Truth-table reductions and minimum sizes of forcing conditions (preliminary draft)

Masahiro Kumabe\(^1\), Toshio Suzuki\(^2\)*, Takeshi Yamazaki\(^3\)

1): University of the Air, 31-1, Ōoka 2, Minami-ku, Yokohama 232-0061, Japan
kumabe@u-air.ac.jp

2): Department of Mathematics and Information Sciences
Tokyo Metropolitan University,
Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan
toshio-suzuki@center.tmu.ac.jp

3): Department of Mathematics,
Tohoku University, Sendai 980-8578, Japan
yamazaki@math.tohoku.ac.jp

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Abstract

This note is a refinement of our former note [KSY05] "Logarithmic truth-table reductions and minimum sizes of forcing conditions (preliminary draft)" Sūrikaiseki-kenkyūsho Kōkyūroku 1442 (2005), 42-47. The current note extends and corrects [KSY05]. In our former works, for a given concept of reduction, we study the following hypothesis: "For a random oracle \(A\), with probability one, the degree of the one-query tautologies with respect to \(A\) is strictly higher than the degree of \(A\)." In our former works, the following three results

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are shown: (1) the hypothesis for polynomial-time Turing reduction is equivalent to the assertion that the probabilistic complexity class \( R \) is not equal to \( \text{NP} \), (2) the hypothesis for polynomial-time truth-table reduction implies that \( \text{P} \) is not \( \text{NP} \), (3) [KSY05] the hypothesis holds for \((\log n)^{O(1)}\)-question truth-table-reduction (without polynomial-time bound). In this note, we show that if \( \varepsilon \) is an enough small positive number, then we can substitute \( \varepsilon \ell \) for \((\log n)^{O(1)}\) in the statement of (3), where \( \ell \) denotes the total number of occurrences of symbols in a relativized formula. We also show the hypothesis holds for monotone truth-table reduction.

1 Preface

In our former works [Su98, Su99, Su00, Su01, Su02, Su05, KSY05], by extending the work of Ambos-Spies [Am86] and related works, we consider the relationships with the canonical product measure of Cantor space and complexity of one-query tautologies. A formula \( F \) of the relativized propositional calculus is called a one-query formula if \( F \) has exactly one occurrence of a query symbol. For example,

\[(q_0 \Leftrightarrow \xi^3(q_1, q_2, q_3)) \Rightarrow (q_1 \Rightarrow q_0)\]

is a one-query formula, where \( q_0, q_1, q_2, q_3 \) are usual propositional variables. We assume that each propositional variable takes the value 0 or 1 (0 denotes false and 1 denotes true). And, \( \xi^3 \) in the above formula is a query symbol. For a given oracle \( A \), a function \( A^3 \) is defined as follows, where \( \lambda \) is the empty string, and the query symbol \( \xi^3 \) is interpreted as the function \( A^3 \).

\[
\begin{align*}
A^3(000) &= A(\lambda), & A^3(001) &= A(0), & A^3(010) &= A(1), & A^3(011) &= A(00), \\
A^3(100) &= A(01), & A^3(101) &= A(10), & A^3(110) &= A(11), & A^3(111) &= A(000).
\end{align*}
\]

Thus, more informally, the following holds for each \( j = 0, 1, \ldots, 2^3 - 1 \), where the order of strings is defined as the canonical length-lexicographic order.

\[
A^3( \text{the } ( j + 1) \text{st } 3\text{-bit string } ) = A( \text{the } ( j + 1) \text{st } 3\text{-bit string } ).
\]

For each \( n \), an \( n \)-ary Boolean function \( A^n \) is defined in the same way, and an interpretation of the query symbol \( \xi^n \) is defined in the same way. For an oracle \( A \), the concept of a tautology with respect to \( A \) is defined in a natural way. If a one-query formula \( F \) is a tautology with respect to \( A \), then we say \( F \) is a one-query tautology with respect to \( A \). The set of all one-query tautologies with respect to \( A \) is denoted by \( \text{1TAUT}^A \).

In [Su02], for a given concept \( \leq \alpha \) of reduction, we study the following hypothesis, where \( \text{1TAUT}^X \) denotes the set of all one-query tautologies with respect to an oracle \( X \).
One-query-jump hypothesis for $\leq_\alpha$: The class $\{X : \text{1TAUT}^X \leq_\alpha X\}$ has measure zero.

For a given reduction $\leq_\alpha$, we denote the corresponding one-query-jump hypothesis by $[\leq_\alpha]$.

In [Su98], it is shown that the one query-jump hypothesis for p-T reduction is equivalent to "$R \neq \text{NP}$.”

And, in [Su02], it is shown that the one query-jump hypothesis for p-tt reduction implies "$P \neq \text{NP}$.”

In [Su05], we show that the one query-jump hypothesis for p-btt reduction holds, where p-btt denotes polynomial-time bounded-truth-table reduction. The anonymous referee of [Su05] noticed that the one query-jump hypothesis holds for bounded-truth-table reduction without polynomial-time bound, and Kumabe independently noticed the same result. The referee’s proof, which may be found in [Su05], uses some concepts of resource-bounded generic oracles in [AM97]. Kumabe’s proof is more simple.

In [KSY05] we show that the one query-jump hypothesis holds for $(\log n)^{O(1)}$-question tt-reduction (without polynomial-time bound).

A Boolean formula is called monotone if every propositional connective in it is either disjunction or conjunction, and it does not have an occurrences of negation symbol. A tt-reduction is called a monotone tt-reduction if its truth table is monotone for every input. In §3, we show that the one query-jump hypothesis holds for monotone tt-reduction (without polynomial-time bound). In §4, we show the following. If $\varepsilon$ is an enough small positive number then the one query-jump hypothesis holds for $\varepsilon\ell$-question tt-reduction (without polynomial-time bound), where $\ell$ denotes the total number of occurrences of symbols in a relativized formula. In §5, we apply the result of §4 to minimum sizes of forcing conditions.

Corrigendum to our former note: Theorem 4 in our former note [KSY05, p.45] has an error in its proof.

2 Notation

Most of our notation is the same as that of [Su02], [Su05] and [KSY05]. Almost all undefined notions may be found in these papers.

$\omega$ stands for $\{0,1,2,3\cdots\}$, while $N$ stands for $\{1,2,3\cdots\}$. In some textbooks, the complexity class $R$ is denoted by $\text{RP}$. For the detail of the class $R$, see for example [BDG88].

The definition of polynomial-time truth-table reduction and its variant may be found in [LLS75].

Monotone tt-reduction

If $A$ is tt-reducible to $B$ via $f$ and, if for any input $x$, propositional connectives used in the truth table (i.e., the $\varphi_x$ of $f(x) = (\varphi_{x_1}, \varphi_{x_2}, \cdots, \varphi_{x_k})$ is conjunction and
disjunction only, and negation is not used, then we say “A is monotone tt-reducible to B via $f$”. If $A$ is monotone tt-reducible to $B$ via some function, then we say “$A$ is monotone tt-reducible to $B$”.

$\ell(F)$, length of a formula
In this note, a given relativised formula $F$, the symbol $\ell(F)$ denotes the total number of occurrences of propositional variables ($q_0, q_1, q_2, \cdots$), propositional connectives ($\land, \lor, \neg, \Rightarrow, \Leftrightarrow$), query symbols ($\xi^1, \xi^2, \xi^3, \cdots$) and punctuation marks (commas, parentheses). In the case of a given string $x$ is not (the binary code of) a relativized formula, the symbol $\ell(x)$ denotes the binary length of $x$.

$\epsilon\ell$-question tt-reduction
Suppose that $\epsilon$ is a positive real number. If $A$ is tt-reducible to $B$ via $f$ and, if for any input $x$ it holds that

$$k \leq \epsilon \ell(x),$$

where $k$ is the norm of $f$ at $x$, then we say “$A$ is $\epsilon\ell$-question tt-reducible to $B$ via $f$”. If $A$ is $\epsilon\ell$-question tt-reducible to $B$ via some function, then we say “$A$ is $\epsilon\ell$-question tt-reducible to $B$”.

3 Monotone truth table reduction

Theorem 1 The Lebesgue measure of the set

$$\{X : 1\text{TAUT}^X \text{ is monotone tt-reducible to } X\}$$

is zero. In other words, one-query jump hypothesis holds for monotone tt-reduction (without polynomial-time bound).

4 The case where norm is linear of length of a formula

Theorem 2 (Main Theorem) Let $\epsilon$ be a positive real number and suppose that $\epsilon$ is enough small. Then the Lebesgue measure of the following class is zero.

$$\{X : 1\text{TAUT}^X \leq_{\epsilon\ell-\text{tt}} X\}$$

In other words, the one-query-jump hypothesis holds for $\epsilon\ell$-question tt-reduction (without polynomial-time bound).

5 Lower bounds for forcing complexity

Theorem 3 Let $\epsilon$ be a positive real number and suppose that $\epsilon$ is enough small. Let $D_{\epsilon t}$ be the class of all oracles $D$ such that there exists a positive integer $c$ ($c$ may
depend on $D$) of the following property. For any $F \in 1\mathrm{TAUT}^D$ such that $\ell(F) \geq c$, there exists a forcing condition $S$ such that $S$ is a subfunction of $D$, $S$ forces $F$ to be a tautology and such that $|\mathrm{dom}S| \leq \epsilon\ell(F)$, where the left-hand side denotes the cardinality of $\mathrm{dom}S$. Then $D_{\epsilon\ell}$ has measure zero.

References


