Borel classes dimensions

1 Introduction and results.

The classes of topological spaces are assumed to be

1. non-empty (we suppose that at least the empty space $\emptyset$ is a member), and

2. monotone with respect to closed subsets.

The letter $\mathcal{P}$ is used to denote a such class and the following classes of spaces satisfy the conditions 1 and 2 above.

- The class of compact metrizable spaces $\mathcal{K}$.
- The class of $\sigma$-compact metrizable spaces $\mathcal{S}$.
- The class of completely metrizable spaces $\mathcal{C}$.
- The class of separable completely metrizable spaces $\mathcal{C}_0$. 
Let $X$ be a space and $A$, $B$ disjoint subsets of $X$. We recall that a closed set $C \subset X$ is said to be a *partition* between $A$ and $B$ in $X$ if there are disjoint open subsets $U$ and $V$ of $X$ such that $A \subset U$, $B \subset V$ and $C = X \setminus (U \cup V)$.

In [4] Lelek introduced the small inductive dimension modulo a class $\mathcal{P}$, $\mathcal{P}$-ind, which is a natural generalization of well known dimension functions such as the small inductive dimension $\text{ind}$ and the small inductive compactness degree $\text{cmp}$.

**Definition 1.1** Let $X$ be a regular $T_1$-space and $\mathcal{P}$ a class of spaces. Then we define the *small inductive dimension modulo a class $\mathcal{P}$, $\mathcal{P}$-ind* $X$, of $X$ as follows.

(i) $\mathcal{P}$-ind $X = -1$ iff $X \in \mathcal{P}$.

(ii) For a natural number $n$, $\mathcal{P}$-ind $X \leq n$ if for any point $x \in X$ and any closed subset $A$ of $X$ with $x \notin A$ there exists a partition $C$ between $x$ and $A$ in $X$ such that $\mathcal{P}$-ind $C < n$.

The small inductive dimension modulo a class $\mathcal{P}$ has a natural transfinite extension.

**Definition 1.2** Let $X$ be a regular $T_1$-space and $\alpha$ either an ordinal number or the integer $-1$. Then the *small transfinite inductive dimension modulo $\mathcal{P}$, $\mathcal{P}$-trind* $X$, of $X$ is defined as follows.

(i) $\mathcal{P}$-trind $X = -1$ iff $X \in \mathcal{P}$;

(ii) $\mathcal{P}$-trind $X \leq \alpha$ if for any point $x \in X$ and any closed subset $A$ of $X$ with $x \notin A$ there exists a partition $C$ between $x$ and $A$ in $X$ such that $\mathcal{P}$-trind $C < \alpha$.

(iii) $\mathcal{P}$-trind $X = \alpha$ if $\mathcal{P}$-trind $X \leq \alpha$ and $\mathcal{P}$-trind $X > \beta$ for any ordinal $\beta < \alpha$;

(iv) $\mathcal{P}$-trind $X = \infty$ if $\mathcal{P}$-trind $X > \alpha$ for any ordinal $\alpha$.

We notice the following.

- $\{\emptyset\}$-trind $X = \text{trind} X$, i.e., the small transfinite dimension.
- \( \mathcal{K}\)-ind \( X = \text{cmp} X \) (and \( \mathcal{K}\)-trind \( X = \text{trcmp} X \)), i.e., the small (transfinite) compactness degree.

- \( C\)-ind \( X = \text{idc} X \) (and \( C\)-trind \( X = \text{tridc} X \)), i.e., the small (transfinite) completeness degree.

- If \( P_2 \subseteq P_1 \), then \( P_1\)-trind \( X \leq P_2\)-trind \( X \); in particular, \( \text{tridc} X \leq \text{trcmp} X \leq \text{trind} X \) holds.

Here, we shall consider on the absolute Borel classes. For each ordinal number \( \alpha \), let \( \mathcal{A}(\alpha) \) and \( \mathcal{M}(\alpha) \) be the absolute additive class \( \alpha \) and the absolute multiplicative classe \( \alpha \), respectively. Further, \( \mathcal{A}(\alpha) \cap \mathcal{M}(\alpha) \) is said to be the absolute ambiguous class \( \alpha \) and we write \( AB = \bigcup \{\mathcal{A}_\alpha : \alpha < \omega_1\} \).

We notice that the absolute Borel classes in the universe of metrizable spaces satisfy the conditions 1 and 2.

Recall that in the universe of separable metrizable spaces, we have the following.

- \( \mathcal{A}(0) = \{\emptyset\} \).
- \( \mathcal{M}(0) = \mathcal{K} \).
- \( \mathcal{A}(1) = S \).
- \( \mathcal{M}(1) = C_0 \).

- A diagram of the hierarchy of absolute Borel classes:

\[
\begin{align*}
\mathcal{A}(1) & = S \\
\{\emptyset\} & \subseteq \mathcal{K} \subseteq \mathcal{A}(1) \cap \mathcal{M}(1) \\
\mathcal{A}(2) & \supseteq \mathcal{A}(1) \cap \mathcal{M}(1) \\
\mathcal{A}(2) & \cap \mathcal{M}(2) \\
\mathcal{M}(1) & = C_0 \\
\mathcal{M}(2) & \supseteq \mathcal{M}(2)
\end{align*}
\]

We have a trivial example which shows the difference between trind and trcmp: The Hilbert cube \( I^\infty \) has \( \text{trind} I^\infty = \infty \) and \( \text{cmp} I^\infty (= \text{idc} I^\infty = S\text{-ind} I^\infty) = -1 \). Furthermore, E. Pol constructed the following example.
Example 1.1 (E. Pol, [5]) There exists a $\sigma$-compact, completely metrizable space $P$ such that $\text{trcmp } P = \infty$ (i.e., $\text{trind } P = \text{trcmp } P = \infty$ and $\text{tricd } P = S\text{-trind } P = A(1) \cap M(1)\text{-trind } P = -1$).

Thus, we may ask whether we can generalize Pol's example to every ordinal number $\alpha < \omega_1$.

It is well known that the small compactness degree $\text{cmp}$ is related to an extension property, i.e., de Groot proved that a separable metrizable space $X$ is rim-compact (i.e., $\text{cmp } X \leq 0$) iff $X$ has a metric compactification $Y$ such that $\dim(Y - X) \leq 0$. Connect with this theorem, we introduce other two dimension-like functions.

Definition 1.3 Let $\mathcal{P}$ be a class of spaces. We recall that a separable metrizable space $Y$ is a $\mathcal{P}$-hull (resp. $\mathcal{P}$-kernel) of a separable metrizable space $X$ if $Y \in \mathcal{P}$ and $X \subset Y$ (resp. $Y \subset X$). Then the small transfinite $\mathcal{P}$-deficiency, $\mathcal{P}\text{-trdef } X$, and the small transfinite $\mathcal{P}$-surplus, $\mathcal{P}\text{-trsur } X$, of a separable metrizable space $X$ are defined by

$$\mathcal{P}\text{-trdef } X = \min\{\text{trind } (Y \setminus X) : Y \text{ is an } \mathcal{P}\text{-hull of } X\},$$

$$\mathcal{P}\text{-trsur } X = \min\{\text{trind } (X \setminus Y) : Y \text{ is an } \mathcal{P}\text{-kernel of } X\},$$

$$\mathcal{P}\text{-trdef } X = \min\{\text{ind } (Y \setminus X) : Y \text{ is an } \mathcal{P}\text{-hull of } X\},$$

$$\mathcal{P}\text{-trsur } X = \min\{\text{ind } (X \setminus Y) : Y \text{ is an } \mathcal{P}\text{-kernel of } X\}.$$ 

It is clear that the functions $\mathcal{P}\text{-trdef}$ and $\mathcal{P}\text{-trsur}$ are transfinite extensions of the functions $\mathcal{P}\text{-def}$ and $\mathcal{P}\text{-sur}$, respectively, which are discussed in [1]. It is also clear that if $\mathcal{P}_2 \subset \mathcal{P}_1$, then $\mathcal{P}_1\text{-trdef } X \leq \mathcal{P}_2\text{-trdef } X$ and $\mathcal{P}_1\text{-trsur } X \leq \mathcal{P}_2\text{-trsur } X$.

Recall also that for the function $\mathcal{K}\text{-def}$ is the well known compact deficiency $\text{def}$. We will denote the transfinite extension $\mathcal{K}\text{-trdef}$ of the compact deficiency $\text{def}$ by $\text{trdef}$.

Facts (cf. [1]). Let $X$ be a separable metrizable space and $\alpha$ an ordinal number. Then we have the following.
1. If $\alpha = 0$, then $\mathcal{M}(0)$-ind $X \leq \mathcal{M}(0)$-def $X \leq \mathcal{M}(0)$-sur $X$ holds and the converse of the inequalities do not hold. (We notice that $\mathcal{M}(0) = \mathcal{K}$ and so $\mathcal{M}(0)$-ind $X = \text{cmp} X$ and $\mathcal{M}(0)$-def $X = \text{def} X$.) We also notice that $\mathcal{A}(0) = \{\emptyset\}$ and hence $\mathcal{A}(0)$-ind $X = \mathcal{A}(0)$-sur $X$ trivially holds and $\mathcal{A}(0)$-def $X$ can not be defined if $X \neq \emptyset$.

2. If $\alpha = 1$, then $\mathcal{A}(1)$-ind $X \leq \mathcal{A}(1)$-def $X = \mathcal{A}(1)$-sur $X$ and $\mathcal{M}(1)$-ind $X = \mathcal{M}(1)$-def $X \leq \mathcal{M}(1)$-sur $X$ hold. The converses of the inequalities above do not hold. (We notice that $\mathcal{A}(1) = \mathcal{S}$ and $\mathcal{M}(1) = \mathcal{C}_0$ and so $\mathcal{M}(1)$-ind $X = \text{icd} X$.)

3. If $\alpha \geq 2$, then $\mathcal{A}(\alpha)$-ind $X = \mathcal{A}(\alpha)$-def $X = \mathcal{A}(\alpha)$-sur $X$ and $\mathcal{M}(\alpha)$-ind $X = \mathcal{M}(\alpha)$-def $X = \mathcal{M}(\alpha)$-sur $X$ hold.

M. Charalambous [2] showed that the equality $\mathcal{M}(\alpha)$-def $X = \mathcal{M}(\alpha)$-ind $X$ can not be extended to the transfinite dimension for the case of $\alpha = 1$.

Example 1.2 (M. Charalambous, [2]) There exists a separable metrizable space $C$ such that $C$-trdef $C (= \mathcal{M}(1)$-trdef $C) = \omega_0$ and $\text{tricd} C (= \mathcal{M}(1)$-trind $C) = \infty$. (We notice that $\mathcal{C}_0$-trdef $\leq \text{tricd} X$ holds for every separable metrizable space.)

Thus, it seems to be natural that we ask whether for each ordinal number $\alpha < \omega_1$ there exits a separable metrizable space $X$ such that $\mathcal{M}(\alpha)$-trdef $X = \omega_0$ and $\mathcal{M}(\alpha)$-trind $X = \infty$ or $\mathcal{A}(\alpha)$-trdef $X = \omega_0$ and $\mathcal{A}(\alpha)$-trind $X = \infty$.

Connect with the questions above, we have the following.

Theorem 1.1 Let $\alpha$ be any ordinal with $1 \leq \alpha < \omega_1$.
(1) There exist separable metrizable spaces $X_\alpha, Y_\alpha$ and $Z_\alpha$ such that
(a) $f X_\alpha, f Y_\alpha, f Z_\alpha \leq \omega_0$, where $f$ is either trdef or $\mathcal{K}$-trsur;
(b) $\mathcal{M}(\alpha)$-trind $X_\alpha = -1$ and $\mathcal{A}(\alpha)$-trind $X_\alpha = \infty$ (and hence $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)$-trind $X_\alpha = \infty$);
(c) $\mathcal{A}(\alpha)$-trind $Y_\alpha = -1$ and $\mathcal{M}(\alpha)$-trind $Y_\alpha = \infty$ (and hence $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)$-trind $X_\alpha = \infty$);
(d) $\mathcal{M}(\alpha)$-trind $Z_\alpha = \mathcal{A}(\alpha)$-trind $Z_\alpha = \infty$ and $\mathcal{A}(\alpha + 1) \cap \mathcal{M}(\alpha + 1)$-trind $Z_\alpha = -1$. 

There does not exist a separable metrizable space $W_\alpha$ such that $A(\alpha)$-trind $W_\alpha \neq \infty$, $\mathcal{M}(\alpha)$-trind $W_\alpha \neq \infty$ and $A(\alpha) \cap \mathcal{M}(\alpha)$-trind $W_\alpha = \infty$.

Theorem 1.2 There exists a separable metrizable space $X$ with trdef $X = K$-trsur $X = \omega_0$ such that for each $1 \leq \alpha < \omega_1$ we have $B$-trind $X = \infty$ and $B$-trdef $X = B$-trsur $X = \omega_0$, where $B = A(\alpha)$, $\mathcal{M}(\alpha)$ or $A(\alpha) \cap \mathcal{M}(\alpha)$.

Remark 1.1 By Theorems 1.1 and 1.2, it follows that the equalities $\mathcal{M}(\alpha)$-def $X = \mathcal{M}(\alpha)$-ind $X$ and $A(\alpha)$-sur $X = A(\alpha)$-ind $X$ can not be extended to transfinite-dimensional cases. For the spaces $X_\alpha$, $Y_{\alpha}$ and $Z_\alpha$ in Theorem 1.1, we additionally have that

- $\mathcal{M}(\alpha)$-trdef $X_\alpha = A(\alpha)$-trsur $Y_\alpha = -1$;
- $\mathcal{M}(\alpha)$-trdef $Y_\alpha = \mathcal{M}(\alpha)$-trdef $Z_\alpha = A(\alpha)$-trsur $X_\alpha = A(\alpha)$-trsur $Z_\alpha = \omega_0$.

We refer the readers to the books [1], [3] and [7] for the dimensions modulo classes, dimension theory and the theory of Borel sets, respectively.

2 Outline of proofs.

All classes of topological spaces considered here are additionally assumed to be finitely additive. We will follow some idea of E. Pol [5]. Let $P$ be a class of topological spaces. A space $X$ is said to have the property $(\ast)_P$ if for every sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint compact subsets of $X$ there exist partitions $L_i$ between $A_i$ and $B_i$ in $X$ and an integer $N$ such that $\cap_{i=1}^{N} L_i \in P$.

It is evident that the property $(\ast)_P$ is closed hereditary.

We have two propositions on the property $(\ast)_P$.

**Proposition 2.1** If a space $X$ is covered by a finite family of closed sets such that each element of this cover possesses property $(\ast)_P$ then $X$ also possesses this property.

**Proposition 2.2** Let $X$ be a space. If $P$-trind $X \neq \infty$ then $X$ possesses property $(\ast)_P$. 
Let $\mathbb{I}^{\infty} = \{(x_i) : 0 \leq x_i \leq 1, i = 1, 2, \ldots\}$ be the Hilbert cube and $Z = \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$ a subspace of the unit interval $\mathbb{I}$. For each $n \geq 1$ we denote the subset $\{(x_i) \in \mathbb{I}^{\infty} : x_j = 0 \text{ for } j \geq n + 1\}$ of $\mathbb{I}^{\infty}$ by $\mathbb{I}^n$. For each $n \geq 1$ and each $i = 1, \ldots, n$, we put

$$A_i^n = \{(x_i) \in \mathbb{I}^n \subset \mathbb{I}^{\infty} : x_i = 0\}, \quad B_i^n = \{(x_i) \in \mathbb{I}^n \subset \mathbb{I}^{\infty} : x_i = 1\}.$$  

Choose for each $n \geq 1$ a subset $E_n$ of $\mathbb{I}^n$ and put

$$X = (\{0\} \times \mathbb{I}^{\infty}) \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \mathbb{I}^n \times E_n\right]. \quad (1)$$

Furthermore, we put $Y = (\{0\} \times \mathbb{I}^{\infty}) \cup (\bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times \mathbb{I}^n)$. It is clear that $X \subset Y \subset Z \times \mathbb{I}^{\infty}$, $Y$ is compact, and $Y \setminus X$ is a subspace of the topological sum $\oplus_{n=1}^{\infty} \mathbb{I}^n$. Thus, $\text{trind} (Y \setminus X) \leq \omega_0$. Observe also that $\text{trind} (X \setminus (\{0\} \times \mathbb{I}^{\infty})) \leq \omega_0$. Hence

$$\text{trdef} X \leq \omega_0 \text{ and } K\text{-trsur} X \leq \omega_0. \quad (2)$$

**Lemma 2.1** If for each $m \geq 1$ there exists an integer $k(m) \geq m + 1$ such that for any $n \geq k(m)$ and any partition $L_i^n$ between $A_i^n$ and $B_i^n$ in $\mathbb{I}^n$, $i \leq m$, we have $E_n \cap \bigcap_{i=1}^{\infty} L_i^n \notin \mathcal{P}$, then $\mathcal{P}$-trind $X = \infty$.

**Proof.** By Proposition 2.2, it suffices to show that $X$ does not have the property $(*)_{\mathcal{P}}$. For each $i = 1, 2, \ldots$ let $L_i$ be a partition between compact sets $A_i = \{(0, (x_j)) \in \{0\} \times \mathbb{I}^{\infty} : x_i = 0\}$ and $B_i = \{(0, (x_j)) \in \{0\} \times \mathbb{I}^{\infty} : x_i = 1\}$ We shall show that $\cap_{i=1}^{N} L_i \notin \mathcal{P}$ for every natural number $N$. Let $N$ be a natural number. For each $i \geq 1$ let us consider a partition $L_i'$ between $A_i$ and $B_i$ in $Y$ such that $L_i = L_i' \cap X$. Note that for every $i$ there exists a natural number $n_i \geq 2$ such that for any $n \geq n_i$ $L_i^n = L_i' \cap \{\frac{1}{n}\} \times \mathbb{I}^n$ is a partition between $\{\frac{1}{n}\} \times A_i^n$ and $\{\frac{1}{n}\} \times B_i^n$ in $\{\frac{1}{n}\} \times \mathbb{I}^n$. Let $n$ a fixed integer with $n \geq \max\{n_1, \ldots, n_N, k(N)\}$. Then $C = (\bigcap_{i=1}^{N} L_i^n) \cap \{\frac{1}{n}\} \times E_n$ is a closed subset of $\bigcap_{i=1}^{N} L_i$, and $C \notin \mathcal{P}$ by the assumption. So $\cap_{i=1}^{N} L_i \notin \mathcal{P}$.

We shall also use the following.

**Lemma 2.2** ([8, Lemma 5.2]) Let $L_{ij}$ be partitions between the opposite faces $A_{ij}^n$ and $B_{ij}^n$ in $\mathbb{I}^n$, where $1 \leq i_1 < i_2 < \ldots < i_p \leq n$ and $1 \leq p < \infty$. Then for any $k \neq i_j, j = 1, \ldots, p$, there is a continuum $C \subset \bigcap_{j=1}^{p} L_{ij}$ meeting the faces $A_{ij}^k$ and $B_{ij}^k$. 

Lemma 2.3 Let $\alpha$ be an ordinal number with $1 \leq \alpha < \omega_1$. Then there exist subsets $Q_\alpha$, $P_\alpha$ and $D_\alpha$ of $I$ such that

1. $Q_\alpha \in A(\alpha) - M(\alpha)$,
2. $P_\alpha \in M(\alpha) - A(\alpha)$,
3. $D_\alpha \in A(\alpha + 1) \cap M(\alpha + 1) - (A(\alpha) \cup M(\alpha))$.

Proof of Theorem 1.1. (1) We shall prove for $Y_\alpha$ only. We put

$$Y_\alpha = (\{0\} \times \mathbb{I}^\infty) \cup \left( \bigcup_{n=2}^{\infty} \left\{ \frac{1}{n} \right\} \times \pi_n^{-1}(Q_\alpha) \right),$$

where $Q_\alpha$ is the subspace $\mathbb{I}$ described in Lemma 2.3 and $\pi_n : \mathbb{I}^n \to \mathbb{I}$ be the projection onto the $n$-th factor. By the construction of $Y_\alpha$, it is clear that $M(\alpha)$-trdef $Y_\alpha \leq \text{trdef} Y_\alpha \leq \omega_0$, and $M(\alpha)$-trsur $Y_\alpha \leq \omega_0$. Since the absolute Borel classes are preserved under perfect preimages, it follows that $\pi_n^{-1}(Q_\alpha) \in A(\alpha)$. Thus, $Y_\alpha \in A(\alpha)$ and hence $A(\alpha)$-trind $Y_\alpha = -1$. Now, it suffices to show that $M(\alpha)$-trind $Y_\alpha = \infty$. To apply Lemma 2.1, for every natural number $m$ let $k(m) = m + 1$. For each $n \geq k(m)$ and each $i \leq n$ let $L_i^n$ be a partition between $A_i^n$ and $B_i^n$ in $\mathbb{I}^n$. By Lemma 2.2, there exists a continuum $C$ such that $C \subset \cap_{i=1}^{n} L_i^n$ and $C \cap A_i^n \neq \emptyset \neq C \cap B_i^n$. Let $\pi_n^C = \pi|C : C \to \mathbb{I}$ be the restriction of the projection $\pi_n$ over $C$. Then $C \cap \pi_n^{-1}(Q_\alpha) = (\pi_n^C)^{-1}(Q_\alpha) \subset \cap_{i=1}^{n} L_i^n \cap \pi_n^{-1}(Q_\alpha)$. Since $C \cap \pi_n^{-1}(Q_\alpha)$ is closed set of $\cap_{i=1}^{n} L_i^n \cap \pi_n^{-1}(Q_\alpha)$ and $(\pi_n^C)^{-1}(Q_\alpha) \notin M(\alpha)$, it follows that $\cap_{i=1}^{n} L_i^n \cap \pi_n^{-1}(Q_\alpha) \notin M(\alpha)$. Thus, it follows from Lemma 2.1 that $M(\alpha)$-trind $Y_\alpha = \infty$. This completes the proof.

(2) The second part of Theorem 1.1 is a direct consequence of the following proposition.

Proposition 2.3 Let $X$ be a separable metrizable space with $A(\alpha)$-trind $X \leq \mu_1$ and $M(\alpha)$-trind $X \leq \mu_2$. Then

$$A(\alpha) \cap M(\alpha)$$

trind $X = \begin{cases} 
\mu_1 + n(\mu_2) + 1, & \text{if } \lambda(\mu_1) = \lambda(\mu_2), \\
\mu_1, & \text{if } \lambda(\mu_1) > \lambda(\mu_2). 
\end{cases}$

Proof. The proposition can be proved by a standard transfinite induction on $\nu = \max\{\mu_1, \mu_2\}$.

Connect with Proposition 2.1, we ask the following question.
Question 2.1 Does there exist a separable metrizable space $X_\alpha$ such that $A(\alpha) \cap \mathcal{M}(\alpha)$-trind $X_\alpha > \max\{A(\alpha)$-trind $X_\alpha, \mathcal{M}(\alpha)$-trind $X_\alpha\}$ for each ordinal number $\alpha$? In particular, does there exist a separable metrizable space $X$ such that $C_0 \cap \mathcal{S}$-ind $X = 1$ and $C_0$-ind $X = \mathcal{S}$-trind $X = 0$?

Recall from M.G. Charalambous ([2]) that we call a subset $A$ of a space $X$ a Bernstein set if $|A \cap B| = |(X \setminus A) \cap B| = c$ for every uncountable Borel set $B$ of $X$, where $c$ denotes the cardinality of the continuum. It is known that every uncountable completely metrizable space $X$ has countably many disjoint Bernstein sets. We notice that $A \notin \mathcal{A}B$ for every Bernstein set $A$ of an uncountable completely metrizable space $X$.

Proof of Theorem 1.2. Let $F$ be a Bernstein set of $\mathbb{I}$. We put $X = (\{0\} \times \mathbb{I}^\infty) \cup (\bigcup_{n=1}^{\infty} \{1\} \times \pi_n^{-1}(F))$. Then, we can show that $X$ is the desired space by an argument similar to Theorem 1.1.

Connect with Theorem 1.1, we may ask the following question.

Question 2.2 For each ordinal numbers $\alpha$ and $\beta$ with $1 \leq \alpha < \omega_1$ and $0 \leq \beta < \omega_1$ do there exist separable metrizable spaces $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ which satisfy the following conditions?

1. $A(\alpha)$-trind $X_{\alpha,\beta} = \beta$,
2. $\mathcal{M}(\alpha)$-trind $Y_{\alpha,\beta} = \beta$, and
3. $\mathcal{M}(\alpha)$-trind $X_{\alpha,\beta} = A(\alpha)$-trind $Y_{\alpha,\beta} = -1$.

References


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