : 0 \leq x_i \leq 1, i = 1, 2, \ldots \}$ be the Hilbert cube and $Z = \{0, \frac{1}{2}, \frac{1}{3}, \ldots \}$ a subspace of the unit interval $\mathbb{I}$. For each $n \geq 1$ we denote the subset \{(x_i) \in \mathbb{I}^\infty : x_j = 0 \text{ for } j \geq n + 1 \} of $\mathbb{I}^\infty$ by $\mathbb{I}^n$. For each $n \geq 1$ and each $i = 1, \ldots, n$, we put
$$A_i^n = \{(x_i) \in \mathbb{I}^n \subset \mathbb{I}^\infty : x_i = 0 \}, \quad B_i^n = \{(x_i) \in \mathbb{I}^n \subset \mathbb{I}^\infty : x_i = 1 \}.$$ Choose for each $n \geq 1$ a subset $E_n$ of $\mathbb{I}^n$ and put
$$X = (\{0\} \times \mathbb{I}^\infty) \cup \left( \bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\} \times E_n \right). \quad (1)$$

Furthermore, we put $Y = (\{0\} \times \mathbb{I}^\infty) \cup \left( \bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\} \times \mathbb{I}^n \right)$. It is clear that $X \subset Y \subset Z \times \mathbb{I}^\infty$, $Y$ is compact, and $Y \setminus X$ is a subspace of the topological sum $\oplus_{n=1}^{\infty} \mathbb{I}^n$. Thus, trind $(Y \setminus X) \leq \omega_0$. Observe also that trind $(X \setminus (\{0\} \times \mathbb{I}^\infty)) \leq \omega_0$. Hence
$$\text{trind} X \leq \omega_0 \text{ and } \mathcal{K}\text{-trsurf} X \leq \omega_0. \quad (2)$$

**Lemma 2.1** If for each $m \geq 1$ there exists an integer $k(m) \geq m + 1$ such that for any $n \geq k(m)$ and any partition $L_i^n$ between $A_i^n$ and $B_i^n$ in $\mathbb{I}^n$, $i \leq m$, we have $E_n \cap \bigcap_{i=1}^{N} L_i^n \not\in \mathcal{P}$, then $\mathcal{P}$-trind $X = \infty$.

**Proof.** By Proposition 2.2, it suffices to show that $X$ does not have the property $(*)_\mathcal{P}$. For each $i = 1, 2, \ldots$ let $L_i$ be a partition between compact sets $A_i = \{(0, (x_j)) \in (0, \mathbb{I}^\infty) \cap \mathbb{I}^\infty : x_i = 0 \}$ and $B_i = \{(0, (x_j)) \in (0, \mathbb{I}^\infty) \cap \mathbb{I}^\infty : x_i = 1 \}$. We shall show that $\cap_{i=1}^{N} L_i \not\in \mathcal{P}$ for every natural number $N$. Let $N$ be a natural number. For each $i \geq 1$ let us consider a partition $L_i$ between $A_i$ and $B_i$ in $Y$ such that $L_i = L_i \cap X$. Note that for every $i$ there exists a natural number $n_i \geq 2$ such that for any $n \geq n_i$ $L_i^n = L_i \cap (\{1/\} \times \mathbb{I}^n)$ is a partition between $\{1/\} \times A_i^n$ and $\{1/\} \times B_i^n$ in $\mathbb{I}^\infty$. Let $n$ a fixed integer with $n \geq \max\{n_1, \ldots, n_N, k(N)\}$. Then $C = (\cap_{i=1}^{N} L_i^n) \cap (\{1/\} \times E_n) = (\cap_{i=1}^{N} L_i) \cap (\{1/\} \times E_n)$ is a closed subset of $\cap_{i=1}^{N} L_i$, and $C \not\in \mathcal{P}$ by the assumption. So $\cap_{i=1}^{N} L_i \not\in \mathcal{P}$.

We shall also use the following.

**Lemma 2.2** ([8, Lemma 5.2]) Let $L_{ij}$ be partitions between the opposite faces $A_i^n$ and $B_i^n$ in $\mathbb{I}^n$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq n$ and $1 \leq p < n$. Then for any $k \neq j$, $j = 1, \ldots, p$, there is a continuum $C \subset \cap_{j=1}^{p} L_{ij}$ meeting the faces $A_k^n$ and $B_k^n$. 
Lemma 2.3 Let \( \alpha \) be an ordinal number with \( 1 \leq \alpha < \omega_1 \). Then there exist subsets \( Q_\alpha, P_\alpha \) and \( D_\alpha \) of \( \mathbb{I} \) such that

1. \( Q_\alpha \in A(\alpha) - \mathcal{M}(\alpha) \),
2. \( P_\alpha \in \mathcal{M}(\alpha) - A(\alpha) \),
3. \( D_\alpha \in A(\alpha + 1) \cap \mathcal{M}(\alpha + 1) - (A(\alpha) \cup \mathcal{M}(\alpha)) \).

Proof of Theorem 1.1. (1) We shall prove for \( Y_\alpha \) only. We put

\[
Y_\alpha = ([0] \times \mathbb{I}^\infty) \cup \left( \bigcup_{n=2}^{\infty} \left( \frac{1}{n} \times \pi_{n}^{-1}(Q_\alpha) \right) \right),
\]

where \( Q_\alpha \) is the subspace \( \mathbb{I} \) described in Lemma 2.3 and \( \pi_{n} : \mathbb{I}^n \to \mathbb{I} \) be the projection onto the \( n \)-th factor. By the construction of \( Y_\alpha \), it is clear that \( \mathcal{M}(\alpha) - \text{trind} Y_\alpha \leq \text{trind} Y_\alpha \leq \omega_0 \), and \( \mathcal{M}(\alpha) - \text{trsur} Y_\alpha \leq \omega_0 \). Since the absolute Borel classes are preserved under perfect preimages, it follows that \( \pi_{n}^{-1}(Q_\alpha) \in A(\alpha) \). Thus, \( Y_\alpha \in A(\alpha) \) and hence \( A(\alpha) - \text{trind} Y_\alpha = -1 \). Now, it suffices to show that \( \mathcal{M}(\alpha) - \text{trind} Y_\alpha = \infty \).

(2) The second part of Theorem 1.1 is a direct consequence of the following proposition.

Proposition 2.3 Let \( X \) be a separable metrizable space with \( A(\alpha) - \text{trind} X \leq \mu_1 \) and \( \mathcal{M}(\alpha) - \text{trind} X \leq \mu_2 \). Then

\[
A(\alpha) \cap \mathcal{M}(\alpha) - \text{trind} X = \begin{cases} \mu_1 + \mu_2 + 1, & \text{if } \lambda(\mu_1) = \lambda(\mu_2), \\ \mu_1, & \text{if } \lambda(\mu_1) > \lambda(\mu_2). \end{cases}
\]

Proof. The proposition can be proved by a standard transfinite induction on \( \nu = \max\{\mu_1, \mu_2\} \).
**Question 2.1** Does there exist a separable metrizable space $X_\alpha$ such that $A(\alpha) \cap \mathcal{M}(\alpha)$-trind $X_\alpha > \max \{A(\alpha)$-trind $X_\alpha, \mathcal{M}(\alpha)$-trind $X_\alpha\}$ for each ordinal number $\alpha$? In particular, does there exist a separable metrizable space $X$ such that $C_0 \cap \mathcal{S}$-ind $X = 1$ and $C_0$-ind $X = \mathcal{S}$-trind $X = 0$?

Recall from M.G. Charalambous ([2]) that we call a subset $A$ of a space $X$ a Bernstein set if $|A \cap B| = |(X \setminus A) \cap B| = c$ for every uncountable Borel set $B$ of $X$, where $c$ denotes the cardinality of the continuum. It is known that every uncountable completely metrizable space $X$ has countably many disjoint Bernstein sets. We notice that $A \notin \mathcal{A}$ for every Bernstein set $A$ of an uncountable completely metrizable space $X$.

**Proof of Theorem 1.2.** Let $F$ be a Bernstein set of $\mathbb{I}$. We put $X = (\{0\} \times \mathbb{I}^\infty) \cup (\bigcup_{n=1}^\infty \{\frac{1}{n}\} \times \pi_n^{-1}(F))$. Then, we can show that $X$ is the desired space by an argument similar to Theorem 1.1.

Connect with Theorem 1.1, we may ask the following question.

**Question 2.2** For each ordinal numbers $\alpha$ and $\beta$ with $1 \leq \alpha < \omega_1$ and $0 \leq \beta < \omega_1$ do there exist separable metrizable spaces $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ which satisfy the following conditions?

1. $A(\alpha)$-trind $X_{\alpha,\beta} = \beta$,
2. $\mathcal{M}(\alpha)$-trind $Y_{\alpha,\beta} = \beta$, and
3. $\mathcal{M}(\alpha)$-trind $X_{\alpha,\beta} = A(\alpha)$-trind $Y_{\alpha,\beta} = -1$.

**References**


(V.A. Chatyrko)
Department of Mathematics, Linköping University, 581 83 Linkeping, Sweden.
vitja@mai.liu.se

(Y. Hattori)
Department of Mathematics, Shimane University, Matsue, Shimane, 690-8504 Japan
hattori@riko.shimane-u.ac.jp