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# Some results on Tsallis entropies in quantum system*

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**Abstract.** In this survey, we review some theorems, properties and applications of Tsallis entropies in classical system without proofs. See our previous papers [21, 15, 16, 17, 22, 23, 53] for the proofs and details.

**Keywords :** Tsallis entropy, Tsallis relative entropy, uniqueness theorem, trace inequality, operator inequality and entanglement

## 1 Tsallis entropies in quantum system

We denote $e_{\lambda}^{x}\equiv(1+\lambda x)^{\frac{1}{\lambda}}$ and its inverse function $\ln_{\lambda}x\equiv\frac{x^{\lambda}-1}{\lambda}$, for $\lambda\in(0,1]$ and $x\geq 0$. The functions $e_{\lambda}^{x}$ and $\ln_{\lambda}x$ uniformly converge to $e^{x}$ and $\log x$ as $\lambda\rightarrow 0$, respectively. Note that we have the following relations:

$$e_{\lambda}^{x+y+\lambda xy}=e_{\lambda}^{x}e_{\lambda}^{y}, \quad \ln_{\lambda}xy=\ln_{\lambda}x+\ln_{\lambda}y+\lambda\ln_{\lambda}x\ln_{\lambda}y. \tag{1}$$

In this survey, we consider the linear bounded operator acting on the Hilbert space $\mathcal{H}$, or simply $n\times n$ complex matrix whose set is denoted by $M_{n}(\mathbb{C})$ in the finite quantum system. A Hermitian matrix $A$ is called a nonnegative matrix (and denoted by $A\geq 0$) if $\langle x, Ax\rangle\geq 0$ for all $x\in \mathbb{C}^{n}$. A nonnegative matrix $A$ is called a positive matrix (and denoted by $A>0$) if it is invertible. The set of all density matrices (quantum states) is represented by $D_{n}\equiv\{X\in M_{n}(\mathbb{C}) : X\geq 0, Tr[X]=1\}$. For $-I\leq X\leq I$ and $\lambda\in(-1,0)\cup(0,1)$, we denote the generalized exponential function by $\exp_{\lambda}(X)\equiv(I+\lambda X)^{1/\lambda}$. As the inverse function of $\exp_{\lambda}(\cdot)$, for $X\geq 0$ and $\lambda\in(-1,0)\cup(0,1)$, we denote the generalized logarithmic function by $\ln_{\lambda}X\equiv\frac{X^{\lambda}-I}{\lambda}$. Then the Tsallis relative entropy and Tsallis entropy for nonnegative matrices $X$ and $Y$ are defined by

$$D_{\lambda}(X|Y)\equiv Tr\left[X^{1-\lambda}(\ln_{\lambda}X-\ln_{\lambda}Y)\right], \quad S_{\lambda}(X)\equiv-D_{\lambda}(X|I).$$

These entropies are generalizations of the von Neumann entropy [36] and the Umegaki relative entropy [50] in the sense that

$$\lim_{\lambda\rightarrow 0}S_{\lambda}(X)=S_{0}(X)\equiv-Tr[X\log X]$$

and

$$\lim_{\lambda\rightarrow 0}D_{\lambda}(X|Y)=D_{0}(X|Y)\equiv Tr[X(\log X-\log Y)].$$

See [37] and [1, 2, 21, 22] for details of the theory of quantum entropy and the Tsallis entropies, respectively. Two Tsallis entropies have non-additivities such that

$$S_{\lambda}(X_{1}\otimes X_{2})=S_{\lambda}(X_{1})+S_{\lambda}(X_{2})+\lambda S_{\lambda}(X_{1})S_{\lambda}(X_{2}), \tag{2}$$

and

$$D_{\lambda}(X_{1}\otimes X_{2}|Y_{1}\otimes Y_{2})=D_{\lambda}(X_{1}|Y_{1})+D_{\lambda}(X_{2}|Y_{2})-\lambda D_{\lambda}(X_{1}|Y_{1})D_{\lambda}(X_{2}|Y_{2}), \tag{3}$$

due to the non-additivity Eq.(1) of the function $\ln_{\lambda}$. Thus the field of the study using these entropies is often called the non-additive statistical physics and many research papers have been published in mainly

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statistical physics [49]. See our previous papers [21, 22] on the mathematical properties of these entropies. See also chapter II written by A.K.Rajagopal in [49], for the quantum version of Tsallis entropies and their applications.

## 1.1 A uniqueness theorem Tsallis entropy in quantum system

As quite similar to the method in [14], we axiomatically characterize the Tsallis entropy in a finite quantum system. Let $T_\lambda$ be a mapping the set $\mathfrak{S}(\mathcal{H})$ of all density operators to the set $\mathbb{R}^+$ of all positive real values.

**Axiom 1.1** ([23]) We give the postulates which the Tsallis entropy should satisfy.

(T1) *Continuity:* For $\rho \in \mathfrak{S}(\mathcal{H})$, $T_\lambda(\rho)$ is a continuous function on the eigenvalues $x_j$ of $\rho$.

(T2) *Invariance:* For unitary transformation $U$, $T_\lambda(U^*\phi U) = T_\lambda(\phi)$.

(T3) *Generalized mixing condition:* For $\rho = \bigoplus_{k=1}^{n} \lambda_k\rho_k$ on $\mathcal{H} = \bigoplus_{k=1}^{n} \mathcal{H}_k$, where $x_k \geq 0, \sum_{k=1}^{n} x_k = 1, \rho_k \in \mathfrak{S}(\mathcal{H})$, we have the additivity:

$$T_\lambda(\rho) = \sum_{k=1}^{n} x_k^{1-\lambda} T_\lambda(\rho_k) + T_\lambda(x_1, \cdots, x_n),$$

where $(x_1, \cdots, x_n)$ represents the diagonal matrix $(x_k\delta_{kj})_{k,j=1,\cdots,n}$.

**Theorem 1.2** ([23]) If $T_\lambda$ satisfies Axiom 1.1, then $T_\lambda$ is uniquely given by the following form

$$T_\lambda(\rho) = \mu_\lambda S_\lambda(\rho),$$

with a positive constant number $\mu_\lambda$ depending on parameter $\lambda$.

## 1.2 Properties of Tsallis relative entropy

Here we review some information-theoretical properties of the Tsallis relative entropy defined in quantum system. For the quantum Tsallis relative entropy $D_\lambda(\rho|\sigma)$ and the quantum relative entropy $U(\rho|\sigma)$, the following relations are known.

**Proposition 1.3** (Ruskai-Stillinger [43] (see also [37])) For density operators $\rho$ and $\sigma$, we have,

(1) $D_{1-\lambda}(\rho|\sigma) \leq U(\rho|\sigma) \leq D_{1+\lambda}(\rho|\sigma)$ for $0 < \lambda \leq 1$.

(2) $D_{1+\lambda}(\rho|\sigma) \leq U(\rho|\sigma) \leq D_{1-\lambda}(\rho|\sigma)$ for $-1 \leq \lambda < 0$.

Note that the both sides in the both inequalities converge to $U(\rho|\sigma)$ as $\lambda \to 0$.

We next consider another relation on the quantum Tsallis relative entropy. In [12], the relative operator entropy was defined by

$$S(A|B) \equiv A^{1/2} \log(A^{-1/2}BA^{-1/2})A^{1/2},$$

for two strictly positive operators $A$ and $B$. If $A$ and $B$ are commutative, then we have $U(A|B) = -Tr[S(A|B)]$. For this relative operator entropy and the quantum relative entropy $U(A|B)$, Hiai and Petz proved the following relation :

$$U(A|B) \leq -Tr[S(A|B)], \tag{4}$$

in [28] (see also [29]).

In our previous papers [53], we introduced the Tsallis relative operator entropy $T_\lambda(A|B)$ as a parametric extension of the relative operator entropy $S(A|B)$ such as

$$T_\lambda(A|B) \equiv \frac{A^{1/2}(A^{-1/2}BA^{-1/2})^\lambda A^{1/2} - A}{\lambda},$$

for $0 < \lambda \leq 1$ and strictly positive operators $A$ and $B$, in the sense that

$$\lim_{\lambda \to 0} T_\lambda(A|B) = S(A|B). \tag{5}$$

Also we now have that

$$\lim_{\lambda \to 0} D_\lambda(A|B) = U(A|B). \tag{6}$$

These relations Eq.(4), Eq.(5) and Eq.(6) naturally lead us to show the following theorem as a parametric extension of Eq.(4).

**Theorem 1.4** **([21])** For $0 < \lambda \leq 1$ and any strictly positive operators $A$ and $B$, we have

$$D_\lambda(A|B) \leq -Tr[T_\lambda(A|B)] \tag{7}$$

Thus the result proved by Hiai and Petz in [28, 29] is recovered as a special case of Theorem 1.4 in the limit $\lambda \to 0$. On the quantum Tsallis relative entropy $D_\lambda(\rho|\sigma)$ for density operators $\rho$ and $\sigma$, we have the following properties.

**Proposition 1.5** **([21, 1, 40])** For $0 < \lambda \leq 1$ and any density operators $\rho$ and $\sigma$, the quantum relative entropy $D_\lambda(\rho|\sigma)$ has the following properties.

(1) (Non-negativity) $D_\lambda(\rho|\sigma) \geq 0$.

(2) (Non-additivity) $D_\lambda(\rho_1 \otimes \rho_2|\sigma_1 \otimes \sigma_2) = D_\lambda(\rho_1|\sigma_1) + D_\lambda(\rho_2|\sigma_2) - \lambda D_\lambda(\rho_1|\sigma_1) D_\lambda(\rho_2|\sigma_2)$.

(3) (Joint convexity) $D_\lambda(\sum_j \lambda_j \rho_j | \sum_j \lambda_j \sigma_j) \leq \sum_j \lambda_j D_\lambda(\rho_j|\sigma_j)$.

(4) (Unitary invariance) The quantum Tsallis relative entropy is invariant under the unitary transformation $U$ :

$$D_\lambda(U\rho U^*|U\sigma U^*) = D_\lambda(\rho|\sigma).$$

(5) (Monotonicity) For any trace-preserving completely positive linear map $\Phi$, any density operators $\rho$ and $\sigma$ and $0 < \lambda \leq 1$, we have

$$D_\lambda(\Phi(\rho)|\Phi(\sigma)) \leq D_\lambda(\rho|\sigma). \tag{8}$$

Putting $\sigma = \frac{1}{n} I$ in Eq.(8), we have the following corollary.

**Corollary 1.6** For any trace-preserving completely positive linear unital map $\Phi$, any density operator $\rho$ and $0 < \lambda \leq 1$, we have

$$S_\lambda(\Phi(\rho)) \geq S_\lambda(\rho).$$

We note that Eq.(8) for the fixed $\sigma$, namely the monotonicity of the quantum Tsallis relative entropy in the case of $\Phi(\sigma) = \sigma$, was proved in [4] to establish Clausius' inequality.

**Remark 1.7** It is known [33] (see also [44]) that there is an equivalent relation between the monotonicity for the quantum relative entropy and the strong subadditivity for the quantum entropy. However in our case, we have not yet found such a relation. Because the non-additivity of $\lambda$-logarithm function

$$\ln_\lambda xy = \ln_\lambda x + \ln_\lambda y + \lambda \ln_\lambda x \ln_\lambda y$$

disturbs us to derive the beautiful relation such as

$$D_\lambda(p(x,y)|p(x)p(y)) = S_\lambda(p(x)) + S_\lambda(p(y)) - S_\lambda(p(x,y))$$

for the Tsallis relative entropy $D_\lambda(p(x,y)|p(x)p(y))$, the Tsallis entropy $S_\lambda(p(x))$, $S_\lambda(p(y))$ and the Tsallis joint entropy $S_\lambda(p(x,y))$, even if our stage is in classical system.

### 1.3 Maximum entropy principle in Tsallis statistics

Here, we review the derivation of the maximum entropy principle for a density matrix in Tsallis statistics.

**Theorem 1.8** ([16]) Let $Y = Z_\lambda^{-1} \exp_\lambda(-H/\|H\|)$, where $Z_\lambda \equiv Tr[\exp_\lambda(-H/\|H\|)]$, for $\lambda \in (-1,0) \cup (0,1)$ and a Hermitian matrix $H$. We denote

$$C_\lambda \equiv \left\{X \in D_n : Tr[X^{1-\lambda}H] \leq Tr[Y^{1-\lambda}H]\right\}.$$

If $X \in C_\lambda$, then $S_\lambda(X) \leq S_\lambda(Y)$.

**Remark 1.9** Since $-x^{1-\lambda} \ln_\lambda x$ is a strictly concave function, $S_\lambda$ is a strictly concave function on the set $C_\lambda$. This means that the maximizer $Y$ is uniquely determined so that we may regard $Y$ as a generalized Gibbs state. Thus we may define a generalized Helmholtz free energy such by

$$F_\lambda(X,H) \equiv Tr[X^{1-\lambda}H] - \|H\|\, S_\lambda(X).$$

This can be also represented by the Tsallis relative entropy such as

$$F_\lambda(X,H) = \|H\|\, D_\lambda(X|Y) + \ln_\lambda Z_\lambda^{-1} Tr[X^{1-\lambda}(\|H\| - \lambda H)].$$

We straightforwardly have the following corollary by taking the limit as $\lambda \to 0$.

**Corollary 1.10** ([46]) Let $Y = Z_0^{-1} \exp(-H/\|H\|)$, where $Z_0 \equiv Tr[\exp(-H/\|H\|)]$, for a Hermitian matrix $H$. If $X \in C_0$, then

$$S_0(X) \leq S_0(Y).$$

## 2 Trace inequalities related to Tsallis entropies

### 2.1 A variational expression for Tsallis relative entropy

In this subsection, we review of the derivation of a variational expression for the Tsallis relative entropy as a parametric extension of that of the relative entropy in Lemma 1.2 of [29]. A variational expression of the relative entropy has been studied in the general setting of von Neumann algebras [41, 32].

In the below, we sometimes relax the condition of the unital trace for the matrices in the definition of the Tsallis relative entropy $D_\lambda(\cdot|\cdot)$, since it is not essential in the mathematical studies of the entropic functionals.

**Theorem 2.1** ([15]) For $\lambda \in (0,1]$, we have the following relations.

(1) If $A$ and $Y$ are nonnegative matrices, then

$$\ln_\lambda Tr[e_\lambda^{A+\ln_\lambda Y}] = \max\left\{Tr[X^{1-\lambda}A] - D_\lambda(X|Y) : X \geq 0, Tr[X] = 1\right\}.$$

(2) If $X$ is a positive matrix with $Tr[X] = 1$ and $B$ is a Hermitian matrix, then

$$D_\lambda(X|e_\lambda^B) = \max\left\{Tr[X^{1-\lambda}A] - \ln_\lambda Tr[e_\lambda^{A+B}] : A \geq 0\right\}.$$

Taking the limit as $\lambda \to 0$, Theorem 2.1 recovers the similar form of Lemma 1.2 in [29] under the assumption of non-negativity of $A$. If $Y = I$ and $B = 0$ in (1) and (2) of Theorem 2.1, respectively, then we obtain the following corollary.

**Corollary 2.2** (1) If $A$ is a nonnegative matrix, then

$$\ln_\lambda Tr[e_\lambda^A] = \max\left\{Tr[X^{1-\lambda}A] + S_\lambda(X) : X \geq 0,\, Tr[X] = 1\right\}.$$

(2) For a density matrix $X$, we have

$$-S_\lambda(X) = \max\left\{Tr[X^{1-\lambda}A] - \ln_\lambda Tr[e_\lambda^A] : A \geq 0\right\}.$$

Taking the limit as $\lambda \to 0$, Corollary 2.2 recovers the similar form of Theorem 1 in [6] under the assumption of non-negativity of $A$.

## 2.2 Generalized logarithmic and exponential trace inequalities

In this subsection, we derive some trace inequalities in terms of the results obtained in the previous section. That is, the results of this subsection are derived by the application of a variational expression for the Tsallis relative entropy. From (1) of Corollary 2.2, we have the generalized thermodynamic inequality:

$$\ln_\lambda Tr[e_\lambda^H] \geq Tr[D^{1-\lambda}H] + S_\lambda(D), \tag{9}$$

for a density matrix $D$ and a nonnegative matrix $H$. Putting $D = \frac{A}{Tr[A]}$ and $H = \ln_\lambda B$ in Eq.(9) for $A \geq 0$ and $B \geq I$, we have the generalized Peierls-Bogoliubov inequality (cf.Theorem 3.3 of [21]):

$$(Tr[A])^{1-\lambda}(\ln_\lambda Tr[A] - \ln_\lambda Tr[B]) \leq Tr[A^{1-\lambda}(\ln_\lambda A - \ln_\lambda B)], \tag{10}$$

for nonnegative matrices $A$ and $B \geq I$.

**Lemma 2.3** The following statements are equivalent.

(1) $F_\lambda(A) = \ln_\lambda Tr[e_\lambda^A]$ is convex in a Hermitian matrix $A$.

(2) $f_\lambda(t) = \ln_\lambda Tr[e_\lambda^{A+tB}]$ is convex in $t \in \mathbb{R}$.

**Corollary 2.4** For nonnegative matrices $A$ and $B$, we have

$$\ln_\lambda Tr[e_\lambda^{A+B}] - \ln_\lambda Tr[e_\lambda^A] \geq \frac{Tr[B(e_\lambda^A)^{1-\lambda}]}{(Tr[e_\lambda^A])^{1-\lambda}}. \tag{11}$$

For nonnegative real numbers $x, y$ and $0 < \lambda \leq 1$, the relations $e_\lambda^{x+y} \leq e_\lambda^{x+y+\lambda xy} = e_\lambda^x e_\lambda^y$ hold. These relations naturally motivate us to consider the following inequalities in the non-commutative case.

**Proposition 2.5** ([15]) For non-negative matrices $X$ and $Y$, and $0 < \lambda \leq 1$, we have

$$Tr[e_\lambda^{X+Y}] \leq Tr[e_\lambda^{X+Y+\lambda Y^{1/2}XY^{1/2}}].$$

Note that we have the matrix inequality :

$$e_\lambda^{X+Y} \leq e_\lambda^{X+Y+\lambda Y^{1/2}XY^{1/2}}$$

for $\lambda \geq 1$ by the application of the Löwner-Heinz inequality [34, 27, 38].

**Proposition 2.6** ([15]) For nonnegative matrices $X, Y$, and $\lambda \in (0, 1]$, we have

$$Tr[e_\lambda^{X+Y+\lambda XY}] \leq Tr[e_\lambda^X e_\lambda^Y]. \tag{12}$$

Notice that Golden-Thompson inequality [26, 47],

$$Tr[e^{X+Y}] \leq Tr[e^X e^Y]$$

which holds for Hermitian matrices $X$ and $Y$, is recovered by taking the limit as $\lambda \to 0$ in Proposition 2.6, in particular case of nonnegative matrices $X$ and $Y$.

Since $Tr[HZHZ] \leq Tr[H^2Z^2]$ for Hermitian matrices $H$ and $Z$ [41, 10], we have for nonnegative matrices $X$ and $Y$,

$$Tr[(I + X + Y + Y^{1/2}XY^{1/2})^2] \leq Tr[(I + X + Y + XY)^2]$$

by easy calculations. This implies the inequality

$$Tr[e_{1/2}^{X+Y+1/2Y^{1/2}XY^{1/2}}] \leq Tr[e_{1/2}^{X+Y+1/2XY}].$$

Thus we have

$$Tr[e_{1/2}^{X+Y}] \leq Tr[e_{1/2}^X e_{1/2}^Y] \tag{13}$$

from Proposition 2.5 and Proposition 2.6. Putting $B = \ln_{1/2} Y$ and $A = \ln_{1/2} Y^{-1/2}XY^{-1/2}$ in (2) of Theorem 2.1 under the assumption of $I \le Y \le X$ and using Eq.(13), we have

$$\begin{aligned} D_{1/2}(X|Y) &= D_{1/2}(X|e_{1/2}^{\ln_{1/2} Y}) \\ &\ge Tr[X^{1/2}A] - \ln_{1/2} Tr[e_{1/2}^{A+B}] \\ &\ge Tr[X^{1/2}A] - \ln_{1/2} Tr[e_{1/2}^{A} e_{1/2}^{B}] \\ &= Tr[X^{1/2} \ln_{1/2} Y^{-1/2}XY^{-1/2}] - \ln_{1/2} Tr[Y^{-1/2}XY^{-1/2}Y] \\ &= Tr[X^{1/2} \ln_{1/2} Y^{-1/2}XY^{-1/2}], \end{aligned} \tag{14}$$

which gives a lower bound of the Tsallis relative entropy in the case of $\lambda = 1/2$ and $I \le Y \le X$.

## 2.3 Hiai-Petz type trace inequalities

In this subsection, we consider an extension of the following inequality:

$$Tr[X(\log X + \log Y)] \le \frac{1}{p} Tr[X \log X^{p/2} Y^p X^{p/2}] \tag{15}$$

for nonnegative matrices $X$ and $Y$, and $p > 0$.

**Theorem 2.7** ([16])

(1) For positive matrices $X$ and $Y$, $p \ge 1$ and $0 < \lambda \le 1$, we have

$$Tr[X^{1-\lambda}(\ln_\lambda X - \ln_\lambda Y)] \le -Tr[X \ln_\lambda (X^{-p/2} Y^p X^{-p/2})^{1/p}]. \tag{16}$$

(2) For positive matrices $X$ and $Y$, $0 < p < 1$ and $0 < \lambda \le 1$, the following inequality dose not hold:

$$Tr[X^{1-\lambda}(\ln_\lambda X - \ln_\lambda Y)] \le -Tr[X \ln_\lambda (X^{-p/2} Y^p X^{-p/2})^{1/p}] \tag{17}$$

**Corollary 2.8** (1) For positive matrices $X$ and $Y$, the trace inequality

$$D_\lambda(X|Y) \le -Tr[X \ln_\lambda (X^{-1/2} Y X^{-1/2})] \tag{18}$$

holds.

(2) For positive matrices $X$ and $Y$, and $p \ge 1$, we have the inequality (15).

## 2.4 A generalized Fannes's inequality

We give an upper bound of the Tsallis entropy. To do so, we state a few lemmas.

**Lemma 2.9** For a density operator $\rho$ on $\mathcal{H}$, we have

$$S_q(\rho) \le \ln_q d,$$

where $d = \dim \mathcal{H} < \infty$.

**Lemma 2.10** If $f$ is a concave function and $f(0) = f(1) = 0$, then we have

$$|f(t+s) - f(t)| \le \max\{f(s), f(1-s)\}$$

for any $s \in [0, 1/2]$ and $t \in [0, 1]$ satisfying $0 \le s + t \le 1$.

**Lemma 2.11** For any real number $u, v \in [0, 1]$ and $q \in [0, 2]$, if $|u - v| \le \frac{1}{2}$, then $|\eta_q(u) - \eta_q(v)| \le \eta_q(|u - v|)$, where $\eta_q(x) \equiv \frac{x^q - x}{1-q}$.

**Theorem 2.12** ([23]) For two density operators $\rho_1$ and $\rho_2$ on $\mathcal{H}$ and $q \in [0,2]$, if $\|\rho_1 - \rho_2\|_1 \le q^{1/(1-q)}$, then

$$|S_q(\rho_1) - S_q(\rho_2)| \le \|\rho_1 - \rho_2\|_1^q \ln_q d + \eta_q(\|\rho_1 - \rho_2\|_1).$$

Where we denote $\|A\|_1 \equiv \mathrm{Tr}\left[(A^*A)^{1/2}\right]$ for a bounded linear operator $A$.

By taking the limit as $q \to 1$, we have the following Fannes's inequality (see pp.512 of [35], also [8, 5, 37]) as a corollary, since $\lim_{q\to 1} q^{1/(1-q)} = \frac{1}{e}$.

**Corollary 2.13** For two density operators $\rho_1$ and $\rho_2$ on $\mathcal{H}$, if $\|\rho_1 - \rho_2\|_1 \le \frac{1}{e}$, then

$$|S_1(\rho_1) - S_1(\rho_2)| \le \|\rho_1 - \rho_2\|_1 \ln d + \eta_1(\|\rho_1 - \rho_2\|_1),$$

where $S_1$ represents von Neumann entropy and $\eta_1(x) = -x \ln x$.

# 3 Operator inequalities related to the Tsallis entropies

## 3.1 Properties of Tsallis relative operator entropy

In this subsection, we give several fundamental properties of Tsallis relative operator entropy defined in Section 1:

$$T_\lambda(A|B) = A^{1/2} \ln_\lambda (A^{-1/2} B A^{-1/2}) A^{1/2}.$$

for two positive operators $A$ and $B$ and $\lambda \in (0,1]$. along the line of the paper [12]. Note that more general operator has been introduced in [13], it was named *solidarity* and then several properties were shown in [13, 11, 9].

**Proposition 3.1** ([11, 9, 13])

(1) (Homogeneity) $T_\lambda(\alpha A|\alpha B) = \alpha T_\lambda(A|B)$ for any positive number $\alpha$.

(2) (Monotonicity) If $B \le C$, then $T_\lambda(A|B) \le T_\lambda(A|C)$.

(3) (Superadditivity) $T_\lambda(A_1 + A_2|B_1 + B_2) \ge T_\lambda(A_1|B_1) + T_\lambda(A_2|B_2)$.

(4) (Joint concavity) $T_\lambda(\alpha A_1 + \beta A_2|\alpha B_1 + \beta B_2) \ge \alpha T_\lambda(A_1|B_1) + \beta T_\lambda(A_2|B_2)$.

(5) (Unitary invariance)

$$T_\lambda(UAU^*|UBU^*) = UT_\lambda(A|B)U^*$$

for any unitary operator $U$.

(6) (Monotonicity) For a unital positive linear map $\Phi$ from the set of the bounded linear operators on Hilbert space to itself, we have

$$\Phi(T_\lambda(A|B)) \le T_\lambda(\Phi(A)|\Phi(B)).$$

(7) (Bounds) For any invertible positive operator $A$ and $B$, $0 < \lambda \le 1$, we have

$$T_{-\lambda}(A|B) \le S(A|B) \le T_\lambda(A|B), \quad T_\lambda(A|B) \le H_\lambda(A) + A^{1-\lambda} \ln_\lambda \|B\|,$$

and

$$\mu A \le B \Longrightarrow T_\lambda(A|B) \ge (\ln_\lambda \mu) A.$$

Moreover we have the following bounds for Tsallis relative operator entropy.

**Lemma 3.2** For any positive real number $x$ and $0 < \lambda \le 1$, the following inequalities hold

$$1 - \frac{1}{x} \le \ln_\lambda x \le x - 1.$$

**Proposition 3.3** ([13]) For any invertible positive operator $A$ and $B$, $0 < \lambda \leq 1$,

$$A - AB^{-1}A \leq T_\lambda(A|B) \leq -A + B. \tag{19}$$

Moreover, $T_\lambda(A|B) = 0$ if and only if $A = B$.

Finally we prove the further bounds of Tsallis relative operator entropy with the $\lambda$-power mean $\sharp_\lambda$.

**Lemma 3.4** For any positive real number $\alpha$ and $x$ and $0 < \lambda \leq 1$, the inequalities hold

$$x^\lambda\left(1 - \frac{1}{\alpha x}\right) + \ln_\lambda \frac{1}{\alpha} \leq \ln_\lambda x \leq \frac{x}{\alpha} - 1 - x^\lambda \ln_\lambda \frac{1}{\alpha}. \tag{20}$$

The equality of the right hand side of the above inequalities hold if and only if $x = \alpha$. The equality of the left hand side of the above inequalities hold if and only if $x = \frac{1}{\alpha}$.

**Theorem 3.5** ([22]) For any invertible positive operators $A$ and $B$, and any positive real number $\alpha$, the following inequality holds

$$A\sharp_\lambda B - \frac{1}{\alpha}A\sharp_{\lambda-1}B + (\ln_\lambda \frac{1}{\alpha})A \leq T_\lambda(A|B) \leq \frac{1}{\alpha}B - A - (\ln_\lambda \frac{1}{\alpha})A\sharp_\lambda B. \tag{21}$$

The equality of the right hand side of the above inequalities holds if and only if $B = \alpha A$. The equality of the left hand side of the above inequalities holds if and only if $A = \alpha B$. We have that $T_\lambda(A|B) = 0$ is equivalent to $A = B$.

**Remark 3.6** We note that Eq.(21) recovers the inequalities shown in [24] :

$$(1 - \log\alpha)A - \frac{1}{\alpha}AB^{-1}A \leq S(A|B) \leq (\log\alpha - 1)A + \frac{1}{\alpha}B$$

as $\lambda \to 0$. Moreover, if we put $\alpha = 1$, then we have

$$A - AB^{-1}A \leq S(A|B) \leq B - A$$

which recover the inequalities of Corollary 5 in [13], cf. Eq.(19).

Taking account for the non-additivity (Eq.(2) and Eq.(3)) which are the typical features of Tsallis entropies, we consider the Tsallis relative operator entropy of two positive operator of the tensor product $A_1 \otimes A_2$ and $B_1 \otimes B_2$. To show our theorem, we state the following lemma for the convenience of the readers.

**Theorem 3.7** ([22]) For any $0 < \lambda \leq 1$ and any strictly positive operators $A_1, A_2, B_1$ and $B_2$, we have

$$T_\lambda(A_1 \otimes A_2|B_1 \otimes B_2) = T_\lambda(A_1|B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2|B_2) + \lambda T_\lambda(A_1|B_1) \otimes T_\lambda(A_2|B_2). \tag{22}$$

Taking the limit as $\lambda \to 0$ in Theorem 3.7, we have the following corollary.

**Corollary 3.8** For any strictly positive operators $A_1, A_2, B_1$ and $B_2$, we have

$$S(A_1 \otimes A_2|B_1 \otimes B_2) = S(A_1|B_1) \otimes A_2 + A_1 \otimes S(A_2|B_2).$$

Since we have $T_\lambda(A|B) \geq 0$ for any $B \geq A$, $T_\lambda(A|B) \leq 0$ for any $B \leq A$ and we have $X \otimes Y \geq 0$ for any $X \geq 0$ and $Y \geq 0$, Theorem 3.7 implies the following corollary.

**Corollary 3.9** (1) For any $0 < \lambda \leq 1$ and $0 < A_i \leq B_i$, $(i = 1, 2)$, we have the following inequalities.

(a) $T_\lambda(A_1 \otimes A_2|B_1 \otimes B_2) \geq \lambda T_\lambda(A_1|B_1) \otimes T_\lambda(A_2|B_2)$.

(b) $T_\lambda(A_1 \otimes A_2|B_1 \otimes B_2) \geq T_\lambda(A_1|B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2|B_2)$.

(2) For any $0 < \lambda \leq 1$ and $0 < B_i \leq A_i$, $(i = 1, 2)$, we have the following inequalities.

(c) $T_\lambda(A_1 \otimes A_2|B_1 \otimes B_2) \leq \lambda T_\lambda(A_1|B_1) \otimes T_\lambda(A_2|B_2)$.

(d) $T_\lambda(A_1 \otimes A_2|B_1 \otimes B_2) \geq T_\lambda(A_1|B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2|B_2)$.

### 3.2 Generalized Shannon inequalities based on Tsallis relative operator entropy

The fundamental properties of $T_\lambda(A|B)$ are shown in the previous subsection. In this subsection we give the Shannon type operator inequality and its reverse one satisfied by Tsallis relative operator entropy. To do so, we need the following lemma.

**Theorem 3.10** ([53]) Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $H$. If $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then

$$0 \geq \sum_{j=1}^{n} T_\lambda(A_j|B_j) \geq \frac{\left(\sum_{j=1}^{n} A_j B_j^{-1} A_j\right)^{-\lambda} - I}{\lambda}.$$

We also obtain the operator version of the Shannon inequality and reverse one given by Furuta [25] as a corollary of Theorem 3.10 in the following.

**Corollary 3.11** ([25]) Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $H$. If $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then

$$0 \geq \sum_{j=1}^{n} A_j^{1/2}\left(\log A_j^{-1/2} B_j A_j^{-1/2}\right) A_j^{1/2} \geq -\log\left(\sum_{j=1}^{n} A_j B_j^{-1} A_j\right).$$

Actually the above Corollary 3.11 is a part of the Corollary 2.4 in [25].

## 4 Applications of Tsallis entropies

Finally we review the application of the Tsallis relative entropy as a measure of entanglements. The concept of entanglement has been important in quantum information theory, especially quantum teleportation and quantum computing and so on. Therefore it is important to quantify the degree of entanglement, in order to scientifically treat the concept of entanglement. Here we give the definition of a separable (disentangled) state and an entangled state in the following [45, 39, 31, 30].

**Definition 4.1** A state $\kappa$ acting on the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called a separable (disentangled) state if it is represented by

$$\kappa = \sum_i p_i \kappa_1^i \otimes \kappa_2^i,\; p_i \geq 0,\; \sum_i p_i = 1,$$

for states $\kappa_1^i$ and $\kappa_2^i$ acting on the subsystems $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. It is also called an entangled state if a state is not a separable state.

When we quantify the entanglement, we should pay the attention whether the considering entangled state is pure or mixed. For pure entangled states, it is easily calculated by the von Neumann entropy of the reduced states. For mixed entangled states, we have several measure of entanglements.

**Definition 4.2** ([7]) For mixed entangled states $\sigma = \sum_i p_i \sigma^{(i)}$, where $\sum_i p_i = 1$, $p_i \geq 0$ and $\sigma^{(i)} = |\phi_i\rangle\langle\phi_i|$ are pure entangled states on $\mathcal{H}_1 \otimes \mathcal{H}_2$, the entanglement of formation is defined by

$$E^F(\sigma) \equiv \min \sum_i p_i S(\sigma_1^{(i)})$$

as a minimum of the average of the von Neumann entropy $S(\sigma_1^{(i)})$ of the reduced states $\sigma_1^{(i)}$ for the pure entangled states $\sigma^{(i)}$, where the minimum is taken over all the possible states $\sigma = \sum_i p_i \sigma^{(i)}$ with $\sigma_1^{(i)} = tr_{\mathcal{H}_2} \sigma^{(i)}$.

**Definition 4.3** ([52]) For mixed entangled states $\sigma$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, the relative entropy of entanglement is defined by

$$E^R(\sigma) \equiv \min_{\rho \in \mathcal{D}} U(\sigma|\rho),$$

where the minimum is taken for all $\rho \in \mathcal{D}$, where $\mathcal{D}$ represents the set of all separable (disentangled) states on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

This measure is a kind of distance (difference) between the entangled states $\sigma$ and the separable (disentangled) states $\rho$. They also proposed the conditions that the entanglement-measure $E(\sigma)$ for any entangled states $\sigma$ on the total system $\mathcal{H}_1 \otimes \mathcal{H}_2$ should satisfy. It is given in [52] by

(E1) $E(\sigma) = 0 \Leftrightarrow \sigma$ is separable.

(E2) $E(\sigma)$ is invariant under the local unitary operations:

$$E(\sigma) = E(U_1 \otimes U_2 \sigma U_1^* \otimes U_2^*),$$

where $U_i, (i = 1, 2)$ represent the unitary operators acting on $\mathcal{H}_i$, $(i = 1, 2)$.

(E3) The measure of entanglement $E(\sigma)$ can not be increased under the trace-preserving completely positive map given by $\Phi$. That is,

$$E(\Phi\sigma) \le E(\sigma).$$

As a measure satisfying the above conditions, a special case of V.Vedral's definition, we introduced the entanglement degree due to the mutual entropy and then applied it to the analysis of the Jaynes-Cummings model in [20, 18, 19].

**Definition 4.4** ([20, 18, 19]) For mixed entangled states $\sigma$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, the entanglement degree due to the mutual entropy is given by

$$E^M(\sigma) \equiv Tr[\sigma(\log \sigma - \log \rho_1 \otimes \rho_2)],$$

where $\rho_1 \equiv tr_{\mathcal{H}_2}\sigma$ and $\rho_2 \equiv tr_{\mathcal{H}_1}\sigma$.

In the above definition, we fixed the separable (disentangled) state such as $\rho = tr_{\mathcal{H}_2}\sigma \otimes tr_{\mathcal{H}_1}\sigma$, because it was difficult to find the separable (disentangled) state attaining the minimum value of the relative entropy of entanglement. The separable (disentangled) state chosen by $\rho = tr_{\mathcal{H}_2}\sigma \otimes tr_{\mathcal{H}_1}\sigma$ is nontrivial state but our measure contains both quantum and classical entanglement. That is, our measure takes greater value than V.Vedral's one. That is, from the definitions, we easily find $E^R(\sigma) \le E^M(\sigma)$. For example, for pure entangled states, by the above Araki-Lieb's triangle inequality, we easily find that our measure is equal to the twice of von Neumann entropy, namely $E^M(\sigma) = 2E^R(\sigma)$ for pure entangled states $\sigma$, since $E^R(\sigma)$ becomes von Neumann entropy for pure entangled states [51]. However it was sufficient to get the rough degree of entanglement for the analysis of the time development of the Janeys-Cummings model.

In the previous paper, we adopted a parametrically extended entanglement-measure due to the Tsallis relative entropy which is a generalization of our previous entanglement-measure.

**Definition 4.5** ([3, 17]) For mixed entangled states $\sigma$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, the entanglement-measure due to the Tsallis relative entropy is given by

$$E_\lambda^T(\sigma) \equiv \frac{Tr[\sigma - \sigma^{1-\lambda}(\rho_1 \otimes \rho_2)^\lambda]}{\lambda},$$

where $0 < \lambda \le 1$, $\rho_1 \equiv tr_{\mathcal{H}_2}\sigma$ and $\rho_2 \equiv tr_{\mathcal{H}_1}\sigma$.

Note that the above entanglement-measure is a special version of the generalized Kullback-Leibler measure of quantum entanglement introduced in [3]. In addition, the above entanglement-measure for a non-trivial example was studied in [3]. From the definition, we easily find that $\lim_{q\to 1} E_q^T(\sigma) = E^M(\sigma)$.

In the below, we show the equality condition of the inequality ((1) of Proposition 1.5) in the properties of the Tsallis relative entropy.

**Lemma 4.6** For $\lambda \in [-1,0) \cup (0,1]$ and density operators $\rho$ and $\sigma$, we have

$$D_\lambda(\rho|\sigma) \geq 0,$$

with equality if and only if $\rho = \sigma$.

**Proposition 4.7** ([17]) For $0 < \lambda \leq 1$, $E_q^T$ satisfies the conditions (E1), (E2) and (E3)

We also have the following proposition.

**Proposition 4.8** ([17])

(1) For any entangled states $\sigma$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, we have $E_0^T(\sigma) = 0$.

(2) There exists $\lambda$ in $(0,1]$ such that $E_\lambda^T(\sigma) = E^R(\sigma)$ for any entangled states $\sigma$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

(3) For any entangled states $\sigma$ and $\sigma'$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and any $0 < \lambda \leq 1$, we have the subadditivity:

$$E_\lambda^T(\sigma \otimes \sigma') \leq E_\lambda^T(\sigma) + E_\lambda^T(\sigma').$$

We note that we have the additivity $E^M(\sigma \otimes \sigma') = E^M(\sigma) + E^M(\sigma')$ as $\lambda \to 0$. In addition, we should note that our measure $E_\lambda^T(\sigma)$ takes 0 when $\lambda = 1$, although $\sigma$ is not separable (disentangled) state.

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