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Matrix functions and unitarily invariant norms
(行列関数とユニタリ不変ノルム)

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1 Introduction

The eigenvalues of an \( n \times n \) Hermitian matrix \( H \) are denoted by \( \lambda_i(H) \) \((i = 1, 2, \cdots, n)\) and arranged in increasing order, that is, \( \lambda_1(H) \leq \lambda_2(H) \leq \cdots \leq \lambda_n(H) \).

\[
\sigma_{(k)}(H) := \sum_{i=1}^{k} \lambda_i(H), \\
\sigma^{(k)}(H) := \sum_{i=n-k+1}^{n} \lambda_i(H).
\]

A norm \( \| \cdot \| \) on the \( n \times n \) matrices is called a unitarily invariant norm if

\[
\|UXV\| = \|X\|
\]

for all \( X \) and for all unitary matrices \( U \) and \( V \).

Example 1.1 The following are typical unitarily invariant norm:

The operator norm \( \|X\| \),

Schatten p-norms \( \|X\|_p := \{\sum_{i=1}^{n} \lambda_i(|X|^p)\}^{1/p}, \ (p \geq 1) \),

Ky Fan k-norms \( \|X\|_{(k)} := \sigma^{(k)}(|X|) \), \((k = 1, 2, \cdots, n)\).

The following useful result is due to Ky Fan:

\( \|X\|_{(k)} \leq \|Y\|_{(k)} \) \((\forall k)\) implies \( \|X\| \leq \|Y\| \) for every unitarily invariant norm \( \| \cdot \| \).

We begin with a simple fact:
Proposition 1.1 Let \( f(t) \) be a concave function on an interval \( I \), and let \( A, B \) be \( n \times n \) Hermitian matrices with the spectra in \( I \). Then for \( R, S \) such that \( R^*R + S^*S = 1 \) and for \( k = 1, 2, \ldots, n \)

\[
\sigma_{(k)}(f(R^*AR + S^*BS)) \geqq \sigma_{(k)}(R^*f(A)R + S^*f(B)S).
\]

Moreover, if \( f(t) \) is monotone, then

\[
\lambda_k(f(R^*AR + S^*BS)) \geqq \lambda_k(R^*f(A)R + S^*f(B)S).
\]

Proof. Let \( \{\lambda_i\}_{i=1}^n \) be the eigenvalues of \( X^*AX + Y^*BY \) so that \( f(\lambda_1) \leqq f(\lambda_2) \leqq \cdots \leqq f(\lambda_n) \), and let \( \{e_i\} \) be the corresponding eigenvectors. Then the left side of \( (??) \) equals \( f(\lambda_1) + \cdots + f(\lambda_k) \). By the concavity of \( f \), we have

\[
\sum_{i=1}^k \langle (X^*f(A)X + Y^*f(B)Y)e_i, e_i \rangle \\
= \sum_{i=1}^k \{||Xe_i||^2(f(A) \frac{Xe_i}{||Xe_i||}, \frac{Xe_i}{||Xe_i||}) + ||Ye_i||^2(f(B) \frac{Ye_i}{||Ye_i||}, \frac{Ye_i}{||Ye_i||})\} \\
\leqq \sum_{i=1}^k \{||Xe_i||^2f(\langle A \frac{Xe_i}{||Xe_i||}, \frac{Xe_i}{||Xe_i||}\rangle) + ||Ye_i||^2f(\langle B \frac{Ye_i}{||Ye_i||}, \frac{Ye_i}{||Ye_i||}\rangle)\} \\
\leqq \sum_{i=1}^k f(\langle (X^*AX + Y^*BY)e_i, e_i \rangle) = \sum_{i=1}^k f(\lambda_i).
\]

Thus, by the min-max theorem, we get the first inequality.

If \( f(t) \) is increasing, we can arrange eigenvalues \( \{\lambda_i\}_{i=1}^n \) as \( \lambda_i \leqq \lambda_{i+1} \) and \( f(\lambda_i) \leqq f(\lambda_{i+1}) \). For any unit vector \( \mathbf{x} \) that is a linear combination of \( e_1, \ldots, e_k \)

\[
\langle (X^*f(A)X + Y^*f(B)Y)x, x \rangle \leqq f(\langle (X^*AX + Y^*BY)x, x \rangle) \\
\leqq f(\lambda_k),
\]

for \( \langle (X^*AX + Y^*BY)x, x \rangle \leqq \lambda_k \). From this, the second inequality follows. It can be similarly shown even if \( f(t) \) is decreasing. \( \square \)
Corollary 1.2 Let $g(t)$ be a convex function on $I$. Then for $1 \leq k \leq n$ and for all $R, S$ such that $R^* R + S^* S = 1$

$$\sigma^{(k)}(g(R^* AR + S^* BS)) \leq \sigma^{(k)}(R^* g(A) R + S^* g(B) S).$$

Moreover, if $g(t)$ is monotone,

$$\lambda_k(g(R^* AR + S^* BS)) \leq \lambda_k(R^* g(A) R + S^* g(B) S).$$

Remark: The case $k = n$ of the first inequality in the corollary had been shown by Brown-Kosaki, Hansen-Pedersen.

The second inequality was shown by Bourin.

The case where $R, S$ are scalars are due to Aujla-Silva.

2 Essential results

It is well known that $tr BA^2 B = tr AB^2 A$. But it is difficult to estimate

$$trCBA^2 BC - trCAB^2 AC.$$  

J. C. Bourin [5] got a nice result to do it. The next special case follows from it, however we can give a simple and direct proof.

Lemma 2.1 Let $A \geq 0$ and $B \geq 0$, and let $Q$ be an orthogonal projection such that $QB = BQ$. If

$$\inf\{\|Bx\| : Qx = x, \|x\| = 1\} \geq \sup\{\|Bx\| : (1-Q)x = x, \|x\| = 1\},$$

then

$$tr QBA^2 BQ \geq tr QAB^2 AQ,$$

$$tr (1-Q)AB^2 A(1-Q) \geq tr (1-Q)BA^2 B(1-Q).$$
Corollary 2.2 Let $A \geqq 0$ and $B \geqq 0$, and let $Q$ be an orthogonal projection such that $QB = BQ$.

Suppose the strict inequality:

$$\inf\{\|Bx\| : Qx = x, \|x\| = 1\} > \sup\{\|Bx\| : (1-Q)x = x, \|x\| = 1\}.$$ 

Then

$$\text{tr} QBA^2BQ = \text{tr} QAB^2AQ \iff QA = AQ.$$ 

Proposition 2.3 Let $h(t)$ be a continuous function on $[0, \infty)$.

If $h(t)$ is decreasing and $th(t)$ is increasing, or

if $h(t)$ is increasing and $th(t)$ is decreasing, then for $A, B \geqq 0$ and for every unitarily invariant norm $\| \cdot \|$,

$$\|A^{1/2}h(A+B)A^{1/2} + B^{1/2}h(A+B)B^{1/2}\| \geqq \|(A+B)h(A+B)\|.$$ 

Corollary 2.4 Let $A$ and $B$ be non-negative Hermitian matrices such that $A + B$ is invertible. Then the following are equivalent:

(i) $H := A^{1/2}(A+B)^{-1}A^{1/2} + B^{1/2}(A+B)^{-1}B^{1/2} \leqq 1$,

(ii) $H = 1$,

(iii) $AB = BA$.

Remark: We give one fact relevant to the above (i).

$$(A + B)^{-1/2}A^{1/2}(A + B)^{-1/2} + (A + B)^{-1/2}B^{1/2}B^{1/2}(A + B)^{-1/2} = 1.$$ 

3 Applications

We can easily give another proof of
Theorem A. (Ando and Zhan)
Let $f(t) \geqq 0$ be an operator monotone function on $[0, \infty)$ such that $f(t)$ is continuous at $t = 0$. Then

$$\|f(A + B)\| \leqq \|f(A) + f(B)\|$$

for every unitarily invariant norm $\| \cdot \|$ and for all $A, B \geqq 0$.

Proof We may assume that $A + B$ is invertible. Then, since $(A + B)^{-1/2}A^{1/2}$ is contractive, by Hansen-Pedersen's inequality [7] we have

$$\varphi(A) = \varphi(A^{1/2}(A + B)^{-1/2}(A + B)(A + B)^{-1/2}A^{1/2})$$

$$\geqq A^{1/2}(A + B)^{-1/2}\varphi(A + B)(A + B)^{-1/2}A^{1/2},$$

$$\varphi(B) \geqq B^{1/2}(A + B)^{-1/2}\varphi(A + B)(A + B)^{-1/2}B^{1/2}.$$  

Since $\varphi(t)$ is increasing and $\varphi(t)/t$ is decreasing, by Proposition 2.3 we get

$$\sigma^{(k)}(\varphi(A) + \varphi(B)) \geqq \sigma^{(k)}(A^{1/2}(\varphi/t)(A + B)A^{1/2} + B^{1/2}(\varphi/t)(A + B)B^{1/2})$$

$$\geqq \sigma^{(k)}(\varphi(A + B)) \quad (1 \leqq k \leqq n).$$

$\square$

Also we can get the following generalization of (1),

For $A_i \geqq 0 \quad (1 \leqq i \leqq k)$

$$\|f(\sum_{i=1}^{k} A_i)\| \leqq \| \sum_{i=1}^{k} f(A_i)\|$$

Let $f(t)$ be a non-negative concave function on $0 \leqq t < \infty$. The following is known:
Theorem B (Rotfel’d [9]) (see also [10, 6, 4, 5])

For $A, B \geq 0$

$$\|f(A + B)\|_1 \leq \|f(A)\|_1 + \|f(B)\|_1.$$  

We will extend this to every unitarily invariant norm.

Theorem 3.1 ([12])

$$\|f(|X + Y|)\| \leq \|f(|X|)\| + \|f(|Y|)\| \ (\forall X, Y).$$

for every unitarily invariant norm $\| \cdot \|$.

Now we can slightly improve this as follows:

Theorem 3.2 Let $f$ be a non-negative (not necessarily continuous) concave function defined on $[0, \infty)$, and let $\{X_i\}(i = 0, 1, \cdots, k)$ be a finite set of matrices. Then there are unitary matrices $U_i(i = 1, \cdots, k)$ such that the inequality

$$\|f(|X_0 + X_1 + \cdots + X_k|)\| \leq$$

$$\|f(|X_0|) + U_1^* f(|X_1|)U_1 + \cdots + U_k^* f(|X_k|)U_k\|$$

holds for every unitarily invariant norm.

References


