数理解析研究所講究録
1535 巻 2007 年 119-124

Order preserving operator inequalities with operator monotone functions

東京理科大・理　柳田 昌宏 (Masahiro Yanagida)
Department of Mathematical Information Science,
Tokyo University of Science

In what follows, an operator means a bounded linear operator on a Hilbert space H. An operator T is positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and strictly positive (denoted by $T > 0$) if T is positive and invertible.

A real-valued continuous function f defined on an interval $I \subseteq \mathbb{R}$ is operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for any self-adjoint operators A and B such that $\sigma(A), \sigma(B) \subseteq I$. Let $\mathbb{P}_+[a,b)$ be the set of all non-negative operator monotone functions defined on $[a,b)$, and $\mathbb{P}_+^{-1}[a,b)$ the set of increasing functions h defined on $[a,b)$ such that $h([a,b)) = [0,\infty)$ and its inverse h^{-1} is operator monotone on $[0,\infty)$. Uchiyama [11] introduces a new concept of majorization, and shows a quite interesting result named "Product theorem" and its applications to operator inequalities.

Definition ([11]). Let h be a non-decreasing function on I and k an increasing function on J. Then h is said to be majorized by k, in symbols $h \preceq k$, if $J \subseteq I$ and the composite $h \circ k^{-1}$ is operator monotone on $k(J)$.

Product theorem ([11]). *Suppose $-\infty < a < b \leq \infty$. Then*

$$\mathbb{P}_+[a,b) \cdot \mathbb{P}_+^{-1}[a,b) \subseteq \mathbb{P}_+^{-1}[a,b), \quad \mathbb{P}_+^{-1}[a,b) \cdot \mathbb{P}_+^{-1}[a,b) \subseteq \mathbb{P}_+^{-1}[a,b).$$

Further, let $h_i \in \mathbb{P}_+^{-1}[a,b)$ for $1 \leq i \leq m$, and let g_j be a finite product of functions in $\mathbb{P}_+[a,b)$ for $1 \leq j \leq n$. Then for $\psi_i, \phi_j \in \mathbb{P}_+[0,\infty)$

$$\prod_{i=1}^{m} h_i(t) \prod_{j=1}^{n} g_j(t) \in \mathbb{P}_+^{-1}[a,b), \quad \prod_{i=1}^{m} \psi_i(h_i(t)) \prod_{j=1}^{n} \phi_j(g_j(t)) \preceq \prod_{i=1}^{m} h_i(t) \prod_{j=1}^{n} g_j(t).$$

Proposition A ([11]). *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g be a finite product of functions in $\mathbb{P}_+[0,\infty)$. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$*

$$A \geq B \geq 0 \Longrightarrow \begin{cases} \varphi(g(B)^{\frac{1}{2}} h(A) g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}} \tilde{h}(A) g(B)^{\frac{1}{2}}, \\ \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}} \tilde{h}(B) g(A)^{\frac{1}{2}}. \end{cases}$$

Theorem B ([11]). *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g_n be a finite product of functions in $\mathbb{P}_+[0,\infty)$ for each n, and let the sequence $\{g_n\}$ converge pointwise to g. Suppose $g \neq 0$ and $g(0+) = g(0)$. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$*

$$A \geq B \geq 0 \Longrightarrow \begin{cases} \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}, \\ \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}. \end{cases}$$

Proposition A and Theorem B are generalizations of the following result.

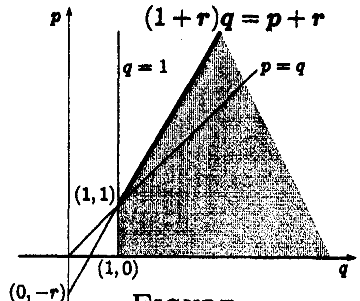
Theorem F (Furuta inequality [4]).
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

We remark that Löwner-Heinz theorem "$A \geq B \geq 0 \Longrightarrow A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0,1]$" is the case $r = 0$ of Theorem F. Other proofs are given in [1][7] and also an elementary one-page proof in [5]. It is shown in [9] that the domain of p, q and r in Theorem F is the best possible for the inequalities (i) and (ii) to hold under the assumption $A \geq B$.

We obtain extensions of Proposition A and Theorem B by weakening their hypotheses from $A \geq B$ to inequalities implied by it.

Proposition 1. *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\hat{h}$ and $\tilde{h}$ be non-negative non-decreasing functions on $[0,\infty)$ such that and $\tilde{h} \preceq h$. Let $g_j(t) = \prod_{i=1}^{j} f_i(t)$ where $f_i \in \mathbb{P}_+[0,\infty)$ such that $f_n(t) \preceq \hat{h}(t)g_{n-1}(t)$. Then for the functions ψ_j and φ_j defined by*

$$\psi_j(h(t)g_j(t)) = \hat{h}(t)g_j(t) \quad \textit{and} \quad \varphi_j(h(t)g_j(t)) = \tilde{h}(t)g_j(t),$$

if $A, B \geq 0$ satisfy

$$\psi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}}) \geq \hat{h}(B)g_{n-1}(B),$$

then

$$\varphi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq f_n(B)^{\frac{1}{2}}\varphi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}})f_n(B)^{\frac{1}{2}}$$

holds. Particularly, $\psi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq \hat{h}(B)g_n(B)$ holds in case $\hat{h} \preceq h$.

Theorem 2. *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g be a finite product of functions in $\mathbb{P}_+[0,\infty)$ and γ_n a finite product of functions in $\{f \in \mathbb{P}_+[0,\infty) \mid f(t) \preceq \tilde{h}(t)g(t)\}$ for each n, and let the sequence*

$\{g(t)\gamma_n(t)\}$ converge pointwise to $\bar{g}(t)$. Suppose $\bar{g} \neq 0$ and $\bar{g}(0+) = \bar{g}(0)$. Then for the functions ψ and $\bar{\psi}$ defined by

$$\psi(h(t)g(t)) = \tilde{h}(t)g(t) \quad \textit{and} \quad \bar{\psi}(h(t)\bar{g}(t)) = \tilde{h}(t)\bar{g}(t),$$

if $A, B \geq 0$ satisfy

$$\psi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq \tilde{h}(B)g(B),$$

then

$$g(B)^{\frac{1}{2}}\bar{\psi}(\bar{g}(B)^{\frac{1}{2}}h(A)\bar{g}(B)^{\frac{1}{2}})g(B)^{\frac{1}{2}} \geq \bar{g}(B)^{\frac{1}{2}}\psi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})\bar{g}(B)^{\frac{1}{2}}$$

and hence $\bar{\psi}(\bar{g}(B)^{\frac{1}{2}}h(A)\bar{g}(B)^{\frac{1}{2}}) \geq \tilde{h}(B)\bar{g}(B)$ hold.

Proof of Proposition 1. Define the function ϕ as $\phi(\hat{h}(t)g_{n-1}(t)) = f_n(t)$, then ϕ is operator monotone by the assumption, so that

$$\phi(\psi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}})) \geq \phi(\hat{h}(B)g_{n-1}(B)) = f_n(B),$$

and there exists a contraction X such that

$$X^*\phi(\psi_{n-1}(D))^{\frac{1}{2}} = \phi(\psi_{n-1}(D))^{\frac{1}{2}}X = f_n(B)^{\frac{1}{2}}$$

by Douglas' theorem [3], where $D = g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}}$. Hence we have

$$\begin{aligned}
\varphi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) &= \varphi_n(f_n(B)^{\frac{1}{2}}g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}}f_n(B)^{\frac{1}{2}}) \\
&= \varphi_n(X^*D\phi(\psi_{n-1}(D))X) \\
&\geq X^*\varphi_n(D\phi(\psi_{n-1}(D)))X \qquad \text{by Hansen's inequality [6]} \\
&= X^*\varphi_{n-1}(D)\phi(\psi_{n-1}(D))X \\
&= f_n(B)^{\frac{1}{2}}\varphi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}})f_n(B)^{\frac{1}{2}}.
\end{aligned}$$

□

In addition, the inequalities in Theorem F are known to be valid in case the parameters are negative under certain conditions.

Theorem C ([2][8][10][12]).

(i) $A \geq B > 0 \Longrightarrow (B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^{\frac{1-t}{p-t}} \geq B^{1-t}$ *for* $1 \geq p > t \geq 0$ *and* $p \geq \frac{1}{2}$.

(ii) $A \geq B > 0 \Longrightarrow (B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^{\frac{-t}{p-t}} \geq B^{-t}$ *for* $1 \geq t > p \geq 0$ *and* $\frac{1}{2} \geq p$.

(iii) $A \geq B > 0 \Longrightarrow (B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^{\frac{2p-t}{p-t}} \geq B^{2p-t}$ *for* $\frac{1}{2} \geq p > t \geq 0$.

(iv) $A \geq B > 0 \Longrightarrow (B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^{\frac{2p-1-t}{p-t}} \geq B^{2p-1-t}$ *for* $1 \geq t > p \geq \frac{1}{2}$.

We also obtain a generalization of (i) and (iii) of Theorem C in similar form to results mentioned before.

Theorem 3. *Let $g(x), h(x), \tilde{h}(x) \in \mathbb{P}_+[0,\infty)$ such that $\frac{h(x)}{g(x)} \in \mathbb{P}_+[0,\infty)$ and $\frac{h(x)^2}{\tilde{h}(x)}$ is a finite product of functions in $\mathbb{P}_+[0,\infty) \cup \mathbb{P}_+^{-1}[0,\infty)$. Then for the function φ defined by*

$$\varphi\left(\frac{h(x)}{g(x)}\right) = \frac{\tilde{h}(x)}{g(x)},$$

if there exists an integer $m \geq 0$ such that $\frac{\varphi(x)}{x^m} \in \mathbb{P}_+[0,\infty)$, then

$$A \geq B > 0 \Longrightarrow \varphi(g(B)^{\frac{-1}{2}}h(A)g(B)^{\frac{-1}{2}}) \geq g(B)^{\frac{-1}{2}}\tilde{h}(A)g(B)^{\frac{-1}{2}} \geq \tilde{h}(B)g(B)^{-1}.$$

Proof. It turns out by results in [11] that

$$\tilde{h}(x)\left(\frac{g(x)}{h(x)}\right)^{\alpha} \in \mathbb{P}_+[0,\infty) \quad \text{for } 0 \leq \alpha \leq m \tag{1}$$

and

$$\frac{g(x)^2}{\tilde{h}(x)}\left(\frac{h(x)}{g(x)}\right)^{\alpha} \in \mathbb{P}_+[0,\infty) \quad \text{for } 2 \leq \alpha \leq m+1. \tag{2}$$

Put $D = g(B)^{\frac{-1}{2}}h(A)g(B)^{\frac{-1}{2}}$. In case $\frac{\varphi(x)}{x^{2n}} \in \mathbb{P}_+[0,\infty)$, we have

$$\begin{aligned}
\varphi(D) &= D^n \frac{\varphi(g(B)^{\frac{-1}{2}}h(A)g(B)^{\frac{-1}{2}})}{\left(g(B)^{\frac{-1}{2}}h(A)g(B)^{\frac{-1}{2}}\right)^{2n}} D^n \\
&\geq D^n \frac{\varphi(g(B)^{\frac{-1}{2}}h(B)g(B)^{\frac{-1}{2}})}{\left(g(B)^{\frac{-1}{2}}h(B)g(B)^{\frac{-1}{2}}\right)^{2n}} D^n \qquad \text{since } h(x), \frac{\varphi(x)}{x^{2n}} \in \mathbb{P}_+[0,\infty) \\
&= D^n g(B)^{\frac{1}{2}} \frac{\tilde{h}(B)}{g(B)^2}\left(\frac{g(B)}{h(B)}\right)^{2n} g(B)^{\frac{1}{2}} D^n \\
&\geq D^n g(B)^{\frac{1}{2}} \frac{\tilde{h}(A)}{g(A)^2}\left(\frac{g(A)}{h(A)}\right)^{2n} g(B)^{\frac{1}{2}} D^n \qquad \text{by (2) for } \alpha = 2n \\
&= D^{n-1} g(B)^{\frac{-1}{2}}\, \tilde{h}(A)\left(\frac{g(A)}{h(A)}\right)^{2(n-1)} g(B)^{\frac{-1}{2}} D^{n-1}. \\
&\geq D^{n-1} g(B)^{\frac{-1}{2}}\, \tilde{h}(B)\left(\frac{g(B)}{h(B)}\right)^{2(n-1)} g(B)^{\frac{-1}{2}} D^{n-1} \qquad \text{by (1) for } \alpha = 2n-2 \\
&= D^{n-1} g(B)^{\frac{1}{2}} \frac{\tilde{h}(B)}{g(B)^2}\left(\frac{g(B)}{h(B)}\right)^{2(n-1)} g(B)^{\frac{1}{2}} D^{n-1} \\
&\geq D^{n-1} g(B)^{\frac{1}{2}} \frac{\tilde{h}(A)}{g(A)^2}\left(\frac{g(A)}{h(A)}\right)^{2(n-1)} g(B)^{\frac{1}{2}} D^{n-1} \qquad \text{by (2) for } \alpha = 2n-2 \\
&= D^{n-2} g(B)^{\frac{-1}{2}}\, \tilde{h}(A)\left(\frac{g(A)}{h(A)}\right)^{2(n-2)} g(B)^{\frac{-1}{2}} D^{n-2} \\
&\ \vdots
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&\geq D^2 g(B)^{\frac{-1}{2}}\ \tilde{h}(A) \left(\frac{g(A)}{h(A)}\right)^4 g(B)^{\frac{-1}{2}} D^2 \\
&\vdots \\
&\geq D g(B)^{\frac{-1}{2}}\ \tilde{h}(A) \left(\frac{g(A)}{h(A)}\right)^2 g(B)^{\frac{-1}{2}} D \\
&\vdots \\
&\geq g(B)^{\frac{-1}{2}} \tilde{h}(A) g(B)^{\frac{-1}{2}} \\
&\geq g(B)^{\frac{-1}{2}} \tilde{h}(B) g(B)^{\frac{-1}{2}} \\
&= \tilde{h}(B) g(B)^{-1}.
\end{aligned}
$$

In case $\frac{\varphi(x)}{x^{2n+1}} \in \mathbb{P}_+[0,\infty)$, we have

$$
\begin{aligned}
\varphi(D) &= D^n \frac{\varphi(g(B)^{\frac{-1}{2}} h(A) g(B)^{\frac{-1}{2}})}{\left(g(B)^{\frac{-1}{2}} h(A) g(B)^{\frac{-1}{2}}\right)^{2n}} D^n \\
&\geq D^n g(B)^{\frac{-1}{2}} g(A)^{\frac{1}{2}} \frac{\varphi(g(A)^{\frac{-1}{2}} h(A) g(A)^{\frac{-1}{2}})}{\left(g(A)^{\frac{-1}{2}} h(A) g(A)^{\frac{-1}{2}}\right)^{2n}} g(A)^{\frac{1}{2}} g(B)^{\frac{-1}{2}} D^n \\
&\qquad\qquad \text{by Hansen's inequality [6]} \\
&= D^n g(B)^{\frac{-1}{2}}\ \tilde{h}(A) \left(\frac{g(A)}{h(A)}\right)^{2n} g(B)^{\frac{-1}{2}} D^n,
\end{aligned}
$$

and the rest of the proof is as same as the former case. □

References

[1] M. Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory **23** (1990), 67–72.

[2] M. Fujii, T. Furuta and E. Kamei, *Complements to the Furuta inequality*, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), 239–242.

[3] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.

[4] T. Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with* $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc. **101** (1987), 85–88.

[5] T. Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci. **65** (1989), 126.

[6] F. Hansen, *An operator inequality*, Math. Ann. **246** (1979/80), 249–250.

[7] E. Kamei, *A satellite to Furuta's inequality*, Math. Japon. **33** (1988), 883–886.

[8] E. Kamei, *Complements to the Furuta inequality, II*, Math. Japon. **45** (1997), 15–23.

[9] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. **124** (1996), 141–146.

[10] K. Tanahashi, *The Furuta inequality with negative powers*, Proc. Amer. Math. Soc. **127** (1999), 1683–1692.

[11] M. Uchiyama, *A new majorization between functions, polynomials, and operator inequalities*, J. Funct. Anal. **231** (2006), 221–244.

[12] T. Yoshino, *Introduction to operator theory*, Pitman Research Notes in Mathematics Series, vol. 300, Longman Scientific & Technical, Harlow, 1993.