<table>
<thead>
<tr>
<th>Title</th>
<th>Matrix functions and unitarily invariant norms (Recent Developments in Theory of Operators and Its Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Uchiyama, Mitsuru</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2007), 1535: 112-118</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58991">http://hdl.handle.net/2433/58991</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Matrix functions and unitarily invariant norms
(行列関数とユニタリ不変ノルム)

Mitsuru Uchiyama (内山 充)
Shimane University ( 島根大学総合理工学部)

1 Introduction

The eigenvalues of an \( n \times n \) Hermitian matrix \( H \) are denoted by \( \lambda_i(H) \) \((i = 1, 2, \cdots, n)\) and arranged in increasing order, that is,
\[
\lambda_1(H) \leq \lambda_2(H) \leq \cdots \leq \lambda_n(H).
\]

\[
\sigma_{(k)}(H) := \sum_{i=1}^{k} \lambda_i(H),
\]
\[
\sigma^{(k)}(H) := \sum_{i=n-k+1}^{n} \lambda_i(H).
\]

A norm \( \| \cdot \| \) on the \( n \times n \) matrices is called a unitarily invariant norm if
\[
\|UXV\| = \|X\|
\]
for all \( X \) and for all unitary matrices \( U \) and \( V \).

Example 1.1 The following are typical unitarily invariant norm:
The operator norm \( \|X\| \),
Schatten \( p \)-norms \( \|X\|_p := \{\sum_{i=1}^{n} \lambda_i(|X|^p)\}^{1/p}, \ (p \geq 1) \),
Ky Fan \( k \)-norms \( \|X\|_{(k)} := \sigma^{(k)}(|X|), \ (k = 1, 2, \cdots, n) \).

The following useful result is due to Ky Fan:
\[
\|X\|_{(k)} \leq \|Y\|_{(k)} \ (\forall k) \text{ implies } \|X\| \leq \|Y\| \text{ for every unitarily invariant norm } \| \cdot \|.
\]

We begin with a simple fact:
Proposition 1.1 Let \( f(t) \) be a concave function on an interval \( I \), and let \( A, B \) be \( n \times n \) Hermitian matrices with the spectra in \( I \). Then for \( R, S \) such that \( R^*R + S^*S = 1 \) and for \( k = 1, 2, \ldots, n \)

\[
\sigma_{(k)}(f(R^*AR + S^*BS)) \geqq \sigma_{(k)}(R^*f(A)R + S^*f(B)S).
\]

Moreover, if \( f(t) \) is monotone, then

\[
\lambda_k(f(R^*AR + S^*BS)) \geqq \lambda_k(R^*f(A)R + S^*f(B)S).
\]

Proof. Let \( \{\lambda_i\}_{i=1}^n \) be the eigenvalues of \( X^*AX + Y^*BY \) so that \( f(\lambda_1) \leqq f(\lambda_2) \leqq \cdots \leqq f(\lambda_n) \), and let \( \{e_i\} \) be the corresponding eigenvectors. Then the left side of (??) equals \( f(\lambda_1) + \cdots + f(\lambda_k) \). By the concavity of \( f \), we have

\[
\sum_{i=1}^k \langle (X^*f(A)X + Y^*f(B)Y)e_i, e_i \rangle = \sum_{i=1}^k \left( ||Xe_i||^2 \langle f(A) \frac{Xe_i}{||Xe_i||}, \frac{Xe_i}{||Xe_i||} \rangle + ||Ye_i||^2 \langle f(B) \frac{Ye_i}{||Ye_i||}, \frac{Ye_i}{||Ye_i||} \rangle \right)
\]

\[
\leqq \sum_{i=1}^k \left( ||Xe_i||^2 f(\langle A \frac{Xe_i}{||Xe_i||}, \frac{Xe_i}{||Xe_i||} \rangle) + ||Ye_i||^2 f(\langle B \frac{Ye_i}{||Ye_i||}, \frac{Ye_i}{||Ye_i||} \rangle) \right)
\]

\[
\leqq \sum_{i=1}^k f(\langle (X^*AX + Y^*BY)e_i, e_i \rangle) = \sum_{i=1}^k f(\lambda_i).
\]

Thus, by the min-max theorem, we get the first inequality.

If \( f(t) \) is increasing, we can arrange eigenvalues \( \{\lambda_i\}_{i=1}^n \) as \( \lambda_i \leqq \lambda_{i+1} \) and \( f(\lambda_i) \leqq f(\lambda_{i+1}) \). For any unit vector \( x \) that is a linear combination of \( e_1, \cdots, e_k \)

\[
\langle (X^*f(A)X + Y^*f(B)Y)x, x \rangle \leqq f(\langle (X^*AX + Y^*BY)x, x \rangle)
\]

\[
\leqq f(\lambda_k),
\]

for \( \langle (X^*AX + Y^*BY)x, x \rangle \leqq \lambda_k \). From this, the second inequality follows. It can be similarly shown even if \( f(t) \) is decreasing. \qed
Corollary 1.2 Let $g(t)$ be a convex function on $I$. Then for $1 \leq k \leq n$ and for all $R, S$ such that $R^*R + S^*S = 1$

$$\sigma^{(k)}(g(R^*AR + S^*BS)) \leq \sigma^{(k)}(R^*g(A)R + S^*g(B)S).$$

Moreover, if $g(t)$ is monotone,

$$\lambda_k(g(R^*AR + S^*BS)) \leq \lambda_k(R^*g(A)R + S^*g(B)S).$$

Remark: The case $k = n$ of the first inequality in the corollary had been shown by Brown-Kosaki, Hansen-Pedersen.
The second inequality was shown by Bourin.
The case where $R, S$ are scalars are due to Aujla- Silva.

2 Essential results

It is well known that $trBA^2B = trAB^2A$. But it is difficult to estimate

$$trCBA^2BC - trCAB^2AC.$$

J. C. Bourin [5] got a nice result to do it. The next special case follows from it, however we can give a simple and direct proof.

Lemma 2.1 Let $A \geq 0$ and $B \geq 0$, and let $Q$ be an orthogonal projection such that $QB = BQ$. If

$$\inf\{\|Bx\| : Qx = x, \|x\| = 1\} \geq \sup\{\|Bx\| : (1-Q)x = x, \|x\| = 1\},$$

then

$$trQBA^2BQ \geq trQAB^2AQ,$$

$$tr(1-Q)AB^2A(1-Q) \geq tr(1-Q)BA^2B(1-Q).$$
Corollary 2.2 Let $A \geqq 0$ and $B \geqq 0$, and let $Q$ be an orthogonal projection such that $QB = BQ$.

Suppose the strict inequality:

$$\inf\{\|Bx\| : Qx = x, \|x\| = 1\} \, > \, \sup\{\|Bx\| : (1 - Q)x = x, \|x\| = 1\}.$$  

Then

$$tr QBA^2BQ = tr QAB^2AQ \iff QA = AQ.$$  

Proposition 2.3 Let $h(t)$ be a continuous function on $[0, \infty)$.

If $h(t)$ is decreasing and $th(t)$ is increasing, or

if $h(t)$ is increasing and $th(t)$ is decreasing,

then for $A, B \geqq 0$ and for every unitarily invariant norm $\| \cdot \|$

$$\|A^{1/2}h(A+B)A^{1/2} + B^{1/2}h(A+B)B^{1/2}\| \geqq \|(A+B)h(A+B)\|.$$  

Corollary 2.4 Let $A$ and $B$ be non-negative Hermitian matrices such that $A + B$ is invertible. Then the following are equivalent:

(i) $H := A^{1/2}(A+B)^{-1}A^{1/2} + B^{1/2}(A+B)^{-1}B^{1/2} \leqq 1,$

(ii) $H = 1,$

(iii) $AB = BA.$

Remark: We give one fact relevant to the above (i).

$$(A + B)^{-1/2}A^{1/2}(A+B)^{-1/2} + (A + B)^{-1/2}B^{1/2}B^{1/2}(A+B)^{-1/2} = 1.$$  

3 Applications

We can easily give another proof of
**Theorem A.** (Ando and Zhan)

Let \( f(t) \geq 0 \) be an operator monotone function on \([0, \infty)\) such that \( f(t) \) is continuous at \( t = 0 \). Then

\[
\|f(A + B)\| \leq \|f(A) + f(B)\| \tag{1}
\]

for every unitarily invariant norm \( \| \cdot \| \) and for all \( A, B \geq 0 \).

**Proof** We may assume that \( A + B \) is invertible. Then, since \((A + B)^{-1/2}A^{1/2}\) is contractive, by Hansen-Pedersen’s inequality [7] we have

\[
\varphi(A) = \varphi(A^{1/2}(A + B)^{-1/2}(A + B)(A + B)^{-1/2}A^{1/2}) \\
\geq A^{1/2}(A + B)^{-1/2}\varphi(A + B)(A + B)^{-1/2}A^{1/2}, \\
\varphi(B) \geq B^{1/2}(A + B)^{-1/2}\varphi(A + B)(A + B)^{-1/2}B^{1/2}.
\]

Since \( \varphi(t) \) is increasing and \( \varphi(t)/t \) is decreasing, by Proposition 2.3 we get

\[
\sigma^{(k)}(\varphi(A) + \varphi(B)) \geq \sigma^{(k)}(A^{1/2}(\varphi/t)(A + B)A^{1/2} + B^{1/2}(\varphi/t)(A + B)B^{1/2}) \\
\geq \sigma^{(k)}(\varphi(A + B)) \quad (1 \leq k \leq n).
\]

Also we can get the following generalization of (1),

For \( A_i \geq 0 \quad (1 \leq i \leq k) \)

\[
\|f(\sum_{i=1}^{k} A_i)\| \leq \|\sum_{i=1}^{k} f(A_i)\|
\]

Let \( f(t) \) be a non-negative concave function on \( 0 \leq t < \infty \). The following is known:
Theorem B (Rotfel'd [9]) (see also [10, 6, 4, 5])

For \(A, B \geq 0\)

\[
\|f(A + B)\|_1 \leq \|f(A)\|_1 + \|f(B)\|_1.
\]

We will extend this to every unitarily invariant norm.

Theorem 3.1 ([12])

\[
\|f(|X + Y|)\| \leq \|f(|X|)\| + \|f(|Y|)\| \quad (\forall X, Y).
\]

for every unitarily invariant norm \(\cdot\|\cdot\|\).

Now we can slightly improve this as follows:

Theorem 3.2 Let \(f\) be a non-negative (not necessarily continuous) concave function defined on \([0, \infty)\), and let \(\{X_i\}(i = 0, 1, \cdots, k)\) be a finite set of matrices. Then there are unitary matrices \(U_i(i = 1, \cdots, k)\) such that the inequality

\[
\|f(|X_0 + X_1 + \cdots + X_k|)\| \leq \|f(|X_0|) + U_1^*f(|X_1|)U_1 + \cdots + U_k^*f(|X_k|)U_k\|
\]

holds for every unitarily invariant norm.

References


