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Kyoto University
Truth-table reductions and minimum sizes of forcing conditions (preliminary draft)

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Abstract

This note is a refinement of our former note [KSY05] "Logarithmic truth-table reductions and minimum sizes of forcing conditions (preliminary draft)" Sürikaiseki-kenkyūsho Kökyūroku 1442 (2005), 42-47. The current note extends and corrects [KSY05]. In our former works, for a given concept of reduction, we study the following hypothesis: "For a random oracle $A$, with probability one, the degree of the one-query tautologies with respect to $A$ is strictly higher than the degree of $A$." In our former works, the following three results

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are shown: (1) the hypothesis for polynomial-time Turing reduction is equivalent to the assertion that the probabilistic complexity class \( R \) is not equal to \( \text{NP} \), (2) the hypothesis for polynomial-time truth-table reduction implies that \( P \) is not \( \text{NP} \), (3) \([\text{KSY05}]\) the hypothesis holds for \( (\log n)^{O(1)} \)-question truth-table-reduction (without polynomial-time bound). In this note, we show that if \( \epsilon \) is an enough small positive number, then we can substitute \( \epsilon \ell \) for \( (\log n)^{O(1)} \) in the statement of (3), where \( \ell \) denotes the total number of occurrences of symbols in a relativized formula. We also show the hypothesis holds for monotone truth-table reduction.

1 Preface

In our former works \([\text{Su98, Su99, Su00, Su01, Su02, Su05, KSY05}]\), by extending the work of Ambos-Spies \([\text{Am86}]\) and related works, we consider the relationships with the canonical product measure of Cantor space and complexity of one-query tautologies. A formula \( F \) of the relativized propositional calculus is called a one-query formula if \( F \) has exactly one occurrence of a query symbol. For example,

\[(q_0 \leftrightarrow \xi^3(q_1, q_2, q_3)) \Rightarrow (q_1 \Rightarrow q_0)\]

is a one-query formula, where \( q_0, q_1, q_2, q_3 \) are usual propositional variables. We assume that each propositional variable takes the value 0 or 1 (0 denotes false and 1 denotes true). And, \( \xi^3 \) in the above formula is a query symbol. For a given oracle \( A \), a function \( A^3 \) is defined as follows, where \( \lambda \) is the empty string, and the query symbol \( \xi^3 \) is interpreted as the function \( A^3 \).

\[
A^3(000) = A(\lambda), \quad A^3(001) = A(0), \quad A^3(010) = A(1), \quad A^3(011) = A(00), \\
A^3(100) = A(01), \quad A^3(101) = A(10), \quad A^3(110) = A(11), \quad A^3(111) = A(000).
\]

Thus, more informally, the following holds for each \( j = 0, 1, \cdots, 2^3 - 1 \), where the order of strings is defined as the canonical length-lexicographic order.

\[A^3( \text{the } (j+1)\text{st } 3\text{-bit string } ) = A( \text{the } (j+1)\text{st } \text{string } ).\]

For each \( n \), an \( n \)-ary Boolean function \( A^n \) is defined in the same way, and an interpretation of the query symbol \( \xi^n \) is defined in the same way. For an oracle \( A \), the concept of a tautology with respect to \( A \) is defined in a natural way. If a one-query formula \( F \) is a tautology with respect to \( A \), then we say \( F \) is a one-query tautology with respect to \( A \). The set of all one-query tautologies with respect to \( A \) is denoted by \( 1\text{TAUT}^A \).

In \([\text{Su02}]\), for a given concept \( \leq_\alpha \) of reduction, we study the following hypothesis, where \( 1\text{TAUT}^X \) denotes the set of all one-query tautologies with respect to an oracle \( X \).
One-query-jump hypothesis for $\leq_\alpha$: The class $\{X : 1\text{TAUT}^X \leq_\alpha X\}$ has measure zero.

For a given reduction $\leq_\alpha$, we denote the corresponding one-query-jump hypothesis by $[\leq_\alpha]$.

In [Su98], it is shown that the one query-jump hypothesis for p-T reduction is equivalent to "$R \neq NP.$" And, in [Su02], it is shown that the one query-jump hypothesis for p-tt reduction implies "$P \neq NP.$"

In [Su05], we show that the one query-jump hypothesis for p-btt reduction holds, where p-btt denotes polynomial-time bounded-truth-table reduction. The anonymous referee of [Su05] noticed that the one query-jump hypothesis holds for bounded-truth-table reduction without polynomial-time bound, and Kumabe independently noticed the same result. The referee's proof, which may be found in [Su05], uses some concepts of resource-bounded generic oracles in [AM97]. Kumabe's proof is more simple.

In [KSY05] we show that the one query-jump hypothesis holds for $(\log n)^{O(1)}$-question tt-reduction (without polynomial-time bound).

A Boolean formula is called monotone if every propositional connective in it is either disjunction or conjunction, and it does not have an occurrences of negation symbol. A tt-reduction is called a monotone tt-reduction if its truth table is monotone for every input. In §3, we show that the one query-jump hypothesis holds for monotone tt-reduction (without polynomial-time bound). In §4, we show the following. If $\epsilon$ is an enough small positive number then the one query-jump hypothesis holds for $\epsilon\ell$-question tt-reduction (without polynomial-time bound), where $\ell$ denotes the total number of occurrences of symbols in a relativized formula. In §5, we apply the result of §4 to minimum sizes of forcing conditions.

**Corrigendum to our former note** Theorem 4 in our former note [KSY05, p.45] has an error in its proof.

## 2 Notation

Most of our notation is the same as that of [Su02], [Su05] and [KSY05]. Almost all undefined notions may be found in these papers.

$\omega$ stands for $\{0, 1, 2, 3\cdots\}$, while N stands for $\{1, 2, 3\cdots\}$. In some textbooks, the complexity class R is denoted by RP. For the detail of the class R, see for example [BDG88].

The definition of polynomial-time truth-table reduction and its variant may be found in [LLS75].

**monotone tt-reduction**

If $A$ is tt-reducible to $B$ via $f$ and, if for any input $x$, propositional connectives used in the truth table (i.e., the $\varphi_x$ of $f(x) = (\varphi_{x}, s_{x,1}, \cdots, s_{x,k})$) is conjunction and
disjunction only, and negation is not used, then we say "A is monotone tt-reducible to B via f". If A is monotone tt-reducible to B via some function, then we say "A is monotone tt-reducible to B".

\( \ell(F) \), length of a formula

In this note, a given relativised formula \( F \), the symbol \( \ell(F) \) denotes the total number of occurrences of propositional variables \( (q_0, q_1, q_2, \cdots) \), propositional connectives \( (\land, \lor, \neg, \Rightarrow, \Leftrightarrow) \), query symbols \( (\xi^1, \xi^2, \xi^3, \cdots) \) and punctuation marks (commas, parentheses). In the case of a given string \( x \) is not (the binary code of) a relativized formula, the symbol \( \ell(x) \) denotes the binary length of \( x \).

\( \epsilon \ell \)-question tt-reduction

Suppose that \( \epsilon \) is a positive real number. If A is tt-reducible to B via \( f \) and, if for any input \( x \) it holds that

\[ k \leq \epsilon \ell(x), \]

where \( k \) is the norm of \( f \) at \( x \), then we say "A is \( \epsilon \ell \)-question tt-reducible to B via \( f \)". If A is \( \epsilon \ell \)-question tt-reducible to B via some function, then we say "A is \( \epsilon \ell \)-question tt-reducible to B".

3 Monotone truth table reduction

**Theorem 1** The Lebesgue measure of the set

\[ \{ X : 1\text{TAUT}^X \text{ is monotone tt-reducible to } X \} \]

is zero. In other words, one-query jump hypothesis holds for monotone tt-reduction (without polynomial-time bound).

4 The case where norm is linear of length of a formula

**Theorem 2** (Main Theorem) Let \( \epsilon \) be a positive real number and suppose that \( \epsilon \) is enough small. Then the Lebesgue measure of the following class is zero.

\[ \{ X : 1\text{TAUT}^X \leq_{\epsilon \ell \text{-tt}} X \} \]

In other words, the one-query-jump hypothesis holds for \( \epsilon \ell \)-question tt-reduction (without polynomial-time bound).

5 Lower bounds for forcing complexity

**Theorem 3** Let \( \epsilon \) be a positive real number and suppose that \( \epsilon \) is enough small. Let \( D_\epsilon \) be the class of all oracles \( D \) such that there exists a positive integer \( c \) (c may
depend on $D$) of the following property. For any $F \in 1\mathrm{TAUT}^D$ such that $\ell(F) \geq c$, there exists a forcing condition $S$ such that $S$ is a subfunction of $D$, $S$ forces $F$ to be a tautology and such that $|\text{dom} S| \leq \epsilon \ell(F)$, where the left-hand side denotes the cardinality of $\text{dom} S$. Then $D_{\epsilon \ell}$ has measure zero.

References


